Solving Linear and Integer Programs

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DAY 2

Dual Simplex Algorithm

Some Motivation

- Dual simplex vs. primal (2002): Dual > 2x faster
- CPLEX 9.0 – 2003
  - Primal 1.2x improvement
  - Dual 1.7x improvement
- Best algorithm of MIP
- There isn't much in books about implementing the dual.

Dual Simplex Algorithm

(Lemke, 1954: Commercial codes ~1990)

Input: A dual feasible basis B and vectors

$$X_B = A_B^{-1}b \quad \text{and} \quad D_N = c_N - A_N^T B^{-T} c_B.$$

- **Step 1:** (Pricing) If $X_B \geq 0$, stop, B is optimal; else let

  $i = argmin\{X_{Bk} : k \in \{1, \ldots, m\}\}.$
- **Step 2:** (BTRAN) Solve $B^T z = e_i$. Compute $\alpha_N = -A_N^T z$.
- **Step 3:** (Ratio test) If $\alpha_N \leq 0$, stop, (D) is unbounded; else, let

  $j = argmin\{D_k/\alpha_k : \alpha_k > 0\}.$
- **Step 4:** (FTRAN) Solve $A_B y = A_j$.
- **Step 5:** (Update) Set $B_i = j$. Update X_B (using y) and D_N (using α_N)

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Implementing the Dual Simplex Algorithm

Implementation Issues for Dual Simplex

1. **Finding an initial feasible basis, or the concluding that there is none**
2. **Pricing:** Dual steepest edge
3. **Solving the linear systems**
   - LU factorization and factorization update
   - BTRAN and FTRAN – exploiting sparsity
4. **Numerically stable ratio test:** Bound shifting and perturbation
5. **Bound flipping:** Exploiting "boxed" variables to combine many iterations into one.

Issue 0
Preparation: Bounds on Variables

In practice, simplex algorithms need to accept LPs in the following form:

$$\begin{aligned} &\textit{Minimize} && c^T x \\ &\textit{Subject to} && Ax = b \qquad (\mathrm{P_{BD}}) \\ & && l \le x \le u \end{aligned}$$

where l is an n-vector of **lower bounds** and u an n-vector of **upper bounds**. l is allowed to have $-\infty$ entries and u is allowed to have $+\infty$ entries. (Note that (P_{BD}) is in standard form if $l_j = 0,\ u_j = +\infty\ \forall\ j$.)

(Issue 0 – Bounds on variables)
Basic Solution

A **basis** for (P_{BD}) is a triple (B,L,U) where B is an ordered m-element subset of $\{1,\ldots,n\}$ (just as before), (B,L,U) is a partition of $\{1,\ldots,n\}$, $l_j > -\infty\ \forall\ j \in L$, and $u_j < +\infty\ \forall\ j \in U$. $N = L \cup U$ is the set of **nonbasic** variables. The associated (**primal**) **basic solution** X is given by $X_L = l_L$, $X_U = u_U$ and

$$X_B = A_B^{-1}(b - A_L l_L - A_U u_U).$$

This solution is **feasible** if

$$l_B \le X_B \le u_B.$$

The associated **dual basic solution** is defined exactly as before: $D_B = 0,\ \Pi^T A_B = c_B^T,\ D_N = c_N - A_N^T \Pi$. It is **dual feasible** if

$$D_L \ge 0 \ \text{ and } \ D_U \le 0.$$

(Issue 0 – Bounds on variables)
The Full Story

- **Modify simplex algorithm**
  - Only the "Pricing" and "Ratio Test" steps much be changed substantially.
  - The complicated part is the ratio test
- **Reference:** See Chvátal for the primal

Issue 1
The Initial Feasible Basis – Phase I

- **Two parts to the solution**
  1. Finding some initial basis (probably not feasible)
  2. Modified simplex algorithm to find a feasible basis

Reference for Primal: **R.E. Bixby (1992). "Implementing the simplex method: the initial basis",** ***ORSA Journal on Computing*** 4**, 267—284.**

(Issue 1 – Initial feasible basis)
Initial Basis

- Primal and dual bases are the same. We begin in the context of the primal. Consider

$$\begin{aligned} &\text{Minimize} && c^T x \\ &\text{Subject to} && Ax = b \qquad (P_{BD}) \\ & && l \le x \le u \end{aligned}$$

- **Assumption:** Every variable has some finite bound**.**
- **Trick:** Add **artificial variables** $x_{n+1}, \ldots, x_{n+m}$:

$$Ax + I \begin{pmatrix} x_{n+1} \\ \cdot \\ \cdot \\ x_{n+m} \end{pmatrix} = b$$

where $l_j = u_j = 0$ for $j = n+1,\ldots,n+m$.

- **Initial basis:** $B = (n+1,\ldots,n+m)$ and for each $j \notin B$, pick some finite bound and place j in L or U, as appropriate.

(Issue 1 – Initial feasible basis)
Solving the Phase I

- If the initial basis is not dual feasible, we consider the problem:

$$\begin{aligned} &\text{Maximize} && \Sigma\,(d_j : d_j < 0) \\ &\text{Subject to} && A^T\pi + d = c \end{aligned}$$

- This problem is "locally linear": Define $\kappa \in \boldsymbol{R}^n$ by $\kappa_j = 1$ if $D_j < 0$, and 0 otherwise. Let

$$K = \{j: D_j < 0\} \;\; \text{and} \;\; \underline{K} = \{j: D_j \ge 0\}$$

Then our problem becomes

$$\begin{aligned} &\text{Maximize} && \kappa^T d \\ &\text{Subject to} && A^T\pi + d = c \\ & && d_K \le 0,\; d_{\underline{K}} \ge 0 \end{aligned}$$

- Apply dual simplex, and whenever d_j for $j \in K$ becomes 0, move it to $\underline{K}$.

Issue 2
Pricing

- The texbook rule: Choose the largest primal violation is **TERRIBLE**: For a problem in standard form

$$j = argmin\{X_{Bi} : i = 1,\ldots,m\}$$

- **Geometry is wrong:** Maximizes rate of change relative to axis; better to do relative to edge.
- Goldfard and Forrest 1992 suggested the following **steepest-edge** alternative

$$j = argmin\{X_{Bi}/\eta_i : i = 1,\ldots,m\}$$

where $\eta_i = \|e_i^T A_B^{-1}\|_2$, and gave an efficient update.

Example: Pricing
Model: dfl001

Pricing: Greatest infeasibility

```
Dual simplex - Optimal:  Objective =    1.1266396047e+07
Solution time = 1339.86 sec.  Iterations = 771647 (0)
```

Pricing: Goldfarb-Forrest steepest-edge

```
Dual simplex - Optimal:  Objective =    1.1266396047e+07
Solution time =   24.48 sec.  Iterations = 18898 (0)
```

Issue 3
Solving FTRAN, BTRAN

- **Computing LU factorization:** See Suhl & Suhl (1990). "Computing sparse LU factorization for large-scale linear programming basis", ORSA Journal on Computing 2, 325-335.
- **Updating the Factorization:** Forrest-Tomlin update is the method of choice. See Chvátal Chapter 24.
- **Exploiting sparsity:** This is the main recent development.

(Issue 3 – Solving FTRAN & BTRAN)

We must solve two linear systems per iterstion:

FTRAN	BTRAN
$A_B y = A_j$	$A_B{}^T z = e_i$

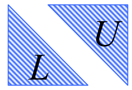
where

A_B = basis matrix (very sparse)
A_j = entering column (very sparse)
e_i = unit vector (very sparse)

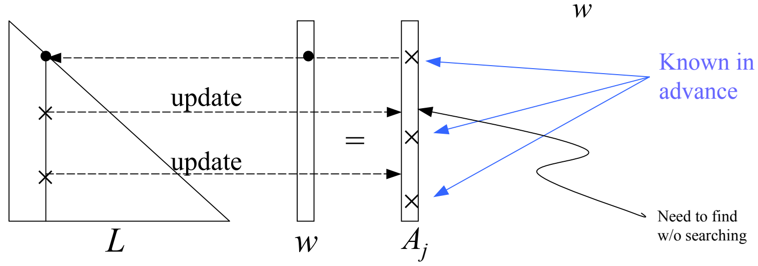
$\Rightarrow y$ an z are typically very sparse

Example: Model pla85900 (from TSP)

Constraints	85900
Variables	144185
Average |y|	15.5

A_B = 

Triangular solve: $Lw=A_j$ $\quad (A_B y = L(\underbrace{Uy}_{w}) = A_j)$



Graph structure: Define an acyclic digraph $D = (\{1,\ldots,m\}, E)$ where $(i,j) \in E \Leftrightarrow l_{ij} \neq 0$ and $i \neq j$.

Solving using *D*: Let $X = \{i \in V: A_{ij} \neq 0\}$. Compute

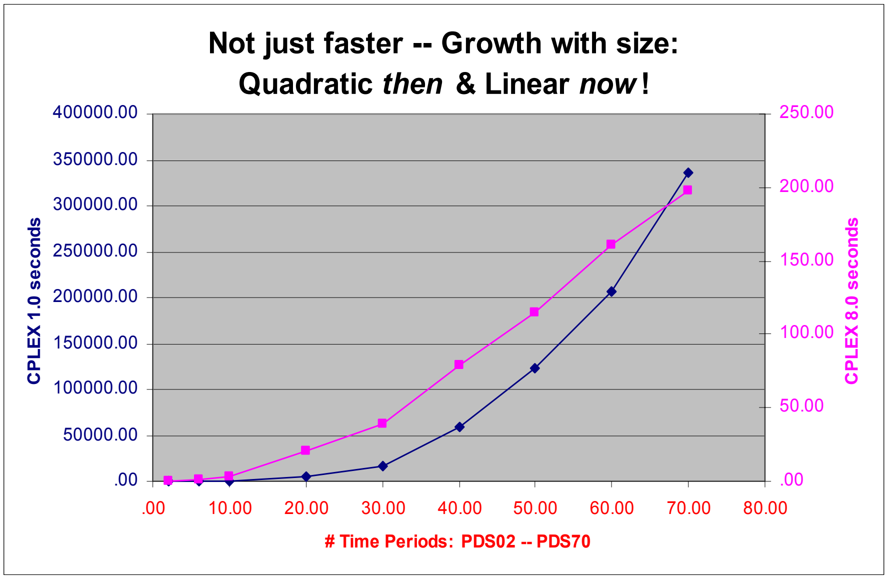
$\underline{X} = \{j \in V: \exists$ a directed path from j to $X\}$.

$\underline{X}$ can be computed in time linear in $|E(\underline{X})|+|\underline{X}|$.

PDS Models

"Patient Distribution System": Carolan, Hill, Kennington, Niemi, Wichmann, *An empirical evaluation of the KORBX algorithms for military airlift applications*, Operations Research 38 **(1990), pp. 240-248**

		CPLEX1.0	CPLEX5.0	CPLEX8.0	SPEEDUP
MODEL	ROWS	1988	1997	2002	1.0→8.0
pds02	2953	0.4	0.1	0.1	4.0
pds06	9881	26.4	2.4	0.9	29.3
pds10	16558	208.9	13.0	2.6	80.3
pds20	33874	5268.8	232.6	20.9	247.3
pds30	49944	15891.9	1154.9	39.1	406.4
pds40	66844	58920.3	2816.8	79.3	743.0
pds50	83060	122195.9	8510.9	114.6	1066.3
pds60	99431	205798.3	7442.6	160.5	1282.2
pds70	114944	335292.1	21120.4	197.8	1695.1
		Primal Simplex	Dual Simplex	Dual Simplex	



Issue 4
Ratio Test and Finiteness

The "standard form" dual problem is

$$\begin{aligned} &\textit{Maximize} && b^T\pi \\ &\textit{Subject to} && A^T\pi + d = c \\ &&& d \geq 0 \end{aligned}$$

Feasibility means

$$\boldsymbol{d \geq 0}$$

However, in practice this condition is replaced by

$$\boldsymbol{d \geq -\varepsilon e}$$

where $e^T=(1,\ldots,1)$ and $\varepsilon = 10^{-6}$. Reason: Degeneracy. In 1972 Paula Harris proposed suggested exploiting this fact to improve numerical stability.

(Issue 4 – Ratio test & finiteness)

STD. RATIO TEST $j_{enter} = argmin\{D_j / \alpha_j : \alpha_j > 0\}$

Motivation: Feasibility ⇒ step length θ satisfies

$$D_N - \theta\alpha_N \geq 0$$

However, the bigger the step length, the bigger the change in the objective. So, we choose

$$\theta_{max} = min\{D_j / \alpha_j : \alpha_j > 0\}$$

Using ε, we have

$$\theta^{\varepsilon}_{max} = min\{(D_j + \varepsilon)/\alpha_j : \alpha_j > 0\} > \theta_{max}$$

HARRIS RATIO TEST $j_{enter} = argmax\{\alpha_j : D_j / \alpha_j \leq \theta^{\varepsilon}_{max}\}$

(Issue 4 – Ratio test & finiteness)

- **Advantages**
  - Numerical stability – α_{jenter} = "pivot element"
  - Degeneracy – Reduces # of 0-length steps
- **Disadvantage**
  - $D_{jenter} < 0$ ⇒ objective goes in wrong direction
- **Solution: BOUND SHIFTING**
  - If $D_{jenter} < 0$, we replace the lower bound on d_{jenter} by something less than its current value.
  - Note that this shift changes the problem and must be removed: 5% of cases, this produces dual infeasibility ⇒ process is iterated.

Example: Bound-Shifting Removal

```
Problem 'pilot87.sav.gz' read.
Reduced LP has 1809 rows, 4414 columns, and 70191 nonzeros.

Iteration log . . .
Iteration:     1   Scaled dual infeas =             0.697540
Iteration:   733   Scaled dual infeas =             0.000404
Iteration:   790   Dual objective     =          -185.892207
...
Iteration: 16326   Dual objective     =           302.786794
Removing shift (3452).
Iteration: 16417   Scaled dual infeas =             0.207796
Iteration: 16711   Scaled dual infeas =             0.000021
Iteration: 16726   Dual objective     =           296.758656
Elapsed time =  104.36 sec. (17000 iterations).
Iteration: 17072   Dual objective     =           300.965492
...
Iteration: 17805   Dual objective     =           301.706409
Removing shift (76).
Iteration: 17919   Scaled dual infeas =             0.000060
Iteration: 17948   Dual objective     =           301.708660
Elapsed time =  114.42 sec. (18000 iterations).
Removing shift (10).
Iteration: 18029   Scaled dual infeas =             0.000050
Iteration: 18039   Dual objective     =           301.710058
Removing shift (1).

Dual simplex - Optimal:  Objective =   3.0171034733e+002
Solution time =  116.44 sec.  Iterations = 18095 (1137)
```

Shift 1: $\varepsilon = 10^{-7}$ (Removing shift (3452))

Shift 2: $\varepsilon = 10^{-8}$ (Removing shift (76))

Shift 3: $\varepsilon = 10^{-9}$ (Removing shift (10))

(Issue 4 – Ratio test & finiteness)

Finiteness: Bound shifting is closely related to the "perturbation" method employed in CPLEX if no progress is being made in the objective.

"No progress" $\Rightarrow$

$$d_j \geq -\varepsilon \qquad j = 1,\ldots,n$$

is replaced by

$$d_j \geq -\varepsilon - \varepsilon_j \qquad j = 1,\ldots,n,$$

where ε_j is random uniform on $[0,\varepsilon]$.

Issue 5
Bound Flipping

- **A basis is given by a triple (B,L,U)**
  - L = non-basics at lower bound: Feasibility $D_L \geq 0$
  - U = non-basics at upper bound: Feasibility $D_U \leq 0$
- **Ratio test:** Suppose $\boldsymbol{X_{Bi}}$ is the leaving variable, and the step length is blocked by some variable $\boldsymbol{d_j, j \in L}$, that is about to become negative and such that $u_j < +\infty$:
  - **Flipping means:** Move j from L to U.
  - **Check:** Do an update to see if X_{Bi} is still favorable
- Can combine many iterations into a single iteration.

Example: Bound Flipping

w/o flipping:

```
Problem 'fit2d.sav.gz' read.
Initializing dual steep norms . . .

Iteration log . . .
Iteration:     1   Dual objective     =        -80412.550000
Perturbation started.
Iteration:   203   Dual objective     =        -80412.550000
Iteration:  1313   Dual objective     =        -80412.548666
Iteration:  2372   Dual objective     =        -77028.548350
Iteration:  3413   Dual objective     =        -71980.245530
Iteration:  4316   Dual objective     =        -70657.605570
Iteration:  5151   Dual objective     =        -68994.477061
Iteration:  5820   Dual objective     =        -68472.659371
Removing perturbation.

Dual simplex - Optimal:  Objective = -6.8464293294e+004
Solution time =   18.74 sec.  Iterations = 5932 (0)
```

w/ flipping:

```
Problem 'fit2d.sav.gz' read.
Initializing dual steep norms . . .

Iteration log . . .
Iteration:     1   Dual objective     =        -77037.550000

Dual simplex - Optimal:  Objective = -6.8464293294e+004
Solution time =    1.88 sec.  Iterations = 201 (0)
```