The CPLEX Library: Mixed Integer Programming

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The Diet Problem Revisited

Nutritional values

- **Bob considered the following foods:**

Food	Serving Size	Energy (kcal)	Protein (g)	Calcium (mg)	Price per serving	
Oatmeal	28 g	110	4	2	$0.30	x_1
Chicken	100 g	205	32	12	$2.40	x_2
Eggs	2 large	160	13	54	$1.30	x_3
Whole milk	237 cc	160	8	285	$0.90	x_4
Cherry pie	170 g	420	4	22	$2.00	x_5
Pork and beans	260 g	260	14	80	$1.90	x_6

- **Bob examines optimal solution and decides he needs more protein**
  - **Adds a protein constraint**
  - **Possible outcome:**
    - Eggs: $x_3 = 0.6203$ in optimal solution

Mixed Integer Programming (MIP)

$$\text{Minimize} \quad c^T x$$

$$\text{Subject to} \quad Ax = b$$

$$l \leq x \leq u$$

$$\text{Some } x_j \text{ are integer}$$

Integrality Restriction

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The "Algorithm" for Solving a MIP

- Base algorithm: branch-and-bound
  - (Land and Doig 1960)
  - Linear programming as a subroutine
  - Provably exponential
- A "bag of tricks" to accelerate the search
  - Most tricks apply to only a subset of models
  - A "barrage" of algorithms

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Branch and Bound

Solution Strategy: Branch & Bound

Key ingredients

- Split the solution space into disjoint subspaces
- Bound the objective value for all solutions in a subspace

Branching

Branch on integer variable

- Choose a *branching variable* x_j
  - Must be an integer variable
- Split the model into two sub-models
  - $x_j \leq i$ or $x_j \geq i+1$
- Binary variable special case:
  - $x_j=0$ or $x_j=1$

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Bounding - Continuous Relaxation

Minimize $c^T x \ (= z_{lp})$

Subject to $Ax = b$

$l \leq x \leq u$

Some x are integer

Lower bound on MIP objective

Relax Integrality Restriction

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Nice Properties of Continuous Rel.

- If relaxation solution satisfies integrality restrictions:
  - No need to further explore subspace
- Natural branching candidates:
  - Integer variables that are fractional in relaxation

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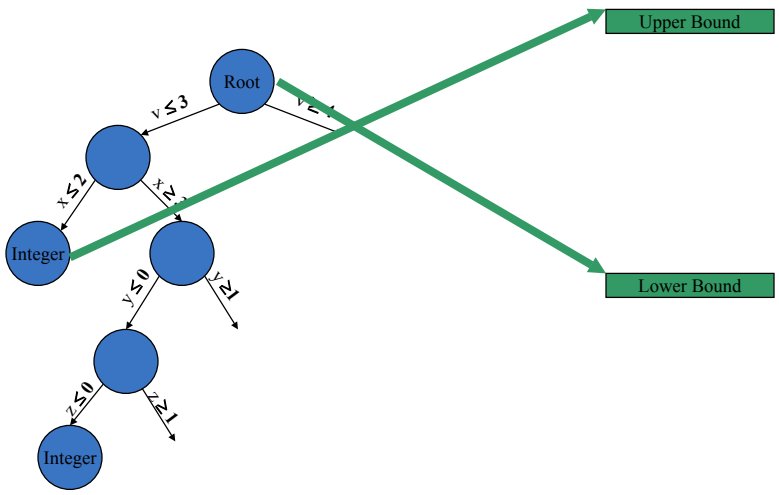
Branch and Bound for MIP



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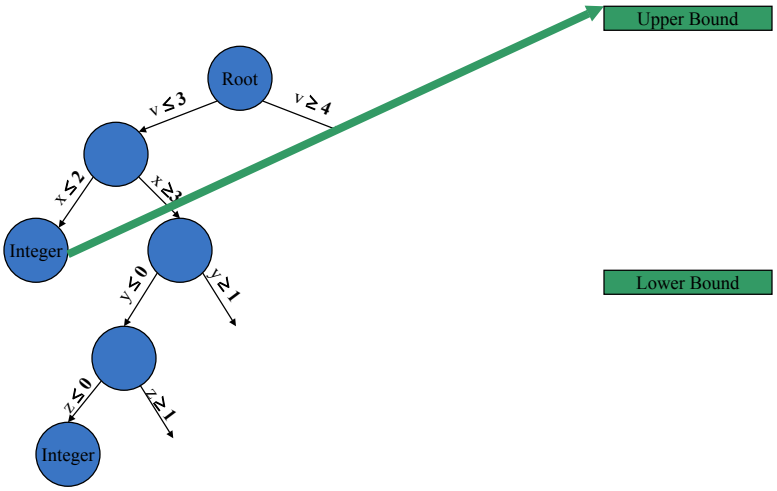
Branch and Bound for MIP



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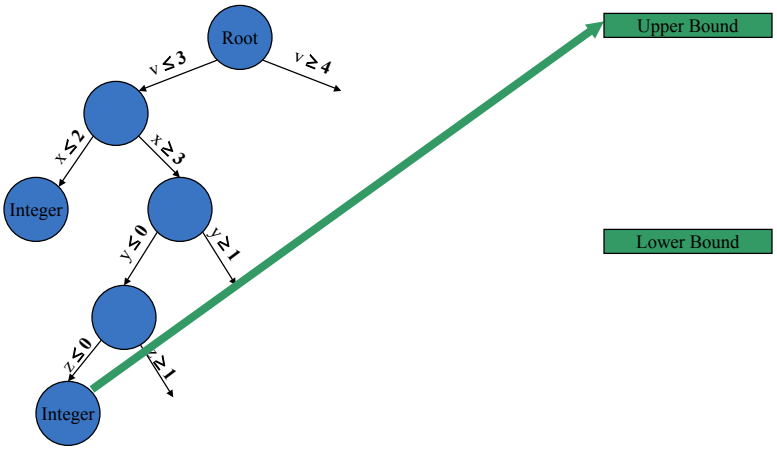
Branch and Bound for MIP



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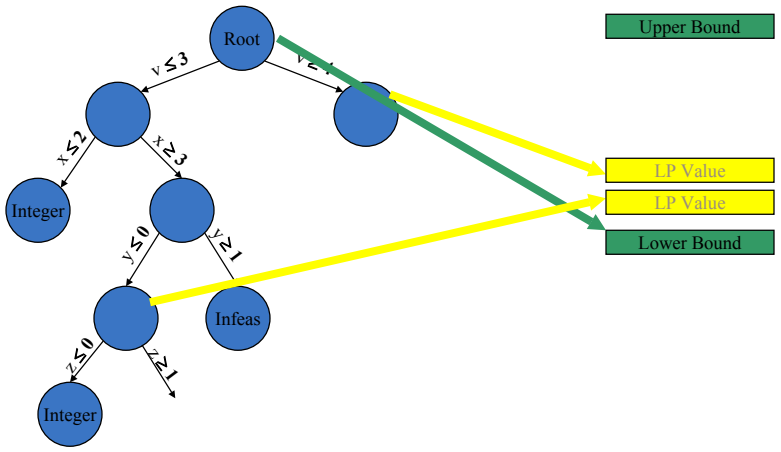
Branch and Bound for MIP



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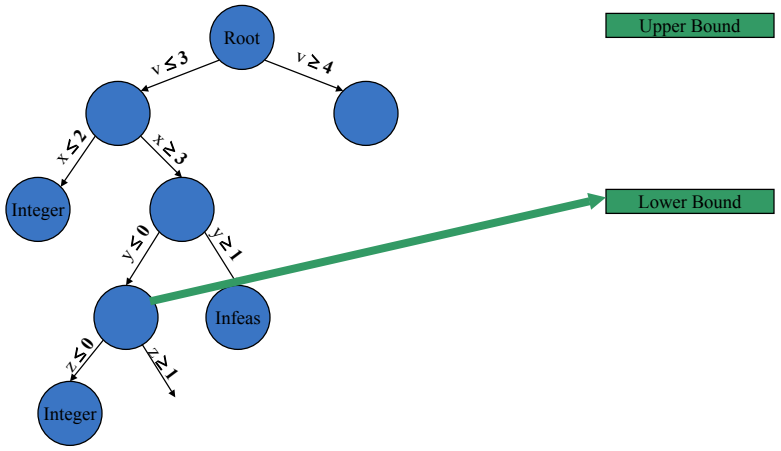
Branch and Bound for MIP



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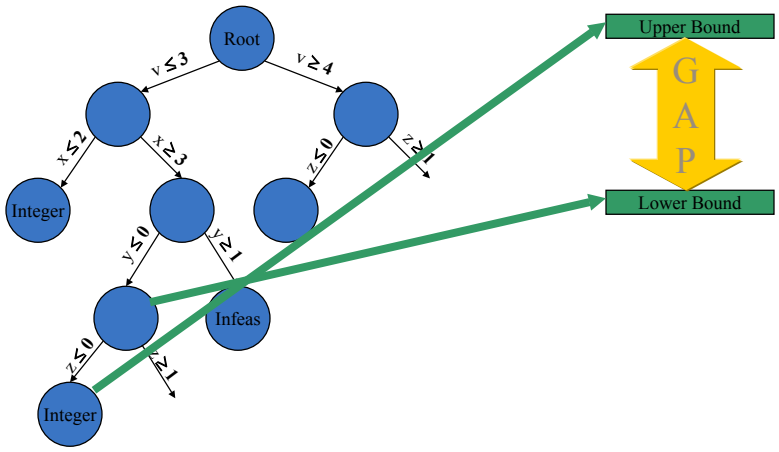
Branch and Bound for MIP



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Important Steps

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Important Steps

The branch and bound loop

- Choose an unexplored node in the tree
- Solve continuous relaxation
- Generate *cutting planes*
- Perform variable fixing
- Find integer feasible solutions that are "similar" to the relaxation solution
- Choose a variable on which to branch
- Explore logical implications of branch
- Repeat

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Important Steps

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- Choose an unexplored node in the tree
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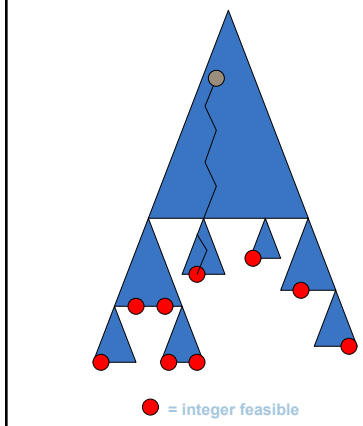
Node Selection



Tradeoff: feasibility versus optimality

- When exploring nodes deep in the search tree…
  - More likely to find integer feasible solutions
  - More likely to explore nodes that would be pruned by later feasible solutions

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Node Selection Options

Options

- Depth first
- Breadth first
- Best first
- Limited discrepancy
- Best estimate

- Plunging (combined with above)
  - Always choose a child of the previously explored node

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Important Steps

The branch and bound loop

- Choose an unexplored node in the tree
- **Solve relaxation**
- Generate *cutting planes*
- Perform variable fixing
- Find integer feasible solutions that are "similar" to the relaxation solution
- Choose a variable on which to branch
- Explore logical implications of branch
- Repeat

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Node Relaxation Solution

Ideally suited to dual simplex

- **Change from parent relaxation is small : a new bound on the branching variable**
  - **Previous basis remains dual feasible**
  - **Solution likely to be "close" to previous basis**
- **A few iterations of dual simplex typically suffice to restore optimality**
- **Cost per node quite low**

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Important Steps

The branch and bound loop

- Choose an unexplored node in the tree
- Solve relaxation
- **Generate *cutting planes***
- Perform variable fixing
- Find integer feasible solutions that are "similar" to the relaxation solution
- Choose a variable on which to branch
- Explore logical implications of branch
- Repeat

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Important Steps

The branch and bound loop

- Choose an unexplored node in the tree
- Solve relaxation
- Generate *cutting planes*
- **Perform variable fixing**
- Find integer feasible solutions that are "similar" to the relaxation solution
- Choose a variable on which to branch
- Explore logical implications of branch
- Repeat

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Reduced Cost Fixing

Use reduced costs to fix variables

- Recall: reduced cost D_N is the marginal cost of moving a variable off of its bound
- If $z_{lp} + |D_j| \geq z^*$
  - z^* = objective of best known feasible solution (incumbent)
- Then x_j can be fixed to its current value in this subtree

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Important Steps

The branch and bound loop

- Choose an unexplored node in the tree
- Solve relaxation
- Generate *cutting planes*
- Perform variable fixing
- **Find integer feasible solutions that are "similar" to the relaxation solution**
- Choose a variable on which to branch
- Explore logical implications of branch
- Repeat

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Important Steps

The branch and bound loop

- Choose an unexplored node in the tree
- Solve relaxation
- Generate *cutting planes*
- Perform variable fixing
- Find integer feasible solutions that are "similar" to the relaxation solution
- **Choose a variable on which to branch**
- Explore logical implications of branch
- Repeat

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Variable Selection

Greatly affects search tree size

- **Guiding principles:**
  - **Make important decisions early**
  - **Both directions of branch should have an impact**
- **Example:**
  - **Decide whether or not to build a factory first**
  - **Decide how many lines to place in the factory later**

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Variable Selection

Predicting impact

- **Question:**
  - **How to predict impact of a branch?**
- **Possible answers:**
  - **Find variables that are furthest from their bounds**
    - Maximum infeasibility
  - **Measure the impact for each branching candidate**
    - Strong branching [Applegate, Bixby, Chvatal, Cook]
  - **Use historical information**
    - Pseudo-costs

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Important Steps

The branch and bound loop

- Choose an unexplored node in the tree
- Solve relaxation
- Generate *cutting planes*
- Perform variable fixing
- Is the relaxation solution near-feasible?
- Choose a variable on which to branch
- **Explore logical implications of branch**
- Repeat

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Logical Propagation

Propagate implications logically

- Simple example:
  - x + 2y + 3z ≤ 3, all variables binary
  - x = 1 (e.g., fixed during tree exploration)
  - z = 2/3 still feasible in LP relaxation
- Use *bound strengthening* to tighten variable bounds

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Later Today

- Brief History of CPLEX MIP
- Heuristic details
- Cutting plane details

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