The CPLEX Library: Presolve and Cutting Planes

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Presolve and Cutting Planes

"Tighter" formulation

- Original MIP formulation can almost always be improved
- What does "improved" mean?
  - Fewer constraints and variables
    - Less data to process
  - Smaller difference between space of feasible continuous and feasible integer solutions
    - Rely less on branching to refine continuous relaxation
- Two techniques:
  - Presolve and cutting planes

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Model Reformulation

"Tighten" formulation

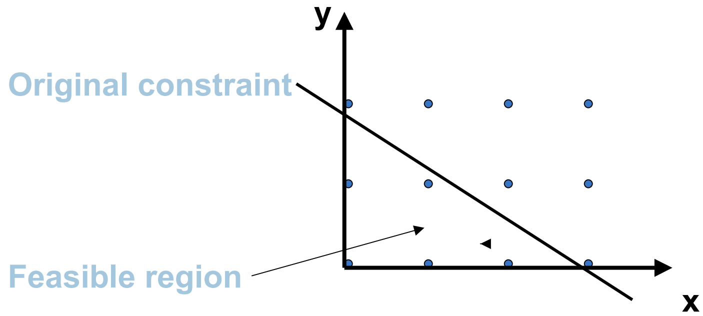
- **Similar steps in both cases:**
  - **Add/replace constraints in model to tighten formulation**
    - Same integer solutions
    - Fewer continuous solutions
- **Important difference:**
  - **Presolve is applied to the original model to create a new model**
  - **Cutting planes are added to an existing model (typically the presolved model) to cut off a relaxation solution**

Model Reformulation

Presolve versus cutting planes

- **Important difference:**
  - **A single constraint can produce an exponential number of tighter constraints**
  - **Presolve introduces tighter constraints that dominate existing constraints**
    - Tighter formulation without creating a larger problem
    - Reformulation is independent of relaxation solution
  - **Cutting planes introduce tighter constraints that cut off a particular relaxation solution**
    - Focused growth in model size

Simple Reformulation Example



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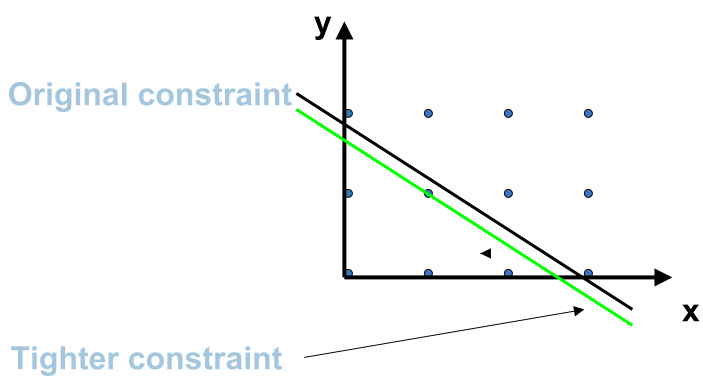
Simple Reformulation Example



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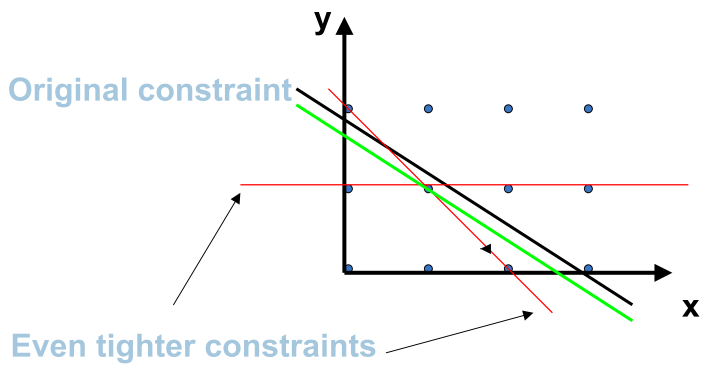
Simple Reformulation Example



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Rounding, Lifting, and Disjunction

- Three powerful, widely used concepts in presolve and cutting planes:
- Rounding
  - Integer multiples of integer variables take integer values
- Lifting
  - Fixing a binary variable at a bound may cause a constraint to go slack
- Disjunction
  - Binary variable must take one of two values

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Rounding

Simplest Form of Rounding

Rounding in presolve

- A fractional bound on an integer variable can be truncated:
  - $x \le 1.5$ implies $x \le 1$
- Effects can become non-trivial when combined with bound strengthening:
  - $x + 2y + 4z = 4$, all variables binary
  - Bound strengthening and rounding together yield:
    - $4z \ge 4 - \sup(x+2y)$; $z \ge 1/4$; $z >= 1$
    - x=0, y=0, z=1

GCD Reduction

More rounding in presolve

- Given a constraint involving all integer variables with integer coefficients
  - $\sum a_j x_j \leq b$
- Divide through by GCD of coefficients (g)
  - $\sum (a_j/g) x_j \leq \lfloor b/g \rfloor$
- LHS is integral, so RHS can be truncated
- Example:
  - $3x + 6y + 9z \leq 11$

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Gomory Rounding Cut

Yet more rounding

- Given a constraint involving non-negative integer variables
  - $\sum a_j x_j \leq b$
- Divide through by some positive constant c
  - $\sum (a_j/c) x_j \leq b/c$
- Truncate coefficients
  - $\sum \lfloor a_j/c \rfloor x_j \leq \sum (a_j/c) x_j \leq b/c$
- LHS is integral, so RHS can be truncated
- Note: does not necessarily dominate original constraint
  - (Probably) not relevant for presolve

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Gomory Rounding Cut

Example

- Given a constraint involving non-negative integer variables
  - $3x + 3y + 5z \leq 8$
- And relaxation solution:
  - $x=1$, $y=1$, $z=2/5$
- Divide through by 3
  - $x + y + 5/3\, z \leq 8/3$
- Truncate coefficients and RHS
  - $x + y + z \leq 2$
- Cuts off relaxation solution:
  - $x + y + z = 12/5$

Lifting

Coefficient Reduction

Lifting in presolve

- Given a constraint involving some binary x_k:
  - $\sum a_j x_j \geq b$
- Will fixing $x_k=1$ cause constraint to go slack?
  - a_k + inf ($\sum_{j!=k} a_j x_j$) > b ?
  - s = a_k + inf ($\sum_{j!=k} a_j x_j$) - b > 0
- If so, we can subtract the following from LHS:
  - $s\ x_k$
- Example:
  - $2x + y \geq 1$ becomes
  - $x + y \geq 1$

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Implied Bound Cuts

Trivial lifting for cutting planes

- Given a continuous variable with an upper bound
  - $y \leq u$
- And a binary variable x that implies a new upper bound on y:
  - e.g., x=0 -> $y \leq u_i$
- Can lift x into '$y \leq u$'
  - $y + (u-u_i)(1-x) \leq u$
- Simple case: $u_i=0$
  - Cut: $y \leq u\ x$

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Implied Bound Cuts

Example

- Given continuous variables with upper bounds
  - $y_1 + y_2 \leq 10\ x$
  - $y_1 \leq 5$ and $y_2 \leq 5$
  - $y_1 = 5,\ y_2 = 0,\ x = 0.5$
- Implied bound cut:
  - $y_1 \leq 5\ x$
- Violated by relaxation solution

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Disjunction

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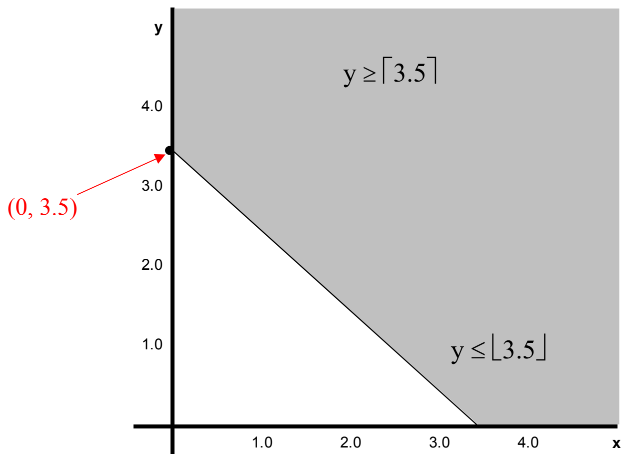
$x + y \geq 3.5, \; x \geq 0, \; y$ integral



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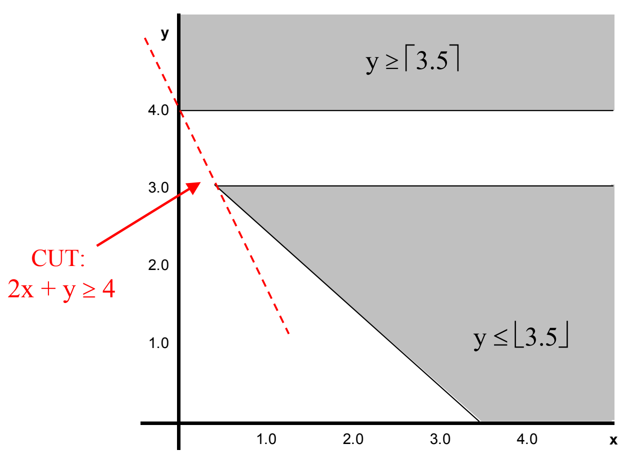
$x + y \geq 3.5, \; x \geq 0, \; y$ integral



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Gomory Mixed Cut

- Given $y, x_j \in Z_+$, and

  $y + \sum a_{ij}x_j = d = \lfloor d \rfloor + f, \ f > 0$
- Rounding: Where $a_{ij} = \lfloor a_{ij} \rfloor + f_j$, define

  $t = y + \sum(\lfloor a_{ij} \rfloor x_j: f_j \le f) + \sum(\lceil a_{ij} \rceil x_j: f_j > f) \in Z$
- Then

  $\sum(f_j\, x_j: f_j \le f) + \sum(f_j-1)x_j: f_j > f) = d - t$
- Disjunction:

  $t \le \lfloor d \rfloor \Rightarrow \sum(f_j x_j : f_j \le f) \ \ge f$

  $t \ge \lceil d \rceil \Rightarrow \sum((1-f_j)x_j: f_j > f) \ge 1-f$
- Combining:

  $\sum((f_j/f)x_j: f_j \le f) \ + \ \sum([(1-f_j)/(1-f)]x_j: \ f_j > f) \ge 1$

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Other Presolve Techniques

Problem Size Reductions

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More Presolve Reductions

- Fixed variables
- Inactive constraints:
  - Example: $x + y \leq 2$, x and y binary
- Redundant constraints:
  - Example: $x + y \leq 2$; $x + y \leq 3$
- Dual fixed reductions:
  - Variable with:
    - Positive objective coefficient
    - Belonging to only less-than-constraints
    - Having all non-negative matrix coefficients
  - …can be fixed to lower bound

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Presolve Summary

- Presolve a vital part of solving a MIP model
- Most models have significant scope for improvement
  - 5X+ problem size reductions are common
  - 10X runtime reductions are typical

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Other Cutting Plane Techniques

Cover (Knapsack) Cuts

- 0-1 Knapsack

  $K = \{x \in B: \sum_{j \in N} a_j x_j \leq b\}$, with $a_j > 0$ and $b > 0$
- The set $C \subseteq N$ is called a cover if

  $\sum_{j \in C} a_j x_j > b$
- The cover inequality

  $\sum_{j \in C} x_j \leq |C| - 1$

  is valid for K

Cover+Lifting: 0-1 Knapsack

- Consider

  $5x_1 + 5x_2 + 5x_3 + 5x_4 + 3x_5 + 8x_6 <= 17$
- Cover inequality

  $x_1 + x_2 + x_3 + x_4 <= 3$
- Lifting x_5 first, then x_6

  $x_1 + x_2 + x_3 + x_4 + \pi_5 x_5 <= 3$

  $\pi_5 = 3 - \max \{x_1 + x_2 + x_3 + x_4\} = 1$

  Similarly, $\pi_6 = 1$, so the lifted cover is

  $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 <= 3$
- Lifting x_6 first, then $x_{5,}$ then the lifted cover is

  $x_1 + x_2 + x_3 + x_4 + 2x_6 <= 3$

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Clique Cuts

- Two binary variables are *incompatible* if they can't both be 1:
  - $x + y \leq 1$ means x and y are incompatible
- A *clique* is a set of pairwise incompatible variables
  - $C = \{x \in B: x_i$ and x_j are incompatible$\}$
  - Clique cut: $\sum_{j \in C} x_j \leq 1$
- Example:
  - $x + y \leq 1$; $x + z \leq 1$; $y + z \leq 1$ implies
  - $x + y + z \leq 1$

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Cutting Plane Summary

Applying Cutting Planes

- Many different varieties of cutting planes
- Number that are valid for a particular model is enormous
- Must identify relevant ones
  - Those that cut off appealing relaxation solutions
- Must solve the *separation* problem to find violated cutting planes
  - Heuristic procedure for each type of cutting plane (not discussed)

Applying Cutting Planes

- **How many cuts should be generated for a relaxation solution?**
  - **One?**
    - Will provide a new relaxation solution
    - Expensive to re-solve relaxation for each cut
  - **As many as possible?**
    - Relaxation solution only needs to be cut off once
    - Cuts increase the size of the model
- **Need to strike a balance**
  - **Multiple rounds of cutting plane generation**
  - **Limited number of cuts per round**

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Sample CPLEX Output

Default settings

```
          Nodes                                          Cuts/
   Node  Left     Objective  IInf  Best Integer     Best Node    ItCnt     Gap

      0     0     4533.5033    40                   4533.5033      125
                  8517.6222    29                  Cuts:  100      236
*     0+    0                   0     9715.0000     8517.6222      236   12.33%
                  8651.9219    10     9715.0000     Cuts:   51      266   10.94%
*     0+    0                   0     8701.0000     8651.9219      266    0.56%
                  8662.8458     4     8701.0000      Cuts:   7      273    0.44%
                  8665.4678     7     8701.0000    Covers:   2      276    0.41%
                  8667.9363     7     8701.0000    Covers:   1      278    0.38%
*     4     3                   0     8691.0000     8688.0000      282    0.03%

GUB cover cuts applied:  23
Clique cuts applied:  10
Cover cuts applied:  31
Implied bound cuts applied:  1
Gomory fractional cuts applied: 30
```

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Performance Impact:Relative to Defaults on our test set with 80 models

-Knapsack Covers	18%
-Cliques	1%
-Flow Covers	5%
-GUB Covers	1%
-Implied Bounds	0%
-Gomory Cuts	22%
-MIR Cuts	5%
-Flow Path Cuts	0%
+Disjunctive Cuts	64%

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CPLEX 8.0 MIP results on 106 models solved by 8.0 but not by 5.0

- Cuts 53.7X
  - Gomory 2.5X
  - MIR 1.8X
  - Knapsack 1.4X
  - Flow covers 1.2X
  - Implied bounds 1.2X
  - …
- Presolve 10.8X
- Heuristics 1.4X
- Node presolve 1.3X
- Probed dives 1.1X

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