Solution for Exercise 5 (Wednesday Session)

Lemma 0.1 *If $f(w)$ is homogeneous of degree α, then, for every $k \geq 0$ and every sequence $1 \leq i_1, \ldots, i_k \leq n$, function $(\nabla^k f(w))_{i_1,\ldots,i_k}$ is homogeneous of degree $\alpha - k$.*

Proof According to Euler's Homogeneity Relation, it suffices to show that

$$(\nabla(\nabla^k f(w))_{i_1,\ldots,i_k})^\top w = (\alpha - k)(\nabla^k f(w))_{i_1,\ldots,i_k} \tag{1}$$

To this end, we proceed by induction. For $k = 0$, (1) collapses to Euler's Homogeneity Relation that holds because f is assumed to be homogeneous of order α. Consider now $k \geq 1$. We may assume inductively that

$$\sum_{i=1}^{n} w_i(\nabla^k f(w))_{i_1,\ldots,i_{k-1},i} = (\nabla(\nabla^{k-1} f(w))_{i_1,\ldots,i_{k-1}})^\top w = (\alpha - k + 1)(\nabla^{k-1} f(w))_{i_1,\ldots,i_{k-1}} \ .$$

Applying operator $\frac{\partial}{\partial w_{i_k}}$ to both sides of this equation, we get

$$\begin{aligned} \sum_{i=1}^{n} w_i(\nabla^{k+1} f(w))_{i_1,\ldots,i_k,i} + (\nabla^k f(w))_{i_1,\ldots,i_k} &= (\nabla(\nabla^k f(w))_{i_1,\ldots,i_k})^\top w + (\nabla^k f(w))_{i_1,\ldots,i_k} \\ &= (\alpha - k + 1)(\nabla^k f(w))_{i_1,\ldots,i_k} \ . \end{aligned}$$

The proof is completed by solving the last equation for $(\nabla(\nabla^k f(w))_{i_1,\ldots,i_k})^\top w$. •