9th Max-Planck Advanced Course on the Foundations of Computer Science (ADFOCS)

Primal-Dual Algorithms for Online Optimization: Lecture 3

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Contents

- The ad-auctions problem
- Caching
  - Relationship with k-server
  - Weighted paging
  - Web caching

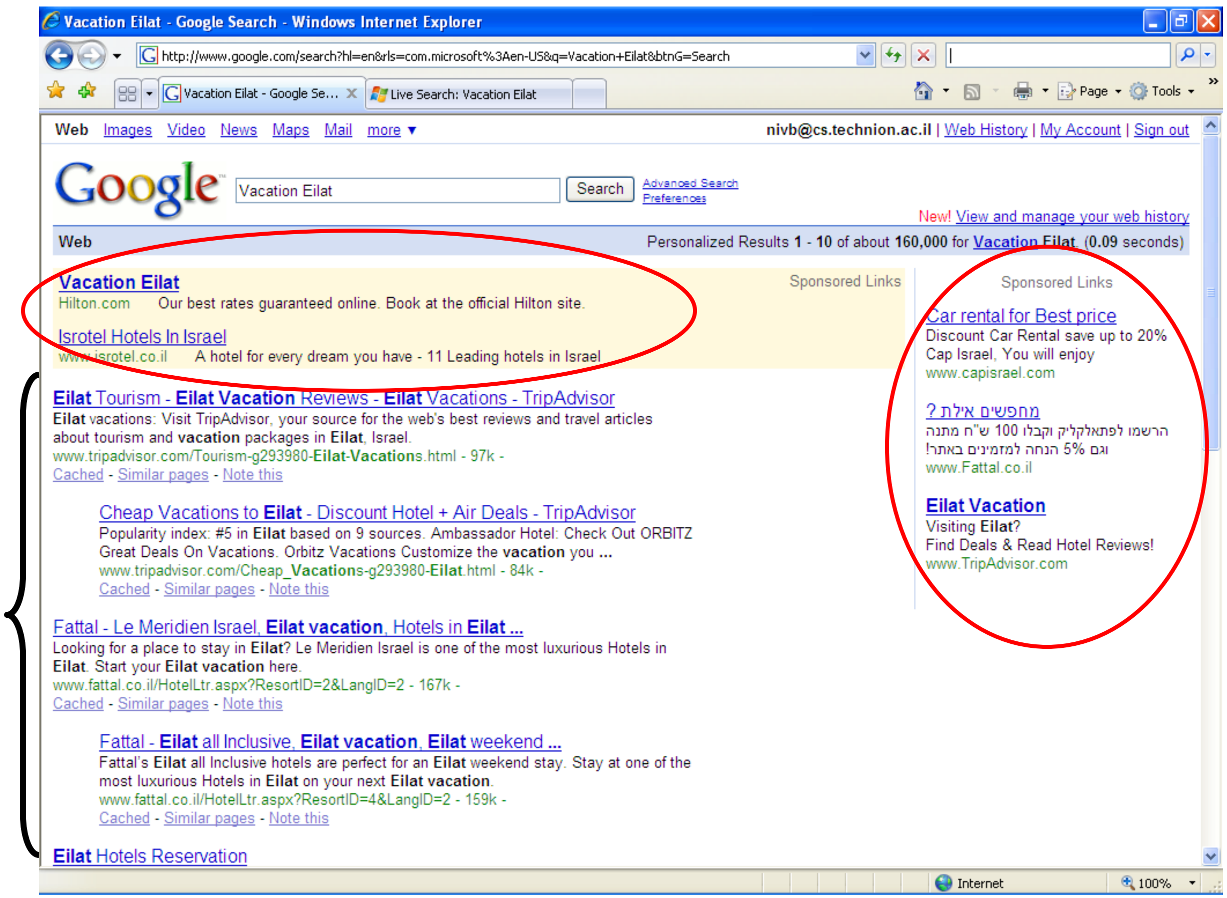
What are Ad-Auctions?

You type in a query: **Vacation Eilat**

You get:

And … Ad-auctions

Algorithmic Search results



How do search engines sell ads?

- Each **advertiser**:
  - Sets a daily budget
  - Provides bids on interesting keywords
- **Search Engine** (on each keyword):
  - Selects ads
  - Advertiser pays bid if user clicks on ad.

Goal (of search engine):

Maximize Revenue

How much does it cost?

Buying keyword like
"divorce lawyer"
may cost as much as
$40 per click

Google AdWords: Keyword Tool - Windows Internet Explorer

https://adwords.google.com/select/KeywordToolExternal?defaultView=1

File Edit View Favorites Tools Help

ALGO 2007 | Google AdWords: Keywor...

Keywords	Estimated Avg. CPC	Estimated Ad Position
luxury vacation	$3.48	1 - 3
golf vacation	$3.23	1 - 3
hotel vacation	$2.66	1 - 3
scuba vacation	$2.62	1 - 3
vacation tours	$2.61	1 - 3
hotels vacation	$2.61	1 - 3
Tel Aviv vacation	**$2.5**	
vacation rentals	$2.40	1 - 3
spa vacation	$2.39	1 - 3
Eilat vacation	**$2.36**	
vacation travel	$2.24	1 - 3
hotels in eilat	$2.21	1 - 3
Red sea vacation	**$2.15**	
vacation	[illegible]	[illegible]
beach vacation	$2.12	1 - 3
Jewish vacation	**$2.05**	

Estimates are for September 30th 2007

Mathematical Model

- Buyer i:
  - has a daily budget B(i)
- Online Setting:
  - items (keywords) arrive one-by-one.
  - buyers bid on the items (bid can be zero)
- **Algorithm:**
  - Assigns each item to an interested buyer.

Assumption:

Each bid is small compared to the daily budget.

Ad-auctions – Linear Program

I - Set of buyers. B(i) – Budget of buyer i
J - Set of items. b(i,j) – bid of buyer i on item j

$y(i,j) = 1 \Rightarrow$ *j*-th adword is sold to buyer i.

$$\max \sum_{i \in I} \sum_{j \in J} b(i,j) y(i,j)$$

s.t:

For each item j: $\sum_{i \in I} y(i,j) \leq 1$

For each buyer i: $\sum_{j \in J} b(i,j) y(i,j) \leq B(i)$

Buyers do not exceed their budget

Ad-auctions: Primal and Dual

P: Primal Covering

$$\min \sum_{i \in I} B(i)x(i) + \sum_{j \in J} z(j)$$

For each item j and buyer i: $b(i,j)x(i) + z(j) \ge b(i,j)$

D: Dual Packing

$$\max \sum_{i \in I} \sum_{j \in J} b(i,j)y(i,j)$$

For each item j: $\sum_{i \in I} y(i,j) \le 1$

For each buyer i: $\sum_{j \in J} b(i,j)y(i,j) \le B(i)$

The Primal-Dual Algorithm

- Initially: for each buyer i: x(i)← 0
- When new **item j arrives**:
- Assign the item to the **buyer i** that **maximizes:**

$$b(i,j)\left[1-x(i)\right]$$

- if x(i)≥1 do nothing, **otherwise**:
  - $y(i,j) \leftarrow 1$
  - $z(j) \leftarrow b(i,j)\left[1-x(i)\right]$
  - $x(i) \leftarrow x(i)\left[1+\frac{b(i,j)}{B(i)}\right]+\frac{b(i,j)}{B(i)\left[c-1\right]}$ - ‘c’ later.

Analysis of Online Algorithm

Proof of competitive factor:

1. Primal solution is feasible.
2. In each iteration, $\Delta P \leq (1 + 1/(c-1))\Delta D$.
3. Dual is feasible.

Conclusion:

Algorithm is **(1+ 1/(c-1))-competitive**

Analysis of Online Algorithm

1. **Primal solution is feasible.**

For each item j and buyer i:

$$b(i,j)x(i) + z(j) \geq b(i,j)$$

If x(i) ≥1 the solution is feasible.

Else, z(j) ← $\max_i$\{ b(i,j)(1-x(i)) \}, and the solution is feasible

Increasing x(i) in the future maintains feasibility

Analysis of Online Algorithm

2. In each iteration, ΔP ≤ (1+ 1/(c-1))ΔD:

If x(i)≥1, ΔP =ΔD=0

Otherwise:

- ΔD = b(i,j)
- ΔP $= B(i)\Delta x(i) + z(j)$

$$= B(i)\left[\frac{b(i,j)x(i)}{B(i)} + \frac{b(i,j)}{B(i)[c-1]}\right] + b(i,j)[1-x(i)] = b(i,j)\left[1+\frac{1}{(c-1)}\right]$$

$$z(j) \leftarrow b(i,j)[1-x(i)] \qquad x(i) \leftarrow x(i)\left[1+\frac{b(i,j)}{B(i)}\right] + \frac{b(i,j)}{B(i)[c-1]}$$

Analysis of Online Algorithm

3. **Dual is feasible:**

- The “last” item assigned to a buyer may exceed his budget
- The online algorithm loses the revenue from such an item
- This where the assumption that each individual bid is small with respect to the budget is used
- The maximum ratio between a bid of any buyer and its total budget:

$$R = \max_{i \in I, j \in M} \left\{ \frac{b(i,j)}{B(i)} \right\}$$

Analysis of Online Algorithm

It is easy to prove by induction that:

$$1 \geq x(i) \geq \frac{1}{c-1}\left[c^{\frac{\sum_j b(i,j)y(i,j)}{B(i)}} - 1 \right]$$

- if x(i)≥1, primal constraints of buyer i are feasible.

➔ No more items are assigned to the buyer.

- simplifying the inequality we get that the dual is almost feasible (up to the “last” item)

Competitive Factor

- Setting $c = (1+R)^{\frac{1}{R}}$

$$c \longrightarrow e \text{ when } R \longrightarrow 0$$

- The competitive factor is

$$\left(1 - \frac{1}{c}\right)(1-R) = \left(1 - \frac{1}{e}\right) \text{ if } R \longrightarrow 0$$

- Result obtained by [MSVV, FOCS 2005]

Extensions – Getting More Revenue

- Seller wants to sell several advertisements
- There are ℓ slots on each page
- Bidders provide bids on keywords which are slot dependent

  b(i,j,k) – bid of buyer i on keyword j and slot k

- A slot can only be allocated to one advertiser

Linear Program

Dual (Packing)	
Maximize:	$\sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{\ell=1}^{k} b(i,j,\ell)y(i,j,\ell)$
Subject to:	
$\forall 1 \le j \le m,\ 1 \le k \le \ell$:	$\sum_{i=1}^{n} y(i,j,k) \le 1$
$\forall 1 \le i \le n$:	$\sum_{j=1}^{m}\sum_{k=1}^{\ell} b(i,j,k)y(i,j,k) \le B(i)$
$\forall 1 \le j \le m, 1 \le i \le n$:	$\sum_{k=1}^{\ell} y(i,j,k) \le 1$

Primal (Covering)	
Minimize :	$\sum_{i=1}^{n} B(i)x(i) + \sum_{j=1}^{m}\sum_{k=1}^{\ell} z(j,k) + \sum_{i=1}^{n}\sum_{j=1}^{m} s(i,j)$
Subject to:	
$\forall i,j,k$:	$b(i,j,k)x(i) + z(j,k) + s(i,j) \ge b(i,j,k)$

Online Allocation Algorithm

Initially, $\forall i, \quad x(i) \leftarrow 0$.

Upon arrival of a new item j:

1. Generate a bipartite graph H: n buyers on one side and ℓ slots on the other side. Edge $(i,k) \in H$ has weight $b(i,j,k)(1 - x(i))$.
2. Find a maximum weight (integral) matching in H, i.e., an assignment to the variables $y(i,j,k)$.
3. Charge buyer i the minimum between $\sum_{k=1}^{\ell} b(i,j,k)y(i,j,k)$ and its remaining budget.
4. For each buyer i, if there exists slot k for which $y(i,j,k) > 0$:

$$x(i) \leftarrow x(i)\left(1 + \frac{b(i,j,k)y(i,j,k)}{B(i)}\right) + \frac{b(i,j,k)y(i,j,k)}{(c-1)\cdot B(i)}$$

Remark: If $\ell = 1$, the maximum weight matching is a single edge maximizing $b(i,j)(1 - x(i))$.

Analysis of Online Algorithm

Proof of competitive factor:

1. Primal solution is feasible.
2. In each iteration, $\Delta P \leq (1 + 1/(c-1))\Delta D$.
3. Dual is feasible.

Conclusion:

Algorithm is **(1+ 1/(c-1))-competitive**

Analysis: Crucial Fact

Dual (Packing)		Primal (Covering)	
$\max \sum_i \sum_k b(i,j,k)\,(1-x(i))\,y(i,j,k)$		$\min \sum_{i=1}^{n} s(i,j) + \sum_{k=1}^{\ell} z(j,k)$	
Subject to:		Subject to:	
$\forall 1 \le k \le \ell$:	$\sum_{i=1}^{n} y(i,j,k) \le 1$	$\forall (i,k)$:	$s(i,j) + z(j,k) \ge b(i,j,k)\,(1-x(i))$
$\forall 1 \le i \le n$:	$\sum_{k=1}^{\ell} y(i,j,k) \le 1$	$\forall i,k$:	$s(i,j), z(j,k) \ge 0$
$\forall i,k$:	$y(i,j,k) \ge 0$		

Figure 1: The LP for the matching problem solved for item j

- Primal variables are the same as in the allocation problem.
- There is an optimal primal solution and a dual **integral** solution satisfying:

$$\sum_{i=1}^{n}\sum_{k=1}^{\ell} b(i,j,k)\,(1-x(i))\,y(i,j,k) = \sum_{i=1}^{n} s(i,j) + \sum_{k=1}^{\ell} z(j,k).$$

- This solution defines the assignmet to the primal and dual variables

Analysis of Online Algorithm

1. **Primal solution is feasible.**

for each buyer I, item j, slot k:

$$b(i,j,k)x(i) + z(j,k) + s(i,j) \geq b(i,j,k).$$

this constraint is satisfied by the primal-dual solution to the weighted matching LP

Increasing x(i) in the future maintains feasibility

Analysis of Online Algorithm

2. In each iteration, ΔP ≤ (1+ 1/(c-1))ΔD:

$$\begin{aligned}
\Delta P &= \sum_{i=1}^{n} z(j,i) + \sum_{k=1}^{\ell} s(i,j) + \sum_{i=1}^{n} B(i)\Delta x(i) \\
&= \sum_{i=1}^{n}\sum_{k=1}^{\ell} b(i,j,k)\,(1-x(i))\,y(i,j,k) \\
&\quad + \sum_{i=1}^{n}\sum_{k=1}^{\ell} B(i)\left(\frac{b(i,j,k)x(i)y(i,j,k)}{B(i)} + \frac{b(i,j,k)y(i,j,k)}{(c-1)\cdot B(i)}\right) \\
&= \sum_{i=1}^{n}\sum_{k=1}^{\ell} b(i,j,k)y(i,j,k)\left(1+\frac{1}{c-1}\right).
\end{aligned}$$

Since $\Delta D = \sum_{i=1}^{n}\sum_{k=1}^{\ell} b(i,j,k)y(i,j,k)$, the claim follows.

$$\sum_{i=1}^{n}\sum_{k=1}^{\ell} b(i,j,k)\,(1-x(i))\,y(i,j,k) = \sum_{i=1}^{n} s(i,j) + \sum_{k=1}^{\ell} z(j,k).$$

Analysis of Online Algorithm

3. Dual is feasible:

- similar to the proof in the single slot case

- the competitive factor is

$$\left(1-\frac{1}{c}\right)(1-R)=\left(1-\frac{1}{e}\right) \quad \text{if } R \rightarrow 0$$

Online Matching in Bipartite Graphs

Input: bipartite graph H=(U,V,E)

Goal: find a maximum matching in H

Online model:

- V is known
- the vertices of U arrive one by one and expose their neighbors in V (upon arrival)
- for each u $\in$ U, upon arrival, online algorithm decides whether to match u to a vertex in V

Online Algorithms for Matching

- any algorithm that matches a vertex, if possible, achieves competitive ratio ½ since

  (maximal matching) ≥ ½ · (maximum matching)
- online algorithm of [KVV 1990]:
  - choose a random permutation π on V
  - assign each vertex u ∈ U to the minimum index vertex in V with respect to π
  - competitive ratio: 1-1/e
- an online primal-dual algorithm can find a fractional matching with competitive ratio 1-1/e
- can an integral matching be computed via the primal-dual method?

The Paging/Caching Problem

- Relationship to the k-Server Problem
- Weighted paging
- Web caching

The Paging/Caching Problem (Reminder)

Universe of of n pages

Cache of size $k \ll n$

Request sequence of pages: 1, 6, 4, 1, 4, 7, 6, 1, 3, …

If requested page is in cache: no penalty.

Else, cache miss! load requested page into cache, evicting some other page.

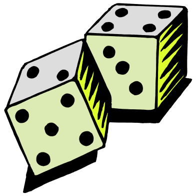
Goal: minimize number of cache misses.

Question: which page to evict in case of a cache miss?

Known Results: Paging

Paging (Deterministic) [Sleator Tarjan 85]:

- Any online algorithm ≥ k-competitive.
- LRU is k-competitive (also other algorithms)
- LRU is k/(k-h+1)-competitive if optimal has cache of size $h \leq k$.



Paging (Randomized):

- Rand. Marking O(log k) [Fiat, Karp, Luby, McGeoch, Sleator, Young 91].
- Lower bound H_k [Fiat et al. 91], tight results known.
- O(log(k/k-h+1))-competitive algorithm if optimal has cache of size $h \leq k$ [Young 91]

The Weighted Paging Problem

One small change:

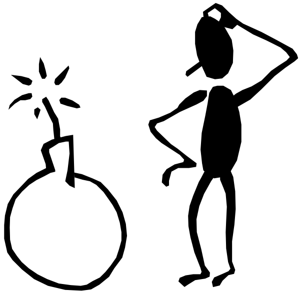
- Each page i has a different fetching cost w(i).
- Models scenarios where cost of loading pages into the cache is not uniform:

Main memory, disk, internet …



Goal

- Minimize the **total cost** of cache misses.



Weighted Paging

	Paging	Weighted Paging
Deterministic	Lower bound k LRU k competitive k/(k-h+1) if opt's cache size h	k-competitive [Chrobak, Karloff, Payne, Vishwanathan 91] k/(k-h+1) [Young 94]
Randomized	O(log k) Randomized Marking O(log k/(k-h+1))	O(log k) for two distinct weights [Irani 02] No o(k) algorithm known even for three distinct weights.

The k-server Problem (1)

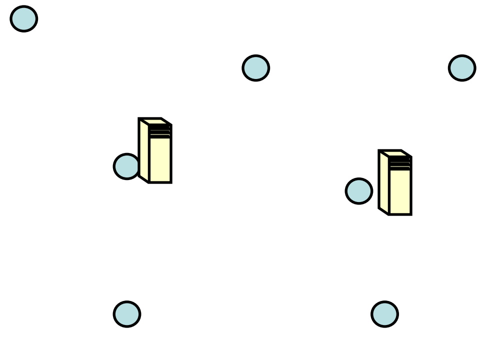
- k servers are placed in an n-point metric space

- requests arrive at points in the metric

- serving a request: move a server to request point

Goal: minimize total distance traveled

by the servers.



The k-server Problem

- Paging = k-server on a uniform metric
  - every page is a point
  - A page is in the cache iff a server is at the point
- Weighted paging = k-server on a weighted star metric

Deterministic Results:

- General metric spaces: (2k-1)-competitive work function algorithm [Koutsoupias-Papadimitriou 95]
- Tree metric: k-competitive algorithm [Chrobak et al. 91]

Randomized Results:

- No $o(k)$ algorithm known (even for very simple spaces).
- Best lower bound $\Omega(\log k)$

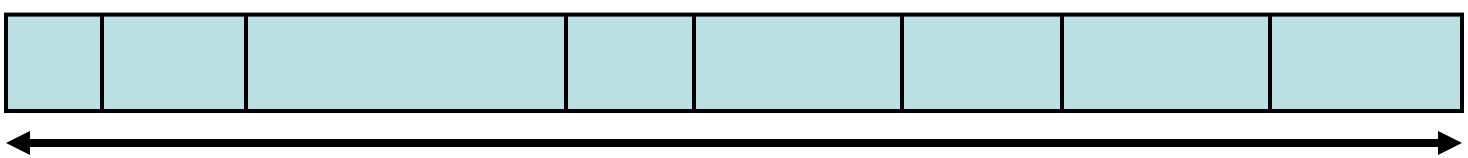
Fractional Weighted Paging

Model:

- Fractions of pages are kept in cache: probability distribution over pages $p_1, \ldots, p_n$

- The total sum of fractions of pages in the cache is at most k.

- If p_i changes by ε, cost = $\varepsilon\, w(i)$



k units of cache

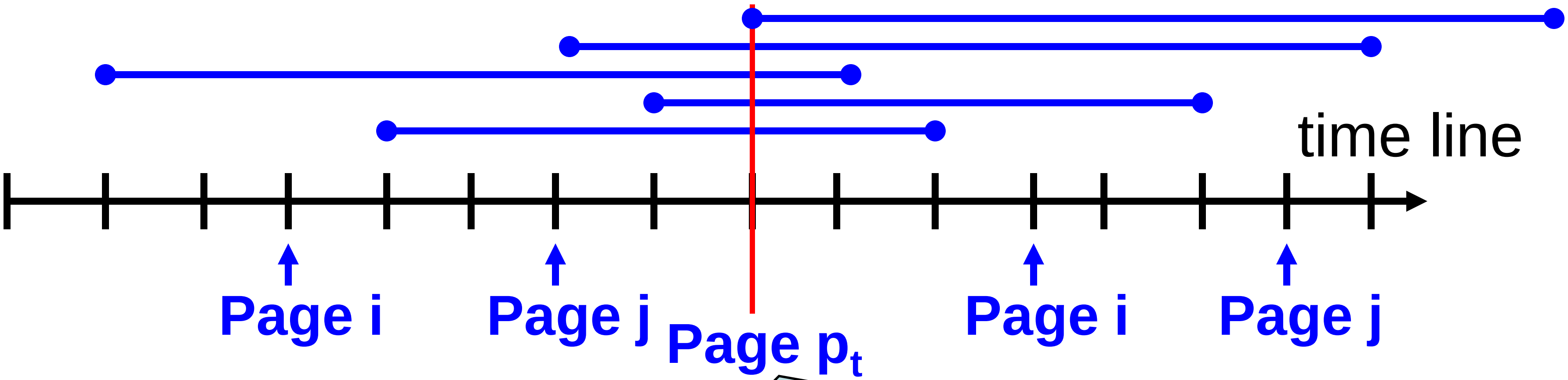
Overview

High level idea:

1. Design a primal-dual **O(log k)-competitive** algorithm for **fractional** weighted paging.

2. Obtain a **randomized algorithm** while losing only a **constant factor**.

Setting up the Linear Program



B(t): Set of pages requested until time t (including p_t)

We can only keep **k pages** out of the **B(t) pages**

Evict ≥ **[|B(t)| - 1 - (k- 1)] = [|B(t)|-k]** pages from B(t)\{p_t}

Weighted paging – Linear Program

$$\min \sum_{i=1}^{n} \sum_{j=1}^{r(i,t)} w(i)x(i,j)$$

$$\forall t \quad \sum_{i \in B(t) \setminus \{p_t\}} x(i, r(i,t)) \geq |B(t)| - k$$

$$0 \leq x(i,j) \leq 1$$

- Idea: charge for evicting pages instead of fetching pages

x(i,j) – indicator for the event that page i is evicted from the cache between the j-th and (j+1)-st times it is requested

r(i,t) - number of times page i is requested till time t, including t

Primal and Dual Programs

P: Primal Covering

$$\min \sum_{i=1}^{n} \sum_{j=1}^{r(i,t)} w(i)x(i,j)$$

$$\forall t \quad \sum_{i \in B(t) \setminus \{p_t\}} x(i, r(i,t)) \geq |B(t)| - k$$

$$0 \leq x(i,j) \leq 1$$

D: Dual Packing

$$\max \sum_{t} \left(|B(t)| - k\right) y(t) - \sum_{i=1}^{n} \sum_{j=1}^{r(i,t)} z(i,j)$$

For each page i and the *j* th time it was asked:

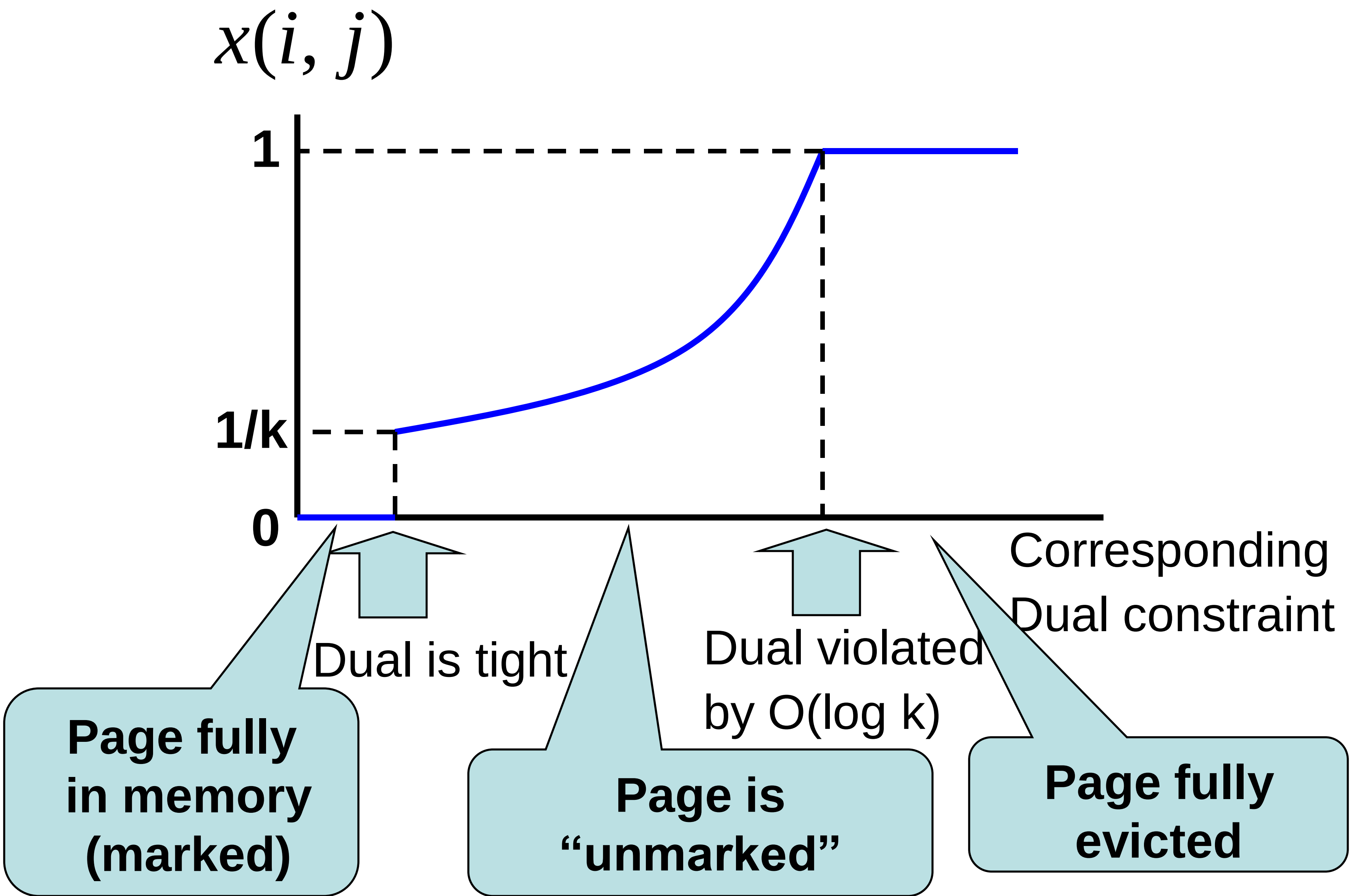
$$\left(\sum_{t=t(i,j)+1}^{t(i,j+1)-1} y(t) \right) - z(i,j) \leq w(i)$$

Fractional Caching Algorithm (1)

At time t, when page p_t is requested:

- Set the new variable: $x(p_t, r(p_t, t)) \leftarrow 0$:
  - this guarantees that p_t is in the cache at time t.
  - this variable can only be increased at times $t' > t$.
- If the primal constraint corresponding to time t is satisfied, then do nothing.
- Else, increase variables $x(i, j)$ as a function of $y(t)$, details follow soon ...

The growth function of x(i,j)



Fractional Caching Algorithm (2)

- Else: increase primal and dual variables, till primal constraint corresponding to time t is satisfied:

  1. Increase variable $y(t)$ continuously; for each variable $x(p,j)$ that appears in the (yet unsatisfied) primal constraint that corresponds to time t:
  2. If $x(p,j) = 1$, then increase $z(p,j)$ at the same rate as $y(t)$.
  3. If $x(p,j) = 0$ and

     $$\left(\sum_{t=t(p,j)+1}^{t(p,j+1)-1} y(t)\right) - z(p,j) = w(p),$$

     then set $x(p,j) \leftarrow 1/k$.
  4. If $1/k \leq x(p,j) < 1$, increase $x(p,j)$ by the following function:

     $$\frac{1}{k} \cdot \exp\left(\frac{1}{w(p)}\left[\left(\sum_{t=t(p,j)+1}^{t(p,j+1)-1} y(t)\right) - z(p,j) - w(p)\right]\right)$$

Analysis of Online Algorithm

Proof of competitive factor:

1. Primal solution is feasible.
2. Primal ≤ 2 · Dual
3. Dual is feasible up to a factor of O(log k)

Conclusion (weak duality):

Algorithm is O**(log k)-competitive**

Analysis of Online Algorithm

1. Primal solution is feasible. At time t:

   – for page p_t, $x(p_t,r(p_t,t))\leftarrow 0$, i.e., p_t is in the cache

   – primal variables $x(q,r(q,t))$ corresponding to other pages q are increased till primal constraint is satisfied

   – for each page q, by the algorithm, $x(q,r(q,t)) \leq 1$ (increase in z balances out increase in y)

Analysis of Online Algorithm

3. Dual is O(log k) feasible:

Consider any dual constraint.

since x(i,j)≤1:

$$1 \geq x(i,j) = \frac{1}{k} e^{\frac{\left(\sum_{t=t(i,j)+1}^{t(i,j+1)-1} y(t)\right) - z(i,j)}{w(i)} - 1}$$

Simplifying, we get that:

$$\left(\sum_{t=t(i,j)+1}^{t(i,j+1)-1} y(t)\right) - z(i,j) \leq w(i)\left[1 + \ln k\right]$$

Analysis of Online Algorithm

2. Primal ≤ **2 · Dual**

This is done in two separate steps:

- C_1 - contribution to the primal cost of the variables x(p,j) when increased from 0 to 1/k
- C_2 - contribution to the primal cost of the variables x(p,j) when increased from 1/k to (at most) 1, according to the exponential function

Each contribution is upper bounded separately by the dual

Bounding C_1

Define: $\tilde{x}(p,j) = \min(x(p,j), \frac{1}{k})$

Primal complementary slackness: if $\tilde{x}(p,j) > 0$,

$$\left(\sum_{t=t(p,j)+1}^{t(p,j+1)-1} y(t) \right) - z(p,j) \geq w(p)$$

Bounding C_1

- $B'(t)$ - set of pages $p \in B(t)$ for which $x(p, r(p,t)) = 1$

Dual complementary slackness (1): if $y(t)$ is being increased at time t then:

$$\sum_{p \in B(t) \setminus (B'(t) \cup \{p_t\})} \tilde{x}(p, r(p,t)) \leq \frac{|B(t)| - 1 - |B'(t)|}{k} \leq |B(t)| - k - |B'(t)|$$

- $|B(t)| - |B'(t)| \geq k + 1$ (else $|B'(t)| \geq |B(t)| - k$, satisfying constraint)
- $\Rightarrow \quad \frac{|B(t)|-1-|B'(t)|}{k} \leq |B(t)| - k - |B'(t)|$

Dual complementary slackness (2): if $z(p,j) > 0$, then $x(p,j) \geq 1$

Bounding C_1

$$\sum_{p=1}^{n} \sum_{j=1}^{r(p,t)} w(p)\tilde{x}(p,j) \leq$$

$$(by\ primal\ complementary\ slackness)$$

$$\leq \sum_{p=1}^{n} \sum_{j=1}^{r(p,t)} \left(\left(\sum_{t=t(p,j)+1}^{t(p,j+1)-1} y(t) \right) - z(p,j) \right) \tilde{x}(p,j) =$$

$$(changing\ order\ of\ summation)$$

$$= \sum_{t} \left(\sum_{i \in B(t) \setminus \{p_t\}} \tilde{x}(p, r(p,t)) \right) y(t) - \sum_{p=1}^{n} \sum_{j=1}^{r(p,t)} \tilde{x}(p,j) z(p,j)$$

Bounding C_1

$$\sum_t \left(\sum_{i \in B(t) \setminus \{p_t\}} \tilde{x}(p, r(p,t)) \right) y(t) - \sum_{p=1}^{n} \sum_{j=1}^{r(p,t)} \tilde{x}(p,j) z(p,j)$$

$$\leq \sum_t (|B(t)| - k)\, y(t) - \sum_{p=1}^{n} \sum_{j=1}^{r(p,t)} z(p,j)$$

- The derivative of the LHS is:

$$\sum_{p \in B(t) \setminus (B'(t) \cup \{p_t\})} \tilde{x}(p, r(p,t)) \leq |B(t)| - k - |B'(t)|$$

since $z(p,j)$ increases at the same rate as $y(t)$ when $x(p, r(p,t)) = 1$

- The derivative of the RHS is $|B(t)| - k - |B'(t)|$

Thus, C_1 is upper bounded by the dual solution

Bounding C_2

Reminder:

If $1/k \leq x(p,j) < 1$, increase $x(p,j)$ by the following function:

$$\frac{1}{k} \cdot \exp\left(\frac{1}{w(p)}\left[\left(\sum_{t=t(p,j)+1}^{t(p,j+1)-1} y(t)\right) - z(p,j) - w(p)\right]\right)$$

Bounding C_2

Variables $y(t)$ and $z(p,j)$ are raised at rate 1 with respect to virtual variable τ.

- $\frac{dy(t)}{d\tau} = 1, \ \frac{dx(p,j)}{dy(t)} = \frac{1}{w(p)} \cdot x(p,j)$

$$\begin{aligned}
\frac{dC_2}{d\tau} &= \sum_{p \in B(t) \setminus \{p_t\}, 1/k \leq x(p,j) < 1} w(p) \cdot \frac{dx(p, r(p,t))}{dy(t)} \cdot \frac{dy(t)}{d\tau} \\
&= \sum_{p \in B(t) \setminus \{p_t\}, 1/k \leq x(p,j) < 1} x(p, r(p,t)) \\
&\leq (|B(t)| - k) - \sum_{p \in B(t) \setminus \{p_t\}, x(p,j) = 1} 1 \\
&= \underbrace{(|B(t)| - k) \frac{dy(t)}{d\tau} - \sum_{p \in B(t) \setminus \{p_t\}, x(p,j) = 1} \frac{dz(p,j)}{d\tau}}_{\text{dual derivative}}
\end{aligned}$$

$$\text{dual objective} = \sum_t (|B(t)| - k)\, y(t) \ - \ \sum_{p=1}^{n} \sum_{j=1}^{r(p,t)} z(p,j)$$

Conclusion

- C_1 is upper bounded by a dual solution
- C_2 is upper bounded by a dual solution

Thus, primal ≤ **2 · dual**

The algorithm is O**(log k)-competitive**

Rounding

Linear program provides a fractional view:

Prob[p is in cache at time t] = 1-x(p,r(p,t))

Randomized alg.: distribution on cache states

Example: pages A,B,C,D k=2

LP state = (1/2,1/2,1/2,1/2)

Consistent distribution = ½ (A,B) + ½ (C,D)

Rounding – Need to be Careful

A,B have wt. 1, C,D have wt. M

LP state = (1/2,1/2,1/2,1/2)

Distribution = ½ (A,B) + ½ (C,D)

LP changes to (1,0,1/2,1/2)

LP cost = ½

randomized algorithm: only consistent distribution =

½(A,C) + ½ (A,D)

cost of randomized algorithm:

(½ (A,B) + ½ (C,D)) ⇒ (½(A,C) + ½ (A,D))

$\Theta(M)$ – either C or D are (partly) evicted

Rounding – Main Ideas

- **Partition** the pages into weight classes:
  - class i pages with size [2i, 2i+1]
- Define a **distribution *D*** on cache states
  - each cache state has ***approximately*** the same number of pages from **each class**.
- Show how to **update** the distribution on the cache states while paying at most **5 times the fractional cost.**

Further Extensions of the Basic Model

First Extension:

- Pages have **different fetching costs**.
- Models scenarios in which the fetching cost is not uniform:

Main memory, disk, internet …



Second (Orthogonal) Extension:

- Pages have **different sizes**.
- Models web-caching problems (Proxy Servers, local cache in browser)

Caching Models

Fetch cost \ Size	Uniform	Non-uniform
Unif		Fault Model
Non-uniform	Weighted Caching	1. General Caching 2. Bit Model (fetching cost= size)

Minimize number of times the user has to wait

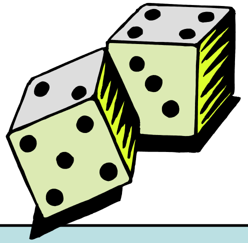
Offline is NP-Hard (Simple reduction from Knapsack/Partition)

Deterministic Algorithms

Any Algorithm ≥ k-competitive

Fetch cost \ Size	Uniform	Non-uniform
Uniform	Basic Caching LRU is k-competitive (also other algorithms)	Fault Model LRU is k-competitive [Irani]
Non-uniform	Weighted Caching k-competitive [Chrobak, Karloff, Payne, Vishwanathan]	1. General Caching k-competitve [Irani,Cao], [Young] 2. Bit Model LRU k-competitve [Irani]

Randomized Algorithms



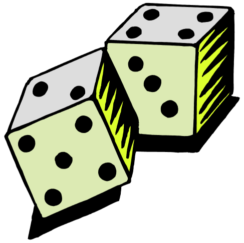
Other algorithms that are optimal with constants

d Algorithm

mpetitive

Si / Fetch cost	Uniform	Non-uniform
Uniform	Basic Caching Randomized Marking $O(\log k)$-competitive [Fiat et al.]	Fault Model $O(\log^2 k)$-competitive algorithm [Irani]
Non-uniform	Weighted Caching $O(\log k)$-competitive algorithm [Bansal, Buchbinder, Naor]	1. General Caching  2. Bit Model $O(\log^2 k)$-competitive algorithm [Irani]

Improved Results



Size / Fetch cost	Uniform	Non-uniform
Uniform	Basic Caching Randomized Marking O(log k)-competitive [Fiat et al.]	Fault Model ~~$O(\log^2 k)$-competitive~~ $O(\log k)$-competitive
Non-uniform	Weighted Caching O(log k)-competitive algorithm [Bansal, Buchbinder, Naor]	1. General Caching $O(\log^2 k)$-competitive 2. Bit Model ~~$O(\log^2 k)$-competitive~~ $O(\log k)$-competitive

Basic Definitions: Generalized Caching

- n pages
- Cache of size k
- Size of page p: $w_p \in [1,k]$
- Fetching cost of page p: c_p (arbitrary)

Fractional solution:

- Algorithm maintains fractions of pages as long as the total size does not exceed k.
- Fetching ε fraction of page p costs εc_p

High level approach

First step:

- General O(log k)-competitive algorithm for the fractional generalized caching.

➔ Maintains fractions on pages.

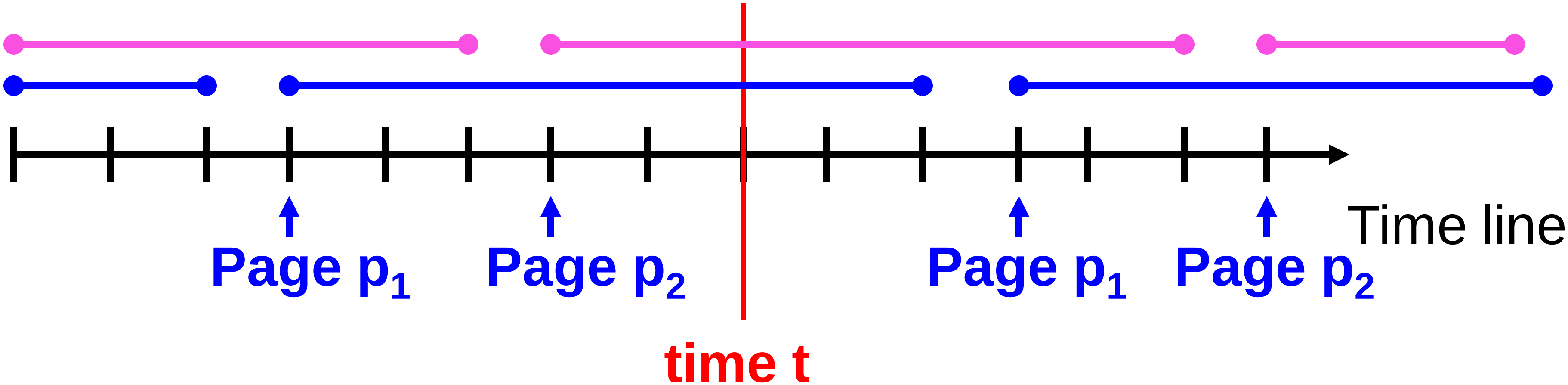
Second Step:

Transform **online** the fractional solution into Randomized algorithm:

- Maintain **distribution on cache states** that is "consistent" with the fractional solution.
- Simulation procedure maps changes in fractions on pages to distribution on cache states (w/ similar cost).
- O(1) simulation for Bit/Fault model

  O(log k) simulation for the general model.

Generalized Caching – Linear Program



- Interval: Keep the page between the jth time it is requested and the (j+1) time it is requested.
- If interval present, no cache miss.
- At any time step t, **total size of intervals** (pages) is at most k.

Generalized Caching: 1st LP formulation

- **x(p,j):** How much of interval (p,j) evicted thus far
- **B(t):** Set of pages requested until time t.
- **W(B(t)):** total size of pages in B(t).
- **r(p,t):** number of times page p requested until time t

P: Primal Covering

$$\min \sum_{p=1}^{n} \sum_{j=1}^{r(p,t)} c_p x(p,j)$$

$$\forall t \quad \sum_{p \in B(t) \setminus \{p_t\}} w_p \cdot x(p, r(p,t)) \geq W(B(t)) - k$$

$$0 \leq x(p,j) \leq 1$$

Problem with LP formulation

The formulation has unbounded integrality gap …

Example:

- Two pages of **size k/2+ε** requested alternately.
- Integral solution: cache miss every turn
- Fractional solution:
  - Keeps almost one unit of each page.
  - Needs to fetch only **O(ε/k)** page every turn

P: Primal Covering

$$\min \sum_{p=1}^{n} \sum_{j=1}^{r(p,t)} c_p x(p,j)$$

$$\forall t \quad \sum_{p \in B(t) \setminus \{p_t\}} w_p \cdot x(p, r(p,t)) \geq W(B(t)) - k$$

$$0 \leq x(p,j) \leq 1$$

Generalized Caching: 2nd LP formulation

Strengthening the LP:

P: Primal Covering

For any time t and set S:

$$\sum_{p \in B(t) \setminus \{p_t\}} \min\{W(S) - k, w_p\} \cdot x(p, r(p,t)) \geq W(S) - k$$

$$0 \leq x(p,j) \leq 1$$ (crossed out)

after strengthening, box constraints are redundant

These are called:
knapsack inequalities

D: Dual Packing $\max \sum_{t} \sum_{S \subseteq B(t), p_t \in S} \left(W(s) - k\right) \cdot y(t, S)$

For each page p and the jth time it is requested:

$$\sum_{t=t(p,j)+1}^{t(p,j+1)-1} \min\{W(s) - k, w_p\} \cdot y(t, S) \leq c_p$$

P

D*=P*

D

Sketch of Primal-Dual algorithm

- While there exists an **unsatisfied primal constraint** of set of pages S and time t:
- Increase the dual variable y(t,S).

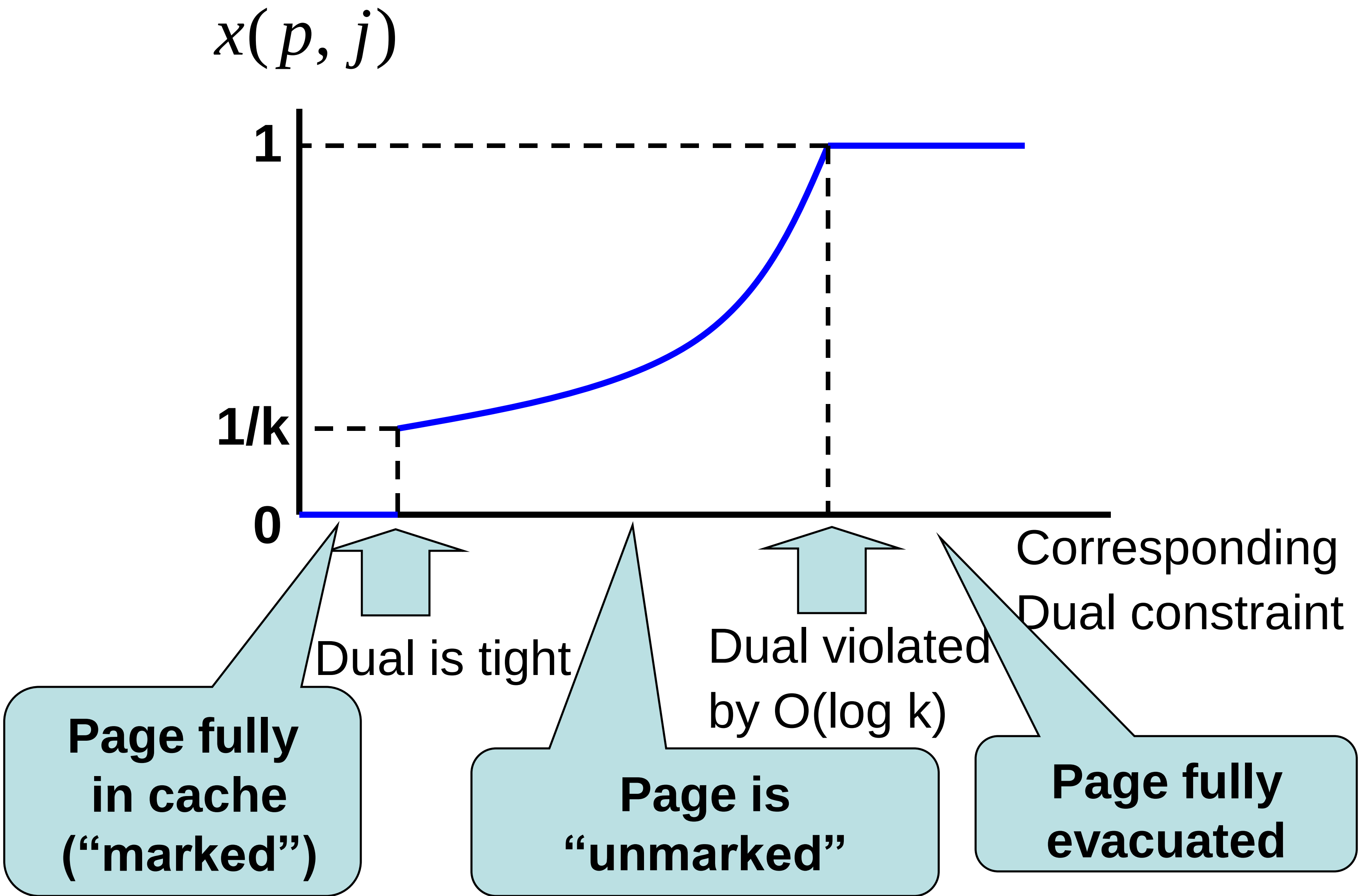
When dual constraint of variable x(p,j) is tight, x(p,j) = 1/k

$$\sum_{t=t(p,j)+1}^{t(p,j+1)-1} \min\{W(s)-k, w_p\} \cdot y(t,S) = c_p$$

From then on, increase x(p,j) exponentially (until x(p,j)=1)

$$x(p,j) = \left(\frac{1}{k}\right) \exp\left[\frac{1}{c_p}\left(\sum_{t=t(p,j)+1}^{t(p,j+1)-1} \min\{W(s)-k, w_p\} \cdot y(t,S)\right) - 1\right]$$

The growth function of x(p,j)



Analysis of Online Algorithm

Proof of competitive factor:

1. Primal solution is feasible.
2. Primal ≤ 2 Dual.
3. Dual is feasible up to O(log k) factor

Conclusion (weak duality):

Algorithm is **O(log k)-competitive**

Analysis - sketch

1. Primal solution is feasible.

   We increase x(p,j)'s until current primal constraint is feasible

2. Primal ≤ 2 Dual:

   a. Setting x(p,j) to 1/k analyzed using **complementary slackness**

   b. During the **exponential** growth the **primal derivative is at most dual derivative**

3. Dual is O(log k) feasible:

$$x(p,j) = \left(\frac{1}{k}\right)\exp\left[\frac{1}{c_p}\left(\sum_{t=t(p,j)+1}^{t(p,j+1)-1} \min\{W(s)-k, w_p\}\cdot y(t,S)\right)-1\right] \le 1$$

$$\sum_{t=t(p,j)+1}^{t(p,j+1)-1} \min\{W(s)-k, w_p\}\cdot y(t,S) \le c_p\left(1+\ln(k)\right)$$

Concluding Remarks

- Primal-dual approach gives simple unifying framework for caching.

Open questions:

1. Improving to O(log k) for the general model.
2. o(k) randomized algorithms for k-server using primal-dual approach.
3. Extend primal-dual framework beyond packing/covering.