ADFOCS 2009: Exercises #1

Tim Roughgarden

Instructions:

(1) You should not necessarily try to not complete all the problems. Rather, pick one you find interesting, and work on/discuss it until you feel like moving on to another (and then repeat).

(2) Collaboration with your fellow attendees is *strongly encouraged*.

Problem 1

Consider the following type of "bicriteria bound": the cost of a Nash equilibrium is at most that of an optimal outcome that has twice as many players. Such a bound holds in nonatomic selfish routing networks (with arbitrary continuous, nondecreasing cost functions). The idea of the original proof is: take a flow at Nash equilibrium f; reset the cost functions to be $\max\{c_e(f_e), c_e(x)\}$ on every edge; prove that, with these new cost functions, every flow with double the traffic has at least double the original cost of f; and prove that these bigger cost functions only increase the cost of any flow by the original cost of f. [Possible exercise: think through the details of this proof.]

What if the cost functions are restricted? In particular, prove the following: in nonatomic selfish routing networks with affine cost functions, the cost of a flow at Nash equilibrium is at most that of an optimal flow with 25% more traffic (for each commodity). Can you state and prove a general result, parametrized by an arbitrary set of cost functions, that interpolates between the affine and unrestricted cases?

[Hint: think along the lines of the geometric proof from lecture that the POA is at most 4/3 in the affine cost function case.]

Problem 2

This problem considers selfish routing in the *atomic splittable* model. The key difference between this model and the usual atomic selfish routing model is that a player i is permitted to route its r_i units of traffic *fractionally* over the s_i-t_i paths of the network. This model is also different from nonatomic selfish routing games; for example, if there is only one player controlling all of the traffic in the network, then the player will minimize its cost by routing this traffic optimally. More generally, a player takes into account the congestion it causes for its own traffic, while ignoring the congestion it creates for other players.

Given an atomic splittable selfish routing game, we can obtain a new game by replacing a player that routes r_i units of traffic from s_i to t_i by two players that each route $r_i/2$ units of traffic from s_i to t_i. This operation does not change the cost of an optimal flow. Intuitively, since it decreases the amount of cooperation in the network, it should only increase the cost of an equilibrium flow. Prove that this intuition is incorrect: in multicommodity atomic splittable selfish routing networks, splitting a player in two can decrease the price of anarchy.

For "extra credit", show that the POA in atomic splittable selfish routing games with affine cost functions is strictly more than 4/3. [A prominent open question is to precisely characterize the worst-case POA in the atomic splittable model; this is open even for affine cost functions.]

Problem 3

In this problem we consider nonatomic selfish routing networks with one source, one sink, one unit of selfish traffic, and affine cost functions (of the form $c_e(x) = a_e x + b_e$ for $a_e, b_e \geq 0$). In parts (a)-(c), we consider the objective of the *maximum cost* incurred by a flow f:

$$\max_{P\,:\,f_P>0} \sum_{e \in P} c_e(f_e).$$

The *price of anarchy* is then defined in the usual way, as the ratio between the maximum cost of an equilibrium flow and that of a flow with minimum-possible maximum cost. (Of course, in an equilibrium flow, all traffic incurs exactly the same cost; this is not generally true in a non-equilibrium flow.)

(a) Prove that in a network of parallel links (each directly connecting the source to the sink), the price of anarchy with respect to the maximum cost objective is 1.

(b) Prove that the price of anarchy with respect to the maximum cost objective can be as large as 4/3 in general networks (with affine cost functions, one source and one sink).

(c) Prove that the price of anarchy with respect to the maximum cost objective is never larger than 4/3 (in networks with affine cost functions, one source and one sink).

(d) A flow that minimizes the average cost of traffic generally routes some traffic on costlier paths than others. Prove that the ratio between the cost of the longest used path and that of the shortest used path in a minimum-cost flow is at most 2 (in networks with affine cost functions, one source and one sink). Prove that this bound can be achieved.

Problem 4

(a) Consider an atomic selfish routing game in which all players have the same source vertex and sink vertex (and each controls one unit of flow). Assume that edge cost functions are nondecreasing, but do not assume that they are affine. Prove that a (pure-strategy) Nash equilibrium (i.e., an equilibrium flow) can be computed in polynomial time.

[Hint: Think about the potential function and the classical minimum-cost flow problem.]

(b) Prove that in an atomic selfish routing network of parallel links, every equilibrium flow minimizes the potential function.

(c) Show by example that (b) does not hold in general networks, even when all players have a common source and sink vertex.

Problem 5

Recall that in a mixed-strategy Nash equilibrium, each player picks a probability distribution over strategies to maximize its expected payoff (equivalently, minimize its expected cost). Exhibit an atomic selfish routing network and a mixed-strategy Nash equilibrium of it that has expected cost strictly larger than the cost of every pure-strategy Nash equilibrium of the network. On the other hand, in the AAE example, show that every mixed-strategy Nash equilibrium has expected cost at most 2.5 times that of an optimal solution (i.e., there are no mixed equilibria strictly worse than the worst pure equilibrium).

ADFOCS 2009: Exercises #2

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Problem 6

Recall our discussion of Bayesian-optimal and prior-free revenue-maximizing auctions.

(a) For prior-free multi-item auctions, prove that the "limited supply" case reduces to that of "unlimited supply", in the following sense. Let k and n denote the number of identical items and of bidders, respectively. Suppose that, for some $c \geq 1$, there is a (possibly randomized) truthful auction for the $k = n$ case with expected revenue at least a $1/c$ fraction of the fixed-price benchmark

$$\mathcal{F}^{(2)}(b) := \max_{2 \leq i \leq n} i \cdot b_i,$$

for every bid vector b (we are assuming without loss that $b_1 \geq b_2 \geq \cdots \geq b_n$).

Using this assumption, prove that, for every $k \in \{2, 3, \ldots, n\}$, there is a (possibly randomized) truthful auction for the case with only k identical goods that has expected revenue at least a $1/c$ fraction of

$$\mathcal{F}^{(2,k)}(b) := \max_{2 \leq i \leq k} i \cdot b_i,$$

the optimal fixed-price revenue subject to the supply constraint.

(b) Use Myerson's Lemma to prove that every *deterministic* truthful auction with identical goods is equivalent to an auction of the following form: given bid vector b, offer each bidder i a posted price (a "take-it-or-leave-it" offer) of $t_i(b_{-i})$, where t_i is an arbitrary function of the other bids, with range $[0, +\infty]$, and ties (when $b_i = t_i(b_{-i})$) broken arbitrarily.

(c) An auction of the form in (b) is *symmetric* if all of the functions $t_1(\cdot), \ldots, t_n(\cdot)$ are a common function $t(\cdot)$, which is itself symmetric (i.e., invariant under permutations of its arguments). Prove that for every constant $c > 1$, no deterministic symmetric auction for digital goods (i.e., with $k = n$) is c-competitive with respect to the fixed-price benchmark $\mathcal{F}^{(2)}(b)$.

[Hint: Consider bid vectors with only "high" and "low" bids.]

(f) Instead of i.i.d. draws from a distribution F, suppose we know that the ith valuation v_i is drawn from the distribution F_i with positive density f_i on $[0, 1]$. Assume that each F_i is regular in the sense of lecture. Describe the Bayesian-optimal auction in this case.

Problem 7

You may have heard of *sponsored search*, meaning the paid links that appear whenever you type a search query into a search engine. These links are chosen using an auction mechanism, and the underlying allocation problem can be thought of as follows. There are k slots and n bidders. Slot i has a "click-through-rate" (or CTR) of α_i. Assume that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$. An auction accepts a bid from each player, assigns at most k players to the slots, and charges various prices to the winners. The assumption is that if a player has valuation v, receives slot i, and has to pay p, then its net utility is $v\alpha_i - p$. (The interpretation is that the bidder has value v for a click, and α_i is the probability that it gets a click given that it is displayed in slot i.)

Argue that ranking bidders by bid (i.e., with the highest bidder in the top slot, the next bidder in the next slot, etc.) is a monotone allocation rule. Apply Myerson's Lemma to this allocation rule to derive an explicit formula for truthful payments. Notice p is the overall price; to express it in "price per click" units, one would divide by α_i.

Contrast the payments above to the more naive approach — which is not truthful but forms the basis for the auctions used in practice — in which for each i, the ith bidder's price per click is the bid (per click) of the next bidder (so the top bidder pays the second-highest bid for each of its clicks, and so on).

Problem 8

Consider the following pricing problem. There is one consumer who wants at most one of n non-identical goods. Assume that the consumer's private valuations $v_1, \ldots, v_n$ for the n goods are i.i.d. draws from a known regular prior distribution F. Our goal is to set prices $p_1, \ldots, p_n$ for the n goods (which can depend on F but not the actual v_i's) to maximize expected revenue, assuming that the consumer responds to prices by picking the good that maximizes $v_i - p_i$ (or picking no good if $p_i > v_i$ for every i).

(a) Prove that the maximum-achievable expected revenue is bounded above by the expected revenue of an optimal single-good auction with n bidders with valuations drawn i.i.d. from F.

(b) Design a simple pricing algorithm that (for every regular distribution F) obtains expected revenue at least a constant fraction of that of an optimal single-good auction with n bidders with valuations drawn i.i.d. from F. Do your best to optimize the constant.

Problem 9

The following open problem is a step toward prior-free revenue-maximizing auctions with non-identical bidders (which corresponds to a Bayesian setting with non-i.i.d. distributions). The setting is digital goods, so there are n bidders and n identical items (each bidder i only wants one item and it has a private valuation v_i for it). For a given input v, define $M(v)$ as the maximum revenue that can be obtained from using a *nonincreasing* vector of take-it-or-leave-it offers. So if the price for bidder #1 (who is not necessarily the highest bidder) is 100, then every other bidder's price is at most 100. Notice that v can be anything, and need *not* be nonincreasing in the bidder index. So for example, if v is increasing (i.e., sorted in the wrong way), then $M(v)$ is equivalent to the maximum revenue that can be obtained from the bidders using a common take-it-or-leave-it offer. (If v is sorted in the right way, then $M(v)$ is just the sum of the valuations.)

The open problem is to design a truthful auction that, for every input v (sorted or not), achieves revenue at least $c \cdot M(v) - h$, where $c > 0$ is a constant and h is the maximum possible valuation. (Actually, it should be possible to do this with h replaced by the difference between the largest and second-largest valuations of v.) [For a weaker result, see Aggarwal-Hartline, SODA 06.]