Approximation algorithms for discrete stochastic optimization problems

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Stochastic Optimization

- Way of modeling uncertainty.
- Exact data is unavailable or expensive – data is uncertain, specified by a probability distribution.

  Want to make the best decisions given this uncertainty in the data.
- Dates back to 1950's and the work of Dantzig.
- Applications in logistics, transportation models, financial instruments, network design, production planning, …

Two-Stage Recourse Model

Given : Probability distribution over inputs.

Stage I : Make some advance decisions – plan ahead or hedge against uncertainty.

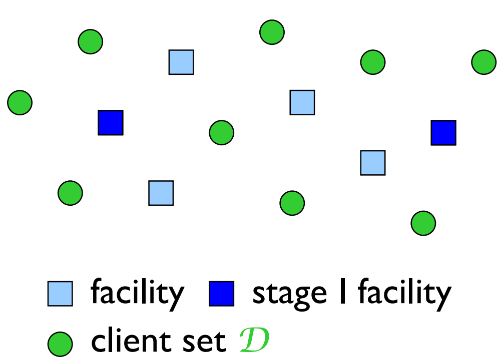
Observe the actual input scenario.

Stage II : Take recourse. Can augment earlier solution paying a recourse cost.

Choose stage I decisions to minimize

(stage I cost) + (expected stage II recourse cost).

2-Stage Stochastic Facility Location

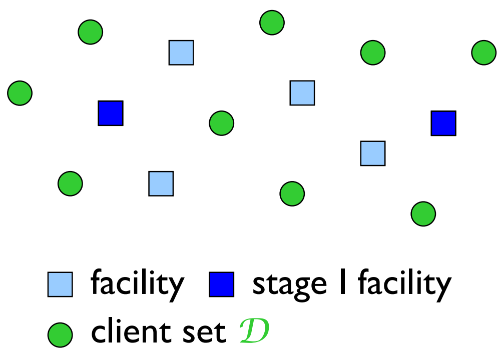


Distribution over clients gives the set of clients to serve.

Stage I: Open some facilities in advance; pay cost f_i for facility i.

Stage I cost = $\sum_{(i \text{ opened})} f_i$.

2-Stage Stochastic Facility Location



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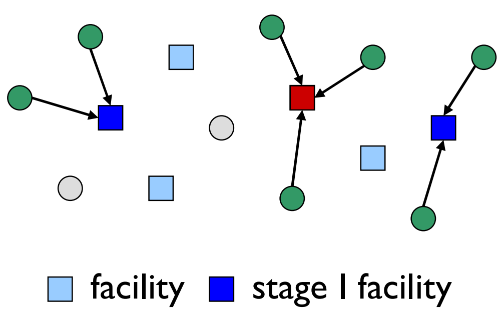
How is the probability distribution on clients specified?

- A short (polynomial) list of possibile scenarios;
- Independent probabilities that each client exists;
- A black box that can be sampled.

2-Stage Stochastic Facility Location



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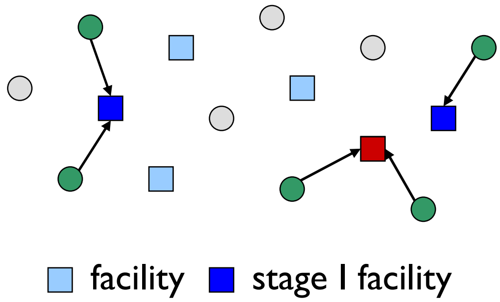
Stage I cost = $\sum_{(i \text{ opened})} f_i$.

Actual scenario A = { ● clients to serve}, materializes.

Stage II: Can open more facilities to serve clients in A; pay cost f_i^A to open facility i. Assign clients in A to facilities.

Stage II cost = $\sum_{\substack{i \text{ opened in} \\ \text{scenario } A}} f_i^A$ + (cost of serving clients in A).

2-Stage Stochastic Facility Location



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Want to decide which facilities to open in stage I.

Goal: Minimize Total Cost =

$$(\text{stage I cost}) + \mathbf{E}_{A \subseteq \mathcal{D}}[\text{stage II cost for } A].$$

We want to prove a worst-case guarantee.

Give an algorithm that "works well" on any instance, and for any probability distribution.

A is an α-approximation algorithm if -

- A runs in polynomial time;
- $A(I) \leq \alpha.OPT(I)$ on all instances I.

α is called the approximation ratio of A.

Goals of this Tutorial

- Focus on techniques of approximation algorithm design
  - LP-rounding
  - Primal-dual algorithms and analysis
  - Random sampling
- Five illustrative problems
  - Set cover problem
  - Facility location problem
  - Steiner tree problem
  - Traveling salesman problem
  - Maximum-weight on-time scheduling

Stochastic Set Cover (SSC)

Universe $U = \{e_1, \ldots, e_n\}$, subsets $S_1, S_2, \ldots, S_m \subseteq U$, set S has weight ω_S.

Deterministic problem: Pick a minimum weight collection of sets that covers each element.

Stochastic version: Set of elements to be covered is given by a probability distribution.

- choose some sets initially paying ω_S for set S
- subset $A \subseteq U$ to be covered is revealed
- can pick additional sets paying W_S for set S.

Minimize (ω-cost of sets picked in stage I) +
$\mathbf{E}_{A \subseteq U}$ [W_S -cost of new sets picked for scenario A].

An LP formulation

p_A : probability of scenario $A \subseteq U$.
x_S : indicates if set S is picked in stage I.
$y_{A,S}$: indicates if set S is picked in scenario A.

Minimize $\sum_S \omega_S x_S + \sum_{A \subseteq U} p_A \sum_S W_S y_{A,S}$
subject to,

$$\sum_{S:e \in S} x_S + \sum_{S:e \in S} y_{A,S} \geq 1 \qquad \text{for each } A \subseteq U,\ e \in A$$

$$x_S,\ y_{A,S} \geq 0 \qquad \text{for each } S, A.$$

Exponential number of variables and exponential number of constraints.

A Rounding Theorem (S & Swamy)

Stochastic Problem: LP can be solved in polynomial time.

Example: polynomial scenario setting

Deterministic problem: α-approximation algorithm A with respect to the LP relaxation, $\mathcal{A}(I) \leq \alpha \cdot \text{LP-OPT}(I)$ for each I.

Example: "the greedy algorithm" for set cover is a $\log n$-approximation algorithm w.r.t. LP relaxation.

Theorem: Can use such an α-approx. algorithm to get a 2α-approximation algorithm for stochastic set cover.

Rounding the LP

Assume LP can be solved in polynomial time.

Suppose we have an α-approximation algorithm wrt. the LP relaxation for the deterministic problem.

Let (x,y) : optimal solution with cost LP-OPT.

$$\sum_{S:e\in S} x_S + \sum_{S:e\in S} y_{A,S} \geq 1 \qquad \text{for each } A \subseteq U,\ e\in A$$

$\Rightarrow$ for every element e, either

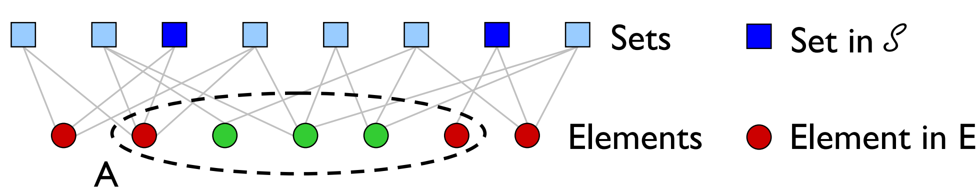
$\sum_{S:e\in S} x_S \geq ½$ OR in each scenario $A : e\in A$, $\sum_{S:e\in S} y_{A,S} \geq ½$.

Let $E = \{e : \sum_{S:e\in S} x_S \geq ½\}$.

So $(2x)$ is a fractional set cover for the set $E \Rightarrow$ can "round" to get an integer set cover $\mathcal{S}$ for E of cost $\sum_{S\in\mathcal{S}} \omega_S \leq \alpha(\sum_S 2\omega_S x_S)$.

$\mathcal{S}$ is the first stage decision.

Rounding (contd.)



Consider any scenario A. Elements in $A \cap E$ are covered.

For every $e \in A\backslash E$, it must be that $\sum_{S:e\in S} y_{A,S} \geq ½$.

So $(2y^A)$ is a fractional set cover for $A\backslash E \Rightarrow$ can round to get a set cover of W-cost $\leq \alpha(\sum_S 2W_S y_{A,S})$.

Using this to augment $\mathcal{S}$ in scenario A, expected cost

$$\leq \sum_{S\in\mathcal{S}} \omega_S + 2\alpha\cdot\sum_{A\subseteq U} p_A \left(\sum_S W_S y_{A,S}\right) \leq 2\alpha\cdot\text{LP-OPT}.$$

A Rounding Theorem

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A Rounding Technique

Assume LP can be solved in polynomial time.

Suppose we have an α-approximation algorithm w.r.t. the LP relaxation for the deterministic problem.

Let (x,y) : optimal solution with cost OPT.

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A Compact Formulation

p_A : probability of scenario $A \subseteq U$.
x_S : indicates if set S is picked in stage I.

Minimize $h(x) = \sum_S \omega_S x_S + f(x)$ s.t. $x_S \geq 0$ for each S

where, $f(x) = \sum_{A \subseteq U} p_A f_A(x)$

and $f_A(x) = \min. \sum_S W_S y_{A,S}$

s.t. $\sum_{S:e \in S} y_{A,S} \geq 1 - \sum_{S:e \in S} x_S$ for each $e \in A$

$y_{A,S} \geq 0$ for each S.

Equivalent to earlier LP.

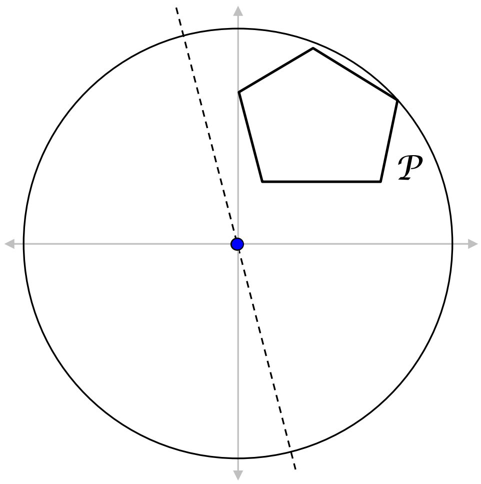
Each $f_A(x)$ is convex, so $f(x)$ and $h(x)$ are convex functions.

Solving the Stochastic LP?

- The LP has exponential number of variables and constraints, but can give other compact formulation as convex program that focuses on Stage I
- Can compute a fractional solution of cost at most $(1+\varepsilon)$LP-OPT with probability at least $1-\delta$ in time polynomial in input size and $\lambda = \max_S W_S/\omega_S$
- Many approaches are possible, including ellipsoid method and sample average approximation [S&Swamy, Nemirovski& Shapiro, Charikar, Chekuri, & Pál]

The Ellipsoid Method

Min c·x subject to x∈ $\mathcal{P}$.



Ellipsoid ≡ squashed sphere

Start with ball containing polytope $\mathcal{P}$.

y_i = center of current ellipsoid.

If y_i is infeasible, use violated inequality to chop off infeasible half-ellipsoid.

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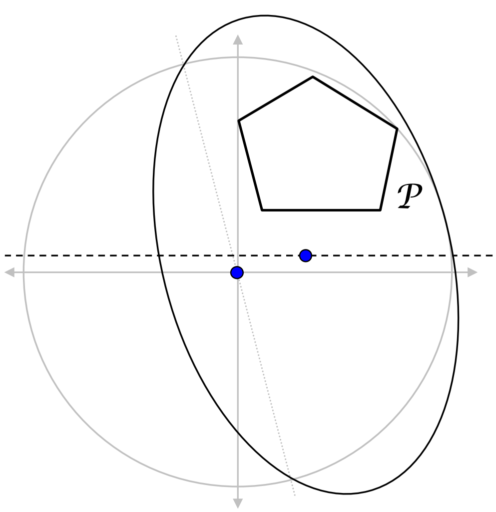

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New ellipsoid = min. volume ellipsoid containing “unchopped” half-ellipsoid.

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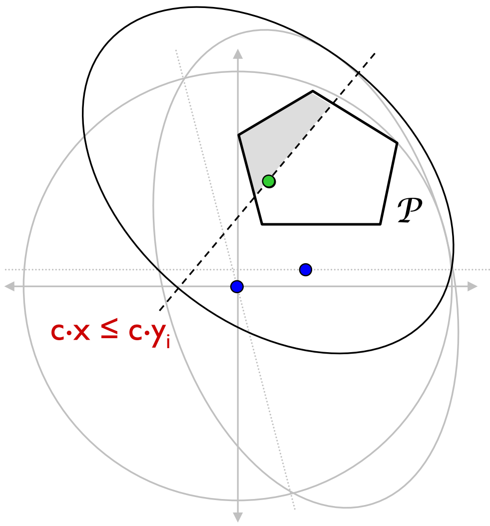

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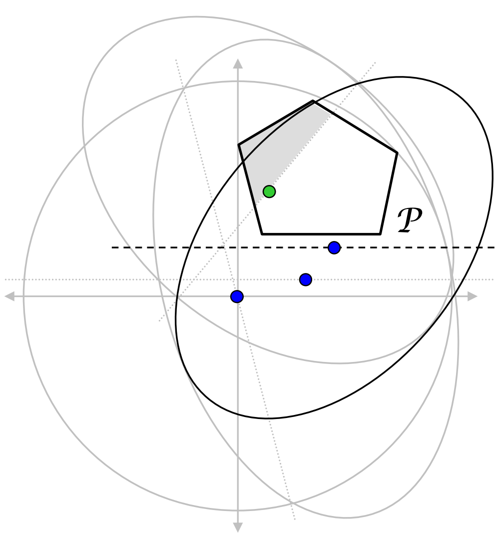

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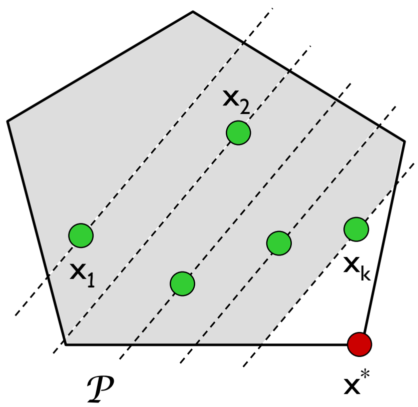

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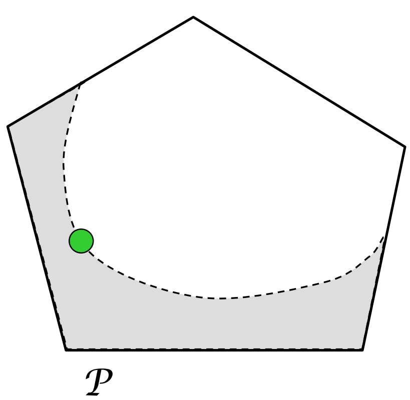
New ellipsoid = min. volume ellipsoid containing "unchopped" half-ellipsoid.

$x_1, x_2, \ldots, x_k$: points lying in $\mathcal{P}$. $c \cdot x_k$ is a close to optimal value.

Ellipsoid for Convex Optimization

Min $h(x)$ subject to $x \in \mathcal{P}$.



Start with ball containing polytope $\mathcal{P}$.

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- If y_i is infeasible, use violated inequality.
- If $y_i \in \mathcal{P}$ – how to make progress?
  add inequality $h(x) \leq h(y_i)$? Separation becomes difficult.

Ellipsoid for Convex Optimization

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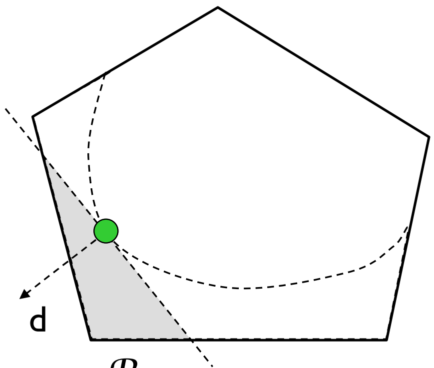

$\mathcal{P}$

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If y_i is infeasible, use violated inequality.

If $y_i \in \mathcal{P}$ – how to make progress?
add inequality $h(x) \leq h(y_i)$? Separation becomes difficult.
Let d = subgradient at y_i.
use subgradient cut $d \cdot (x - y_i) \leq 0$.

Generate new min. volume ellipsoid.

$d \in \Re^n$ is a subgradient of $h(.)$ at u, if for every v, $h(v)-h(u) \geq d \cdot (v-u)$.

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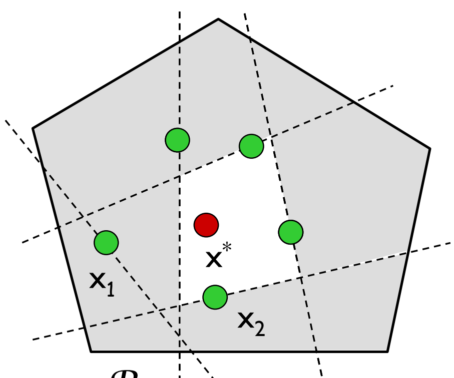

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$x_1, x_2, \ldots, x_k$: points in $\mathcal{P}$. Can show, $\min_{i=1\ldots k} h(x_i) \leq OPT+\rho$.

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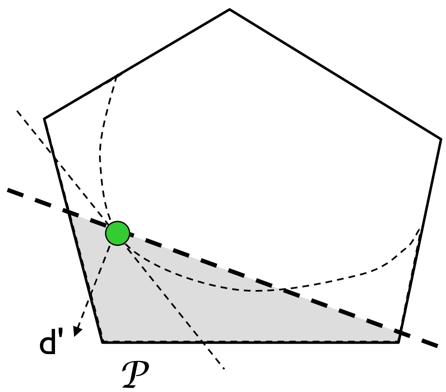

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If $y_i \in \mathcal{P}$ – how to make progress?
add inequality $h(x) \le h(y_i)$? Separation becomes difficult.
subgradient is difficult to compute.
Let d' = ε-subgradient at y_i.
use ε-subgradient cut $d' \cdot (x - y_i) \le 0$.

$d' \in \Re^n$ is a ε-subgradient of $h(.)$ at u, if $\forall v \in \mathcal{P}$, $h(v)-h(u) \ge d' \cdot (v-u) - \varepsilon \cdot h(u)$.

$x_1, x_2, \ldots, x_k$: points in $\mathcal{P}$. Can show, $\min_{i=1\ldots k} h(x_i) \le OPT/(1-\varepsilon) + \rho$.

Subgradients and ε-subgradients

Vector d is a subgradient of $h(.)$ at u,
if for every v, $h(v) - h(u) \ge d \cdot (v-u)$.

Vector d' is an ε-subgradient of $h(.)$ at u,
if for every $v \in \mathcal{P}$, $h(v) - h(u) \ge d' \cdot (v-u) - \varepsilon \cdot h(u)$.

$\mathcal{P} = \{ x : 0 \le x_S \le 1$ for each set $S \}$.

$h(x) = \sum_S \omega_S x_S + \sum_{A \subseteq U} p_A f_A(x) = \omega \cdot x + \sum_{A \subseteq U} p_A f_A(x)$

Lemma: Let d be a subgradient at u, and d' be a vector such that $d_S - \varepsilon\omega_S \le d'_S \le d_S$ for each set S. Then, d' is an ε-subgradient at point u.

Getting a “nice” subgradient

$h(x) = \omega \cdot x + \sum_{A \subseteq U} p_A f_A(x)$

$f_A(x) =$ min. $\sum_S W_S y_{A,S}$

s.t. $\sum_{S: e \in S} y_{A,S} \geq 1 - \sum_{S: e \in S} x_S \quad \forall e \in A$

$y_{A,S} \geq 0 \quad \forall S$

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s.t. $\sum_{S: e \in S} y_{A,S} \geq 1 - \sum_{S: e \in S} x_S \quad \forall e \in A$ s.t. $\sum_{e \in A \cap S} z_{A,e} \leq W_S \quad \forall S$

$y_{A,S} \geq 0 \quad \forall S$ $z_{A,e} \geq 0 \quad \forall e \in A$

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$y_{A,S} \geq 0 \quad \forall S \qquad z_{A,e} = 0 \quad \forall e \notin A, \quad z_{A,e} \geq 0 \quad \forall e$

Consider point $u \in \Re^n$. Let $z_A \equiv$ optimal dual solution for A at u.

Lemma: For any point $v \in \Re^n$, we have $h(v) - h(u) \geq d \cdot (v-u)$ where $d_S = \omega_S - \sum_{A \subseteq U} p_A \sum_{e \in S} z_{A,e}$.

$\Rightarrow$ d is a subgradient of h(.) at point u.

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Consider point $u \in \Re^n$. Let $z_A \equiv$ optimal dual solution for A at u. So

$$f_A(u) = \sum_e (1 - \sum_{S: e \in S} u_S) z_{A,e}.$$

For any other point v, z_A is a feasible dual solution for A. So

$$f_A(v) \geq \sum_e (1 - \sum_{S: e \in S} v_S) z_{A,e}.$$

Get that $h(v) - h(u) \geq \sum_S (\omega_S - \sum_{A \subseteq U} p_A \sum_{e \in S} z_{A,e})(v_S - u_S) = d \cdot (v-u)$ where $d_S = \omega_S - \sum_{A \subseteq U} p_A \sum_{e \in S} z_{A,e}$. So d is a subgradient of h(.) at point u.

Computing an ε-Subgradient

Given point $u \in \Re^n$. $z_A \equiv$ optimal dual solution for A at u.

Subgradient at u: $d_S = \omega_S - \sum_{A \subseteq U} p_A \sum_{e \in S} z_{A,e}$.

Want: d' such that $d_S - \varepsilon\omega_S \le d'_S \le d_S$ for each S.

For each S, $-W_S \le d_S \le \omega_S$. Let $\lambda = \max_S W_S / \omega_S$.

Sample once from black box to get random scenario A.

Compute X with $X_S = \omega_S - \sum_{e \in S} z_{A,e}$.

$\mathbf{E}[X_S] = d_S$ and $\mathbf{Var}[X_S] \le W_S^2$.

Sample $O(\lambda^2/\varepsilon^2 \cdot \log(n/\delta))$ times to compute d' such that

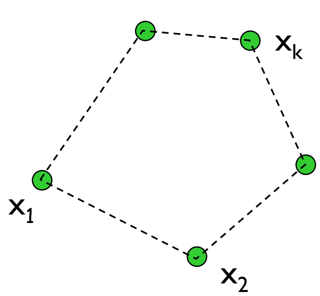
$\mathbf{Pr}[\forall S,\ d_S - \varepsilon\omega_S \le d'_S \le d_S] \ge 1\text{-}\delta$.

$\Rightarrow$ d' is an ε-subgradient at u with probability $\ge 1\text{-}\delta$.

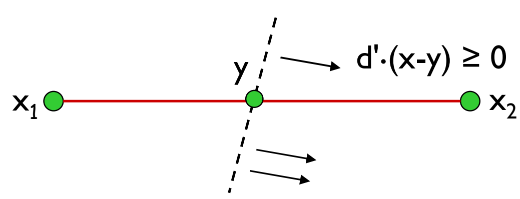
One last hurdle

Cannot evaluate $h(.)$ – how to compute $\bar{x} = \text{argmin}_{i=1\ldots k}\ h(x_i)$?

Will find point $\bar{x}$ in the convex hull of $x_1, \ldots, x_k$ such that $h(\bar{x})$ is close to $\min_{i=1\ldots k} h(x_i)$.



Take two points x_1 and x_2. Find point $\bar{x}$ on x_1–x_2 line segment with value close to $\min(h(x_1), h(x_2))$ using bisection search.



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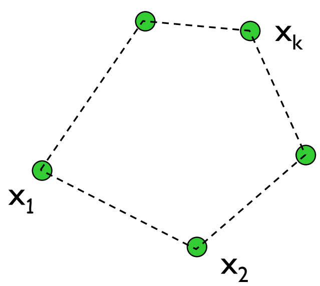

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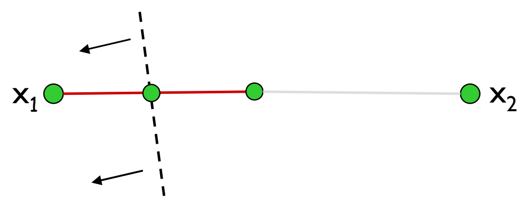
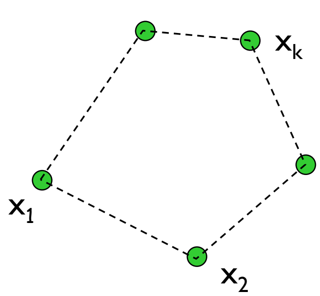


One last hurdle

Cannot evaluate $h(.)$ – how to compute $\bar{x} = \text{argmin}_{i=1\ldots k}\ h(x_i)$?

Will find point $\bar{x}$ in the convex hull of $x_1, \ldots, x_k$ such that $h(\bar{x})$ is close to $\min_{i=1\ldots k}\ h(x_i)$.



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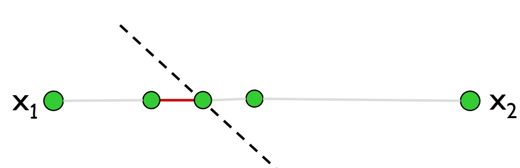

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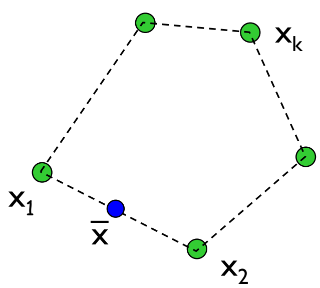

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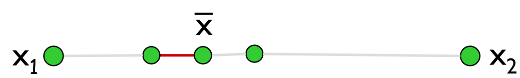

Stop when search interval is small enough.
Set $\bar{x}$ = either end point of remaining segment.

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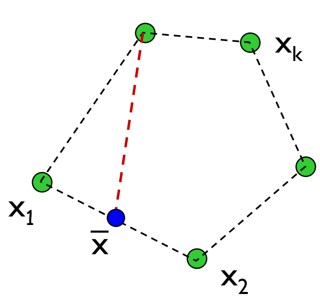

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Iterate using $\bar{x}$ and $x_3, \ldots, x_k$ updating $\bar{x}$ along the way.

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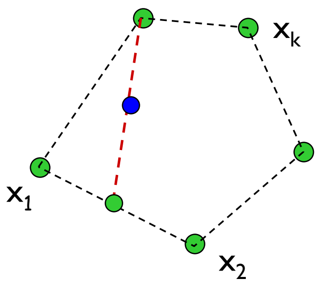

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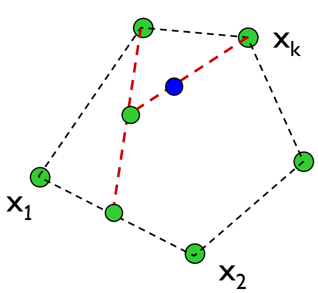

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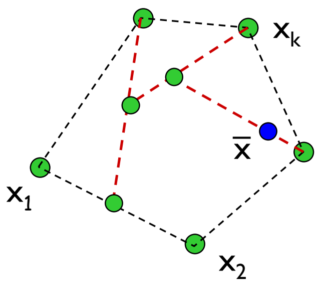

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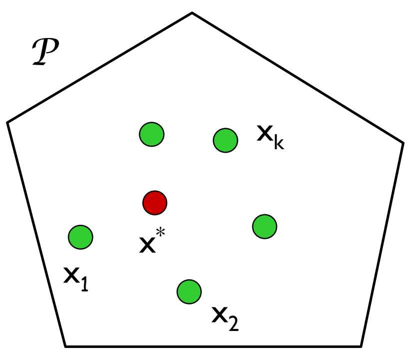
Iterate using $\bar{x}$ and $x_3,\ldots, x_k$ updating $\bar{x}$ along the way.

Can show that $h(\bar{x}) \le (\min_{i=1\ldots k} h(x_i) + k\cdot\rho)/(1-\varepsilon)^{kN}$.

Putting it all together

Min $h(x)$ subject to $x \in \mathcal{P}$.



✓ Can compute ε-subgradients.

Run ellipsoid algorithm.

Given y_i = center of current ellipsoid.

If y_i is infeasible, use violated inequality as a cut.

If $y_i \in \mathcal{P}$ use ε-subgradient cut.

Continue with smaller ellipsoid.

Generate points $x_1, x_2, \ldots, x_k$ in $\mathcal{P}$. Return $\bar{x} = \text{argmin}_{i=1\ldots k}\, h(x_i)$.

Get that $h(\bar{x}) \le \text{OPT}/(1-\varepsilon) + \rho$.

Finally,

Get solution x with $h(x)$ close to OPT.

Sample initially to detect if $\text{OPT} = \Omega(1/\lambda)$ – this allows one to get a $(1+\varepsilon)\cdot\text{OPT}$ guarantee.

Theorem: Compact convex program can be solved to within a factor of $(1+\varepsilon)$ in polynomial time, with high probability.

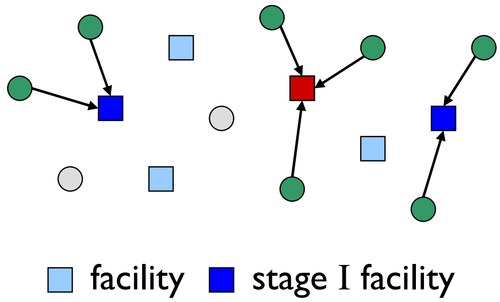
Gives a $(2\log n+\varepsilon)$-approximation algorithm for the stochastic set cover problem.

A Solvable Class of Stochastic LPs

$$
\begin{aligned}
\text{Minimize } h(x) &= w.x + \textstyle\sum_{A\subseteq U} p_A f_A(x) \\
\text{s.t.} \quad & x \in \Re^n,\ x \geq 0,\ x \in \mathcal{P} \\
\text{where} \quad f_A(x) &= \min.\ w^A.y_A + c^A.r_A \\
\text{s.t.} \quad & Br_A \geq j^A \\
& Dr_A + Ty_A \geq \ell^A - Tx \\
& y_A \in \Re^n,\ r_A \in \Re^m,\ y_A \geq 0,\ r_A \geq 0.
\end{aligned}
$$

Theorem: Can get a $(1+\varepsilon)$-optimal solution for this class of stochastic programs in polynomial time.

2-Stage Stochastic Facility Location



Distribution over clients gives the set of clients to serve.

Stage I: Open some facilities in advance; pay cost f_i for facility i.

Stage I cost = $\sum_{(i \text{ opened})} f_i$.

Actual scenario A = { ● clients to serve}, materializes.

Stage II: Can open more facilities to serve clients in A; pay cost f_i^A to open facility i. Assign clients in A to facilities.

Stage II cost = $\sum_{i \text{ opened in scenario } A} f_i^A$ + (cost of serving clients in A).

A Convex Program

p_A : probability of scenario $A \subseteq \mathcal{D}$.

y_i : indicates if facility i is opened in stage I.

$y_{A,i}$: indicates if facility i is opened in scenario A.

$x_{A,ij}$: whether client j is assigned to facility i in scenario A.

Minimize $h(y) = \sum_i f_i y_i + g(y)$ s.t. $y_i \geq 0$ for each i (SUFL-P)

where, $g(y) = \sum_{A \subseteq \mathcal{D}} p_A g_A(y)$

and $g_A(y) = \min. \sum_i F_i y_{A,i} + \sum_{j,i} c_{ij} x_{A,ij}$

$$\begin{aligned} \text{s.t.} \quad \sum_i x_{A,ij} &\geq 1 && \text{for each } j \in A \\ x_{A,ij} &\leq y_i + y_{A,i} && \text{for each } i,j \\ x_{A,ij}, y_{A,i} &\geq 0 && \text{for each } i,j. \end{aligned}$$

Lecture #2

A priori optimization (no recourse)

Given: Probability distribution over inputs.

In advance: Compute master plan.

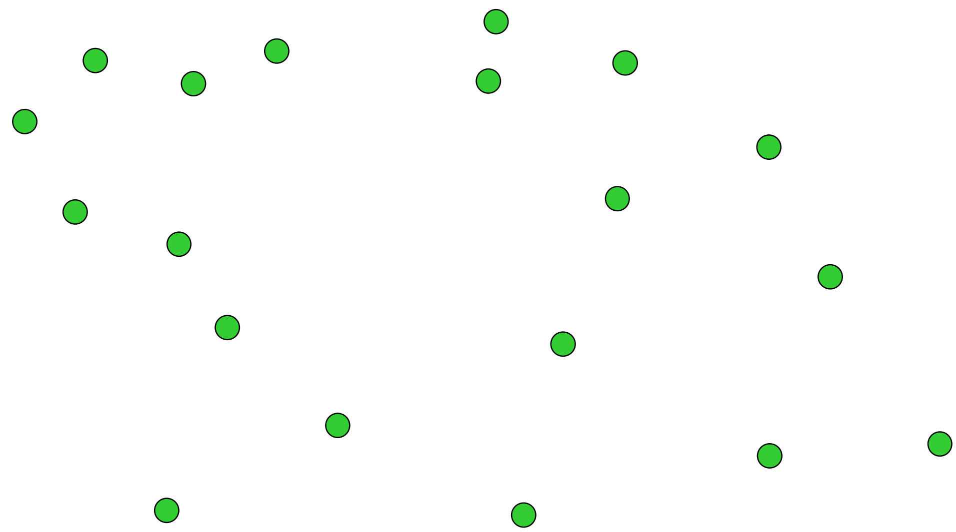
Observe the actual input scenario.

In real time: Adapt master plan to scenario.

Compute master plan to minimize

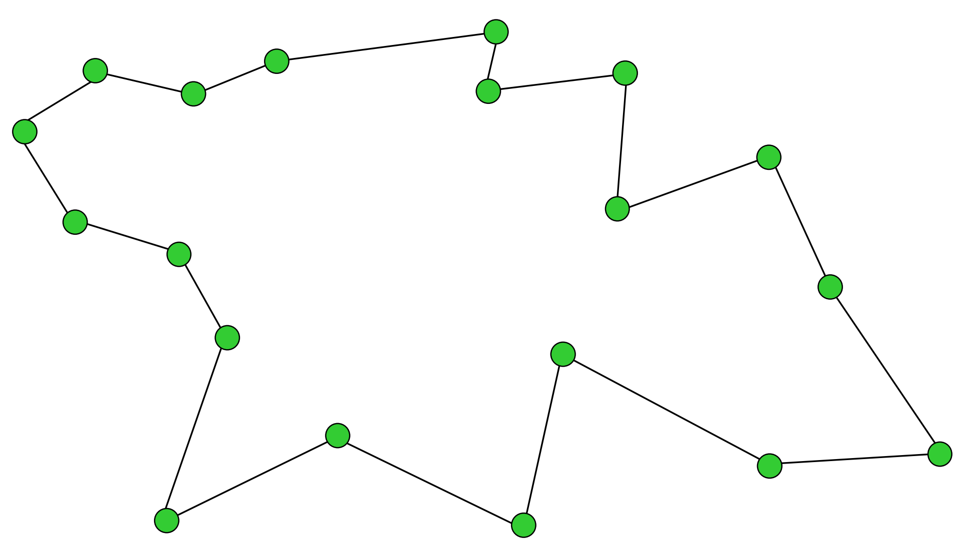
expected real time cost.

The Traveling Salesman Problem (TSP)



Given input points, compute tour τ to minimize total length $c(\tau)$

The Traveling Salesman Problem (TSP)



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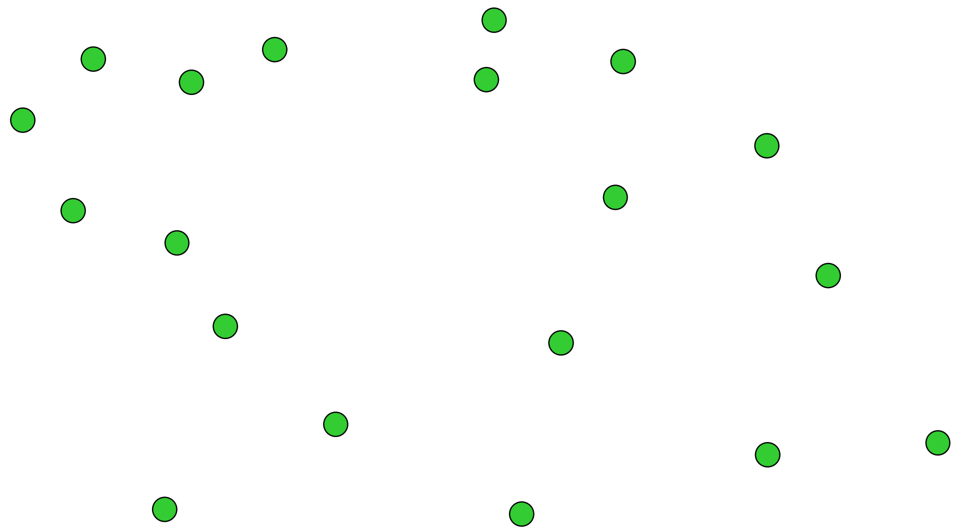
The *A Priori* TSP



Given input points N and a distribution Π of active sets A 2 2^N

Need to specify the probability that a given set A is active

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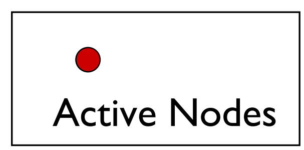

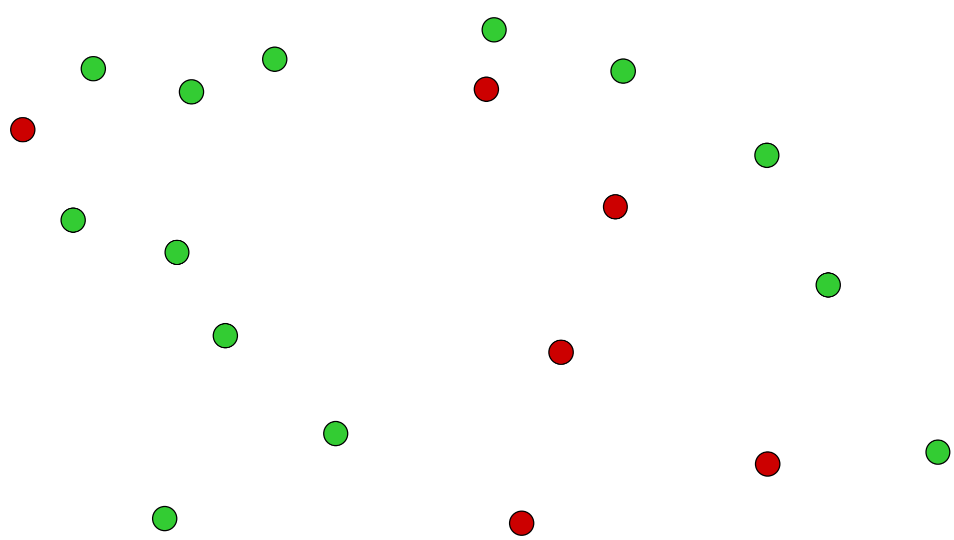

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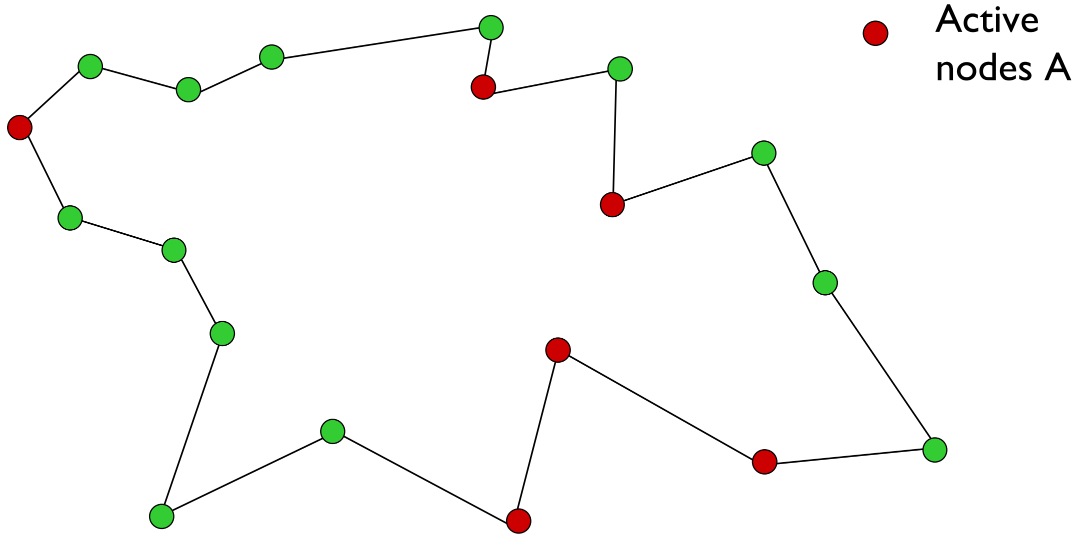

Given input points N and a distribution Π of active sets A 2 2^N, compute master tour τ to minimize expected length of the tour τ shortcut to serve only A

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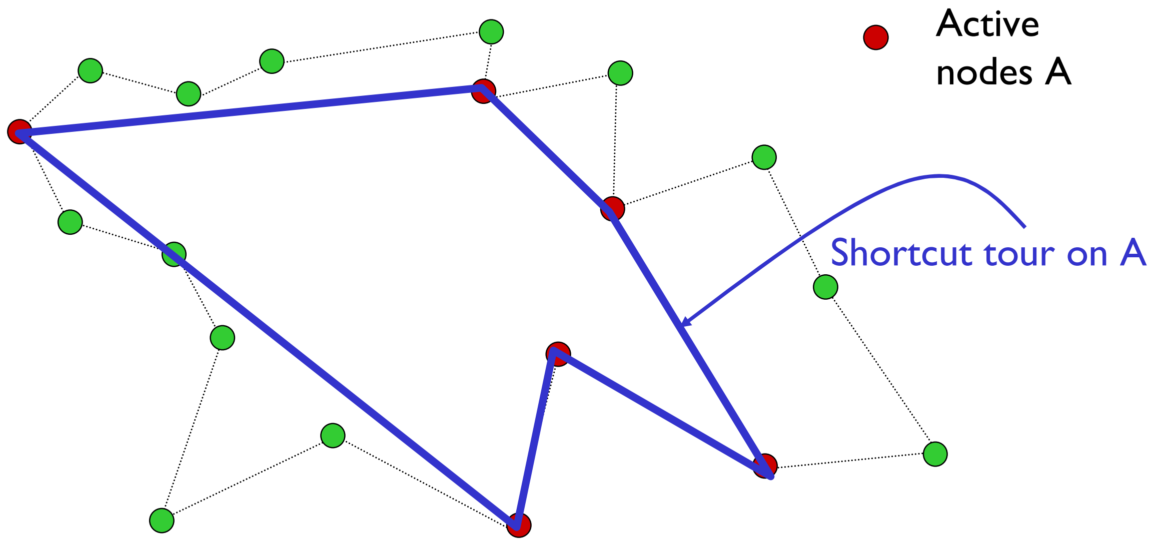

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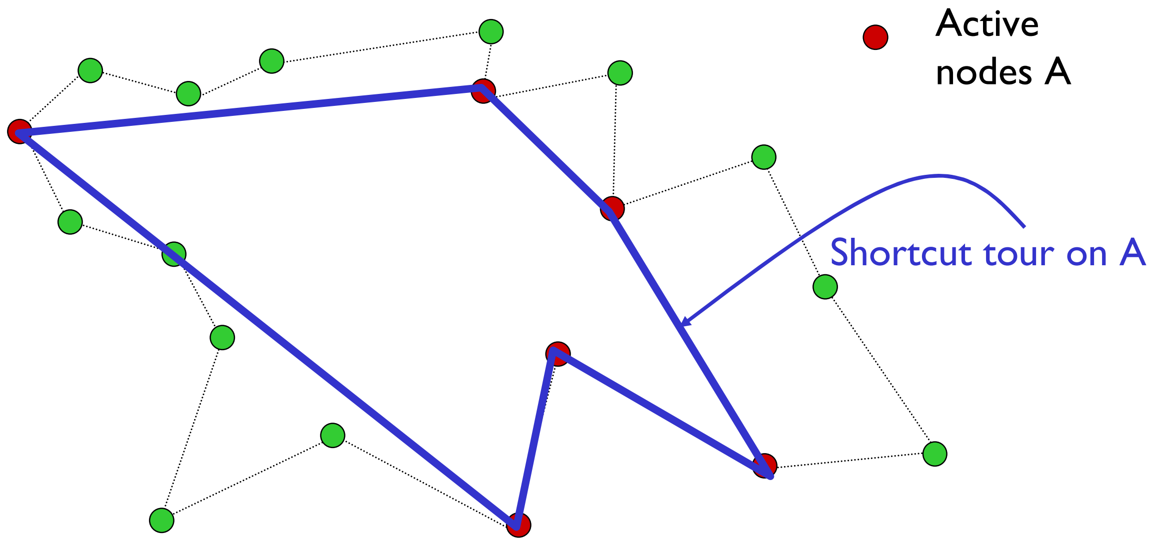
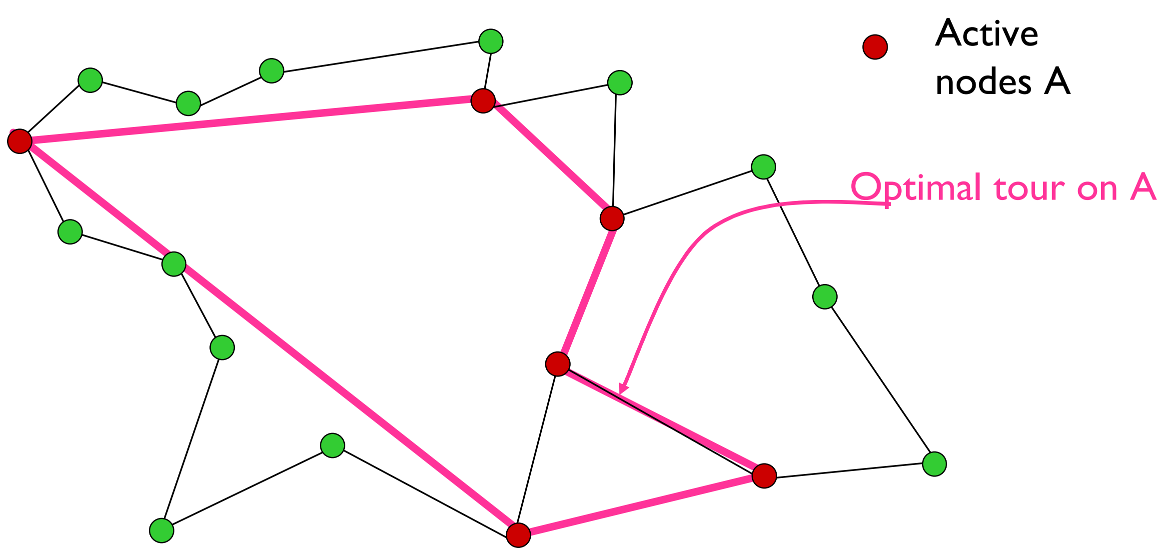


Given input points N and a distribution Π of active sets A 2 2^N, compute tour τ to minimize expected length E_A [c(τ_A)], where τ_A is the tour τ shortcut to serve only A

The *A Priori* TSP



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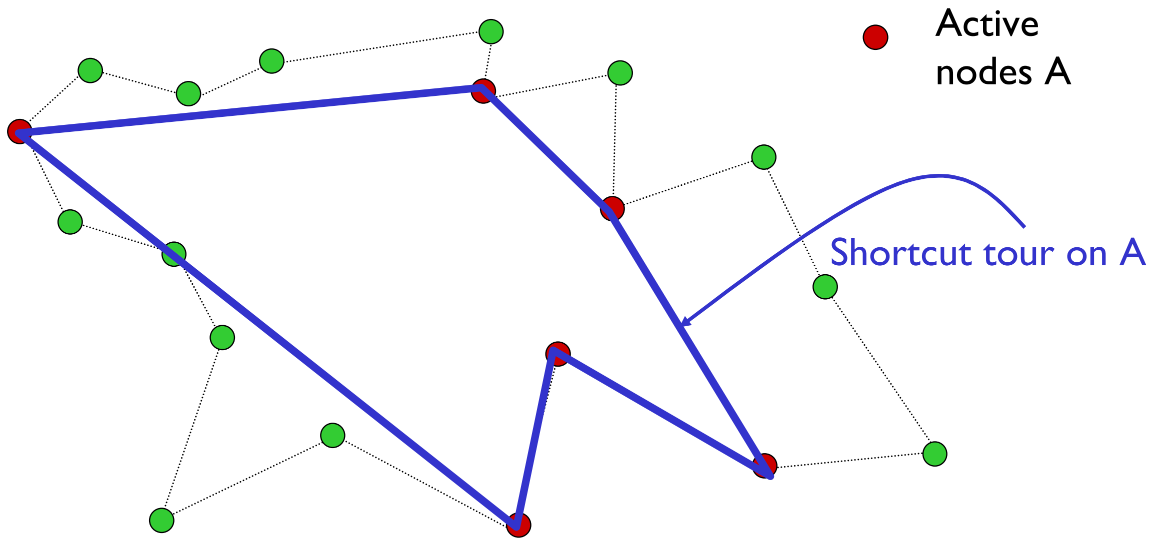

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Given input points N and a distribution Π of active sets A 2 2^N, compute tour ¿ to minimize expected length E_A [c(τ_A)], where τ_A is the tour τ shortcut to serve only A $\Rightarrow \tau^*$ (optimal solution)

Goal: Find tour τ such that E_A [c(τ_A)] $\leq$ ®E_A [c(τ^*_A)] $\Rightarrow$ ®OPT

(This is an ®-approximation algorithm for the *a priori* TSP.)

How is the probability distribution on active set specified?

- A short (polynomial) list of possibile scenarios;
- Independent probabilities that each point is active;
- A black box that can be sampled.

The *A Priori* TSP

Given input points N and a distribution Π of active sets A 2 2^N, compute tour ¿ to minimize expected length $E_A [c(\tau_A)]$, where τ_A is the tour τ shortcut to serve only A $\Rightarrow \tau^*$ (optimal solution)

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Some relevant history for *a priori* TSP

- Jaillet (1985, 1988), Bertsimas (1988), Jaillet, Bertsimas, & Odoni (1990) introduce problem – analyze with probabilistic assumptions on distances
- Schalekamp & S (2007) randomized O(log n)-approximation
- Maybecast problem Karger & Minkoff (2000)
- Rent-or-buy problem Gupta, Kumar, Pál, Roughgarden (2007)
- Stochastic Steiner Tree variants Gupta, Pál, Ravi, Sinha (2004)
  Gupta, Ravi, Sinha ('04), Hayraptian, Swamy, Tardos ('05)
  Garg, Gupta, Leonardi, Sankowski (2008)
- Universal TSP Bartholdi & Plazman (1989), Jia, Lin, Noubir, Rajaraman & Sundaram, (2005), Hajiaghayi, Kleinberg & Leighton (2006), Gupta, Hajiaghayi, Räcke (2006)

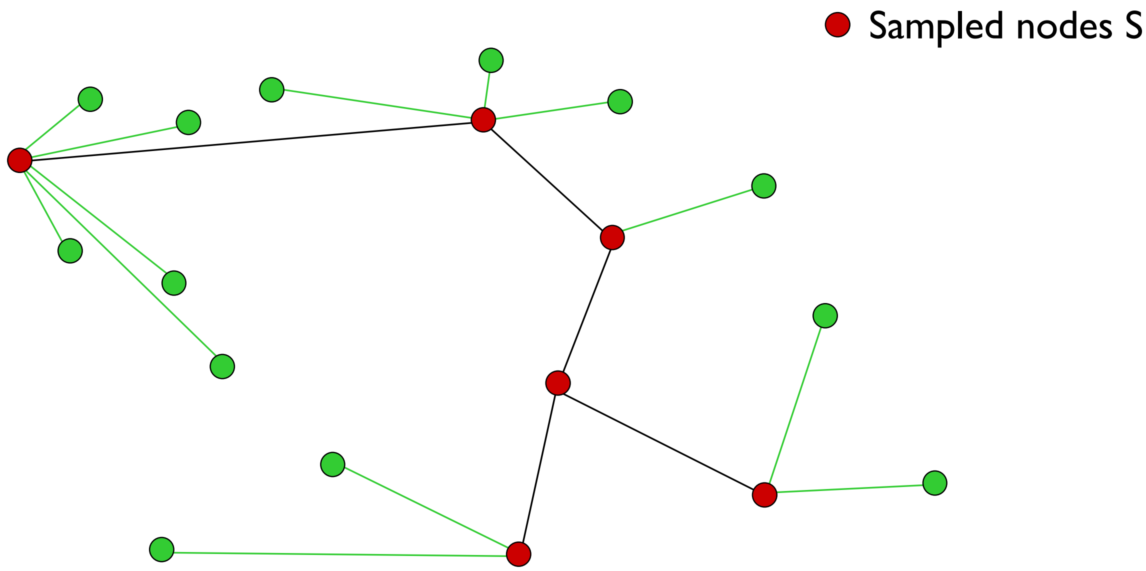
The One Random Sample Algorithm

1. Draw sample $S \subseteq N$ according to Π (i.e., pick each point j independently with probability p_j)
2. Build minimum spanning tree on S
3. For each $j \notin S$, connect j to its nearest neighbor in S
4. Build "double tree" tour of this tree $\Rightarrow \tau$

Simplifying Assumption: $\exists$ node r with $p_r = 1$ (wlog)

Theorem The one random sample algorithm is a 4-approximation algorithm for the *a priori* TSP.

Running the Algorithm



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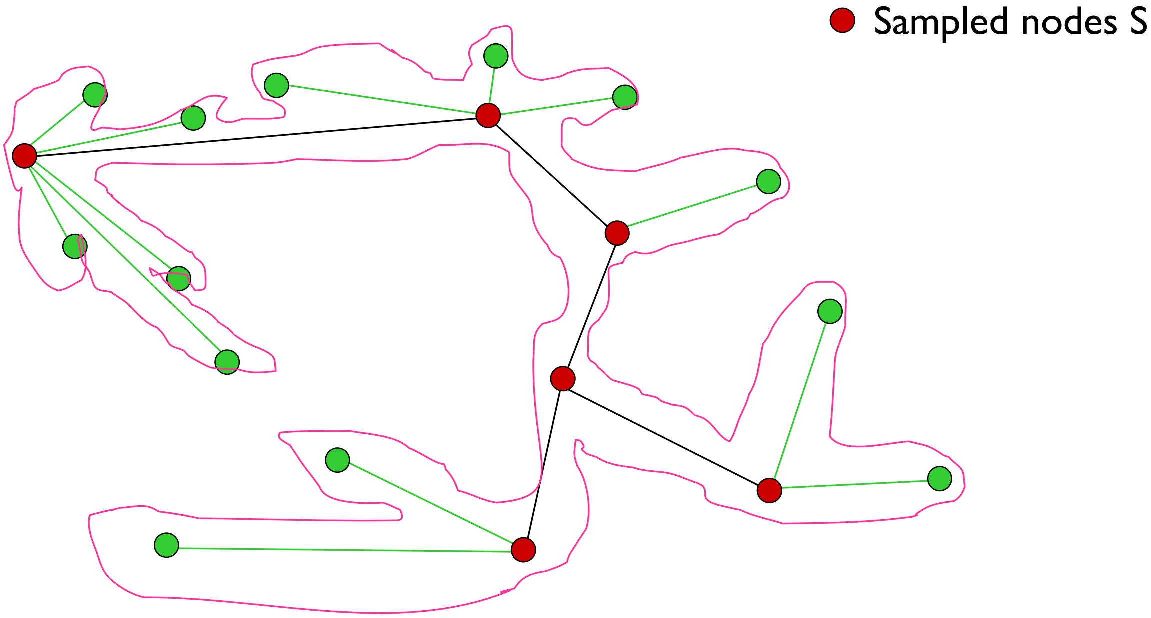

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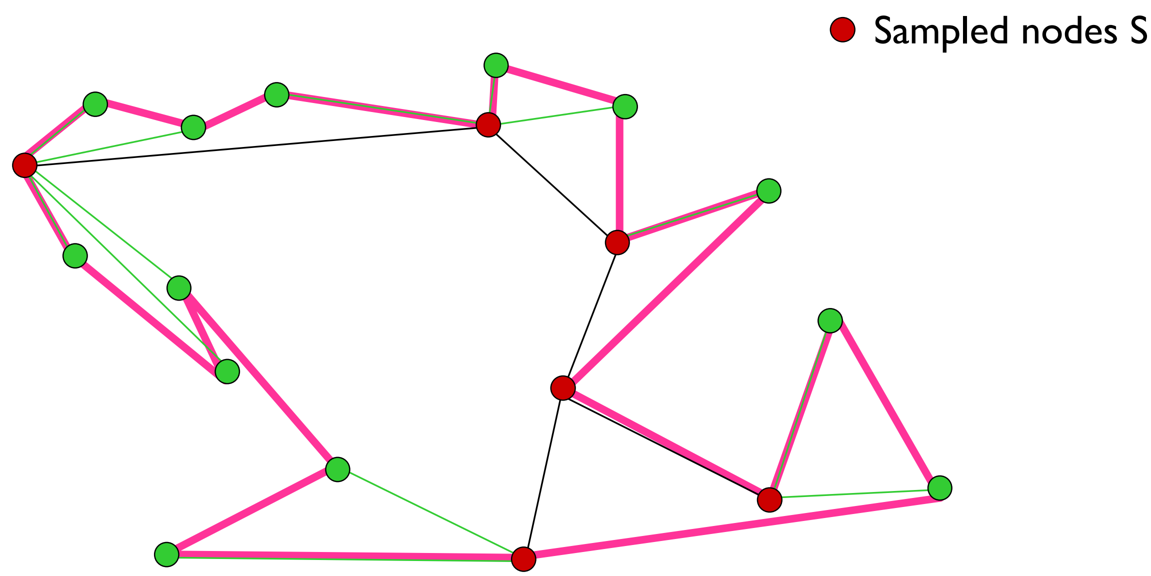

Analyzing the Algorithm

Let $D_j(S)$ be the distance from j to its nearest neighbor in S-{j}

Let MST(S) be the length of the minimum spanning tree on S

Goal: Analyze $E_S [E_A [c(\tau_A)]]$

Fact 1. $E_S [D_j(S)] = E_S [D_j | j \notin S] = E_S [D_j | j \in S] = E_A [D_j(A) | j \in A]$

Why? Choice of S-{j} is independent of whether j 2 S, and
S and A are independent draws from same distribution

Fact 2. MST(A) ≤ c(¿ $^*_A$) for each A µ N

Why? Tour ¿* shortcut to A still contains spanning tree

Fact 3. $\sum_{j \neq r} \mathbb{1}(j \in A) D_j(A) \leq c(\tau^*_A)$ for all A

Why? Any tour on A "leaves" each node i by some edge

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Key Idea: always pay for backbone built on S (for any active A)

$$
\begin{aligned}
E_S[E_A[c(\tau_A)]] &\leq E_S[2MST(S)]+E_S [E_A [\textstyle\sum_{j \neq r} \mathbb{1}(j \in A) \mathbb{1}(j \notin S) 2D_j(S)]] \\
&= E_S [2MST(S)] + \textstyle\sum_{j\neq r} E_{S,A} [\mathbb{1}(j \in A)\mathbb{1}(j \notin S) 2D_j (S)] \\
&= E_S [2MST(S)] + 2\textstyle\sum_{j \neq r} p_j (1-p_j) E_S[D_j(S)] \\
&\leq 2(E_S [MST(S)] + \textstyle\sum_{j \neq r} p_j E_S[D_j(S)]) \\
&\leq 2 (OPT+OPT) \\
&= 4OPT
\end{aligned}
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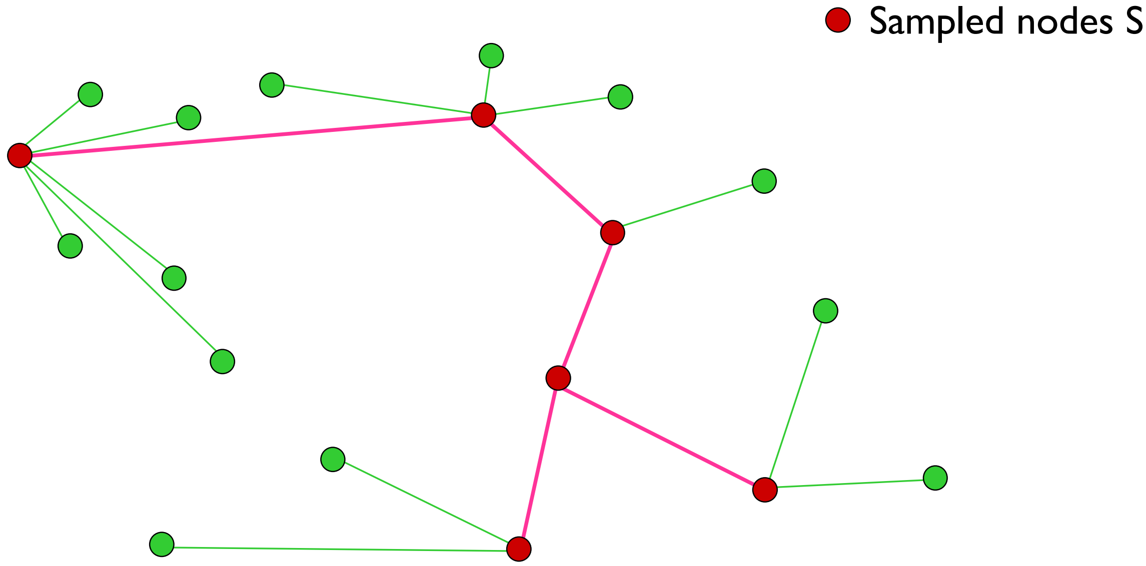
Analyzing the Algorithm



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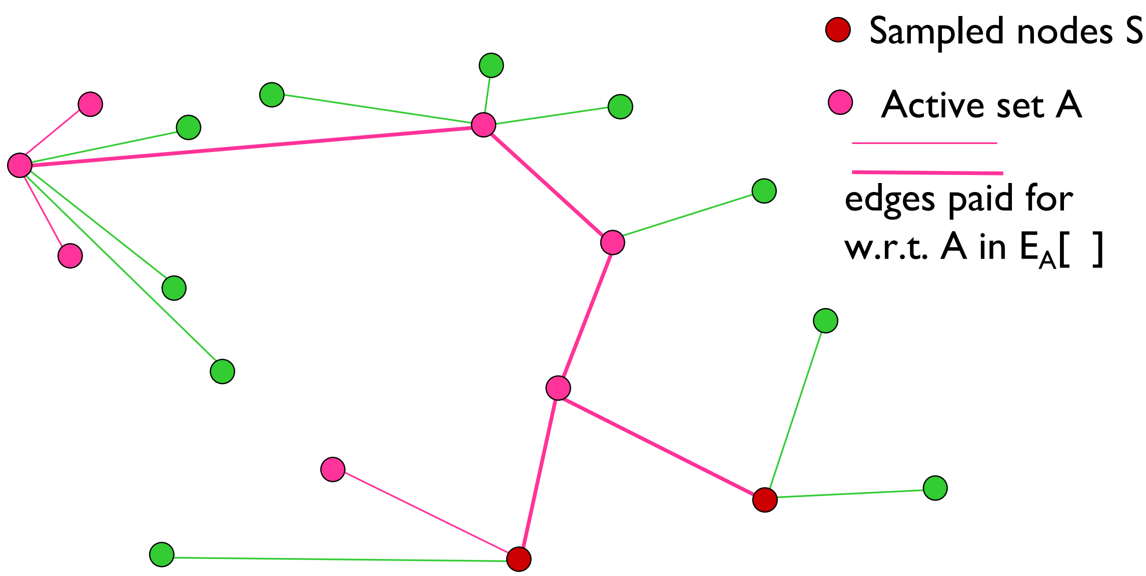

Always pay for all of backbone and just those attached leaves you need

Cost of shortcut tour for A is at most twice the cost of these edges

Let $D_j(S)$ be the distance from j to its nearest neighbor in S-{j}
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Simplifying Assumption: $\exists$ node r with $p_r = 1$ (wlog)

Theorem (S & Talwar) The one random sample algorithm is a 4-approximation algorithm for the *a priori* TSP.

Two Footnotes

Can be derandomized -

Let $D_j(S)$ be the distance from j to its nearest neighbor in S-{j}

Let MST(S) be the length of the minimum spanning tree on S

Goal: analyze $E_S [E_A [c(\tau_A)]]$

Fact 1. $E_S [D_j(S)] = E_S [D_j | j \notin S] = E_S [D_j | j \in S] = E_A [D_j(A) | j \in A]$

Fact 2. $MST(A) \leq c(\tau^*_A)$ for all A

Fact 3. $\sum_{j \neq r} \mathbb{1}(j \in A) D_j(A) \leq c(\tau^*_A)$ for all A

Key Idea: always pay for backbone built on S (for any active A)

$$
\begin{aligned}
E_S[E_A[c(\tau_A)]] &\leq E_S[2MST(S)]+E_S [E_A [\textstyle\sum_{j \neq r} \mathbb{1}(j \in A) \mathbb{1}(j \notin S) 2D_j(S)]] \\
&= E_S [2MST(S)] + \textstyle\sum_{j\neq r} E_{S,A} [\mathbb{1}(j \in A)\mathbb{1}(j \notin S) 2D_j (S)] \\
&= E_S [2MST(S)] + 2\textstyle\sum_{j \neq r} p_j (1-p_j) E_S[D_j(S)] \\
&\leq 2(E_S [MST(S)] + \textstyle\sum_{j \neq r} p_j E_S[D_j(S)]) \\
&\leq 2 (OPT+OPT) \\
&= 4OPT
\end{aligned}
$$

Two Footnotes

Can be derandomized – Williamson & van Zuylen (2007) show how to deterministically achieve twice guarantee for rent-or-buy/connected facility location problem by the method of conditional probabilities (by an LP estimate)

Assumption that $p_r = 1$ is not needed;

Need only that $D_j(S)$ is well defined.

Modify Π to condition on that each set has cardinality ≥ 2

Can sample according to this new distribution also, and this just rescales things (any tour has cost 0 restricted to 0 or 1 points) but must be careful about dependence

Theorem (S & Talwar) There is a deterministic 8-approximation algorithm for the a priori TSP in the independent activation model

What about the black box model?

Recent work of Gorodezky, R. Kleinberg, S, & Spencer shows that for a (slightly) restricted class of algorithms can embed a universal computation in an a priori one, and thereby show a non-constant lower bound on performance guarantees possible with a polynomial number of samples

---

2-Stage Steiner Tree Problem

Given a set of points N (with root) in a metric space, integer inflation factor λ, and distribution over 2^N

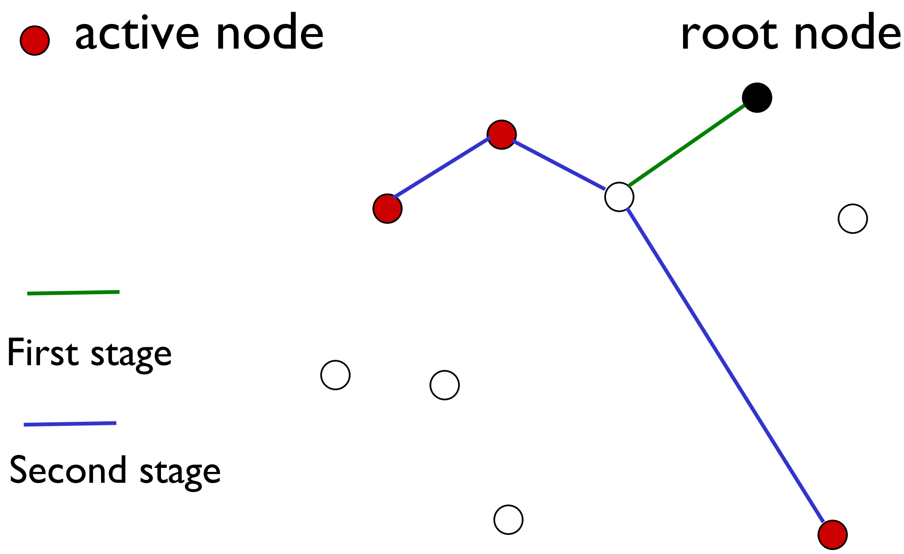
Stage I: install edges A_I – cost of e is c_e

Set of active terminals $T \subseteq N$ is selected (including root)

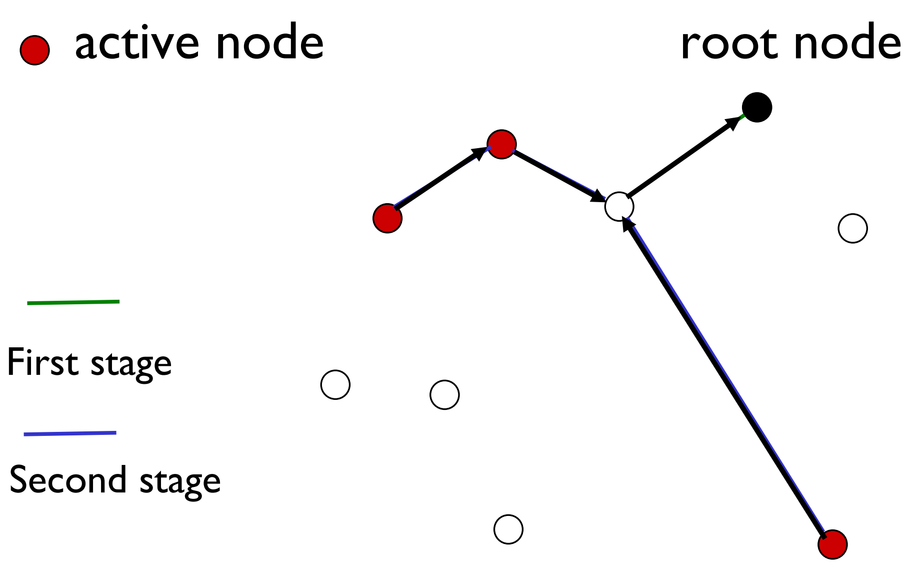
Stage II: install edges A_{II} s.t. $A_I \cup A_{II}$ is Steiner tree on T - cost of edge e is λc_e

Goal: Minimize
(cost of edges installed in stage I) +
$\lambda \, \mathbf{E}_{T \subseteq N}$ [cost of edges installed for scenario T].

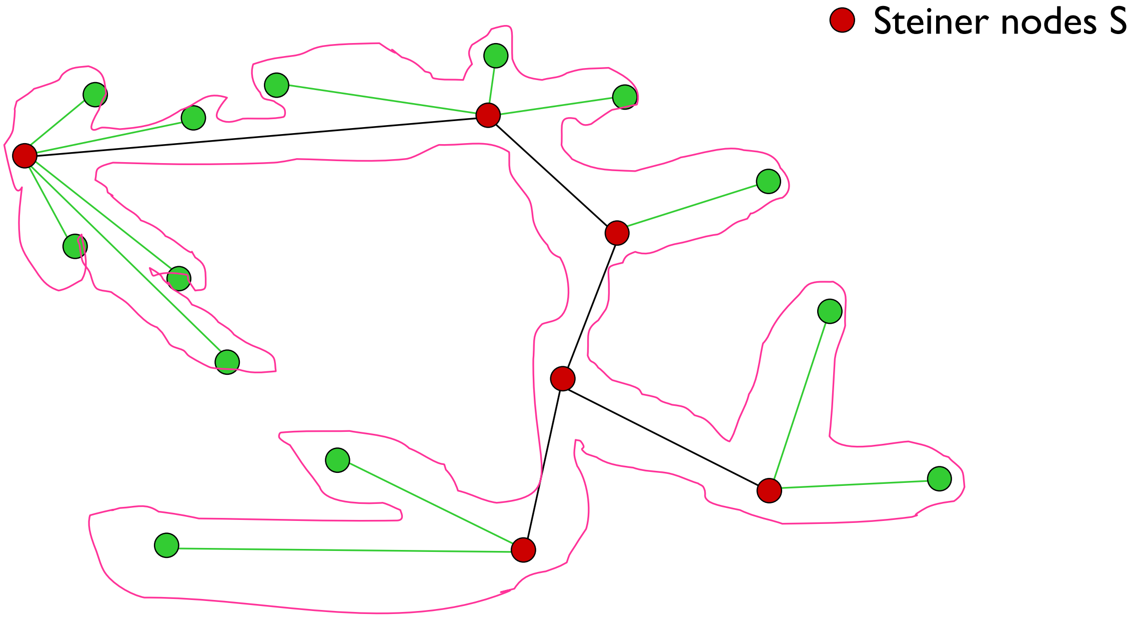
An Example



An Example

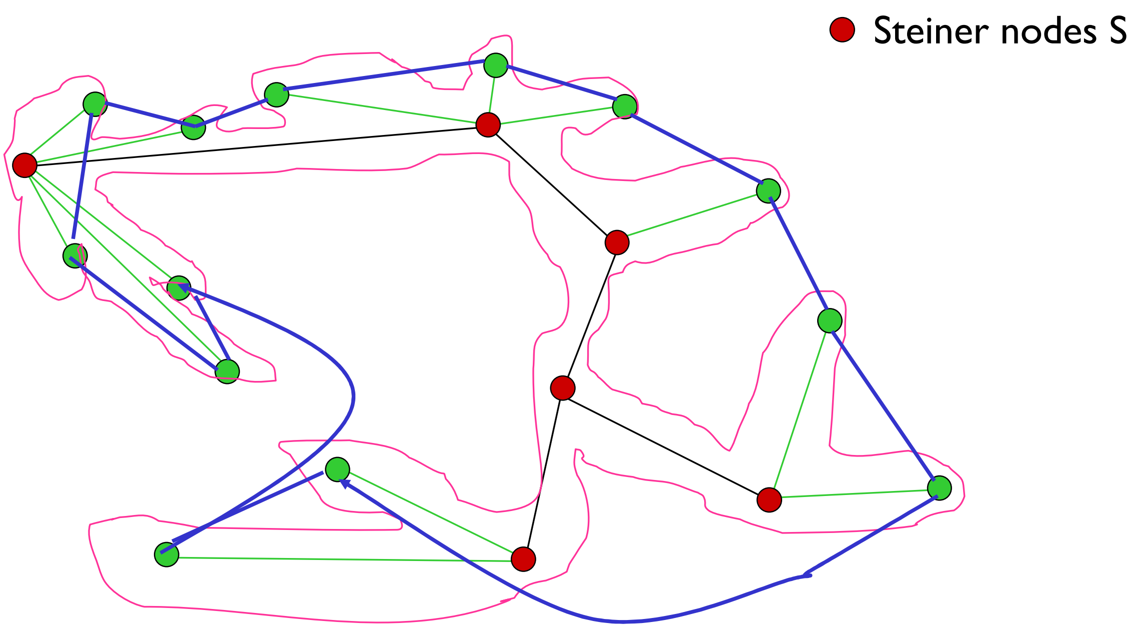


Deterministic Steiner Tree



Walking “around” optimal Steiner tree gives connected graph, so can view as connected graph on just terminal nodes (by shortcuts)

Deterministic Steiner Tree



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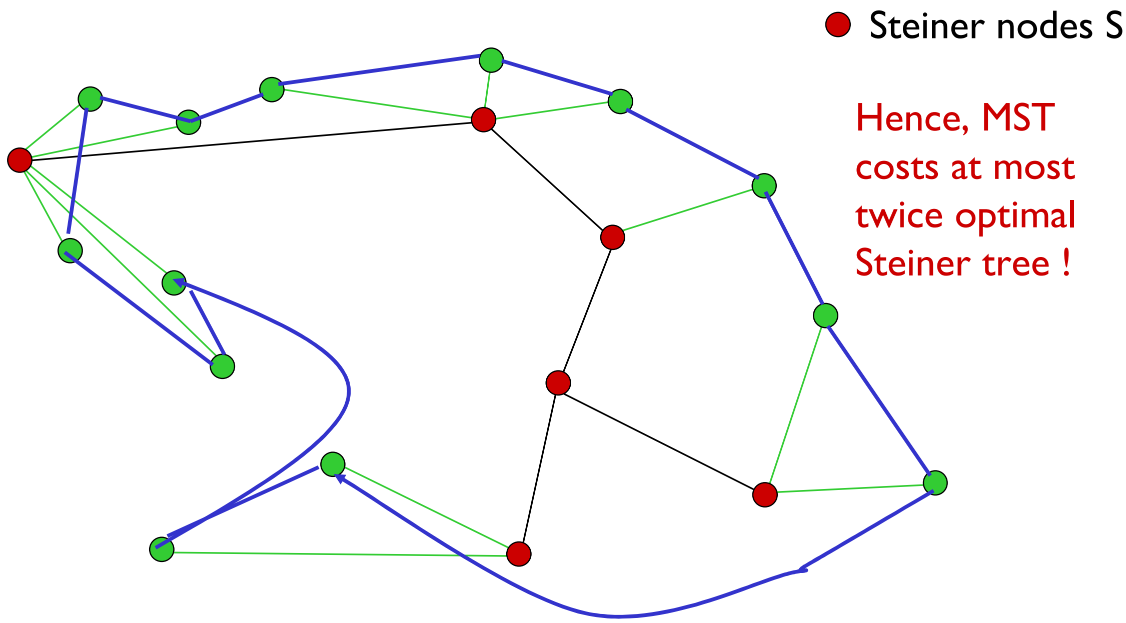
Deterministic Steiner Tree



Walking “around” optimal Steiner tree gives connected graph, so can view as connected graph on just terminal nodes (by shortcuts)

Boosted Sampling Algorithm
(Gupta, Pál, Ravi, Sinha)

- Draw λ independent samples $S_1, S_2, \ldots, S_\lambda \rightarrow S$
- First stage decision: compute minimum spanning tree (MST) for S (including root), and install those edges $\rightarrow \text{Alg}_I$
- Observe scenario T (independently drawn from same dist)
- Compute (rooted) minimum spanning tree on $S \cup T$,
  (but make cost of edges Alg_I all 0)
  and let e[j] be edge from j to its parent
- Let $\text{Alg}_{II} \leftarrow \{ e[j] : j \in T \}$

First Stage Cost

Optimal cost $Z^* = c(Opt_I) + \lambda\, E_{T \subseteq N} [\, c(Opt_{II}(T))\,]$

We compute MST on $S \leftarrow S_1 \cup \ldots \cup S_\lambda$ for Stage I

(this is 2-approximation for S)

How expensive is it to connect S? Could use

$$Opt_I \cup Opt_{II}(S_1) \cup \ldots \cup Opt_{II}(S_\lambda)$$

Each S_i is identical random T so its expected cost is

$$c(Opt_I) + \lambda\, E_{T \subseteq N} [\, c(Opt_{II}(T))] \rightarrow Z^*$$

Since MST is 2-approximation algorithm $\Rightarrow$

expected Stage I cost is at most $2Z^*$

Cost sharing role of parental edge

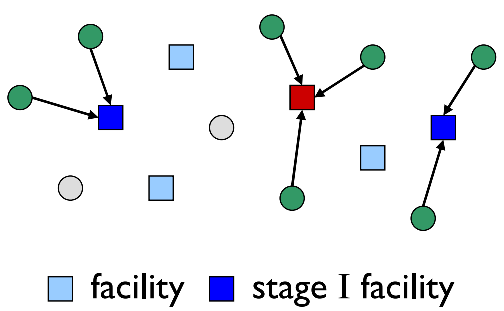
- Build a MST on a set $S \cup T$ (plus root)
- Focus on parental edge e[j] for each $j \in S \cup T$
- Total edge cost is $\sum_{j \in S \cup T} c_{e[j]}$
- But this is $\leq$ twice cost of optimal Steiner tree on $S \cup T$
- Attribute share $c_{e[j]}/2$ of optimal cost to j
- Total share cost is $\leq$ optimal Steiner tree cost

Second Stage Cost

- Algorithm computes Steiner tree for $S_I \cup \ldots \cup S_\lambda \cup T$
- Consider $\mathcal{T} \leftarrow Opt_I \cup Opt_{II}(S_I) \cup \ldots \cup Opt_{II}(S_\lambda) \cup Opt_{II}(T)$
- Role of $\lambda+1$ sets, $S_I, \ldots, S_\lambda$, T is symmetric
- $E[c(\mathcal{T})] \leq c(Opt_I) + (\lambda+1)\, E[c(Opt_{II}(S_i))] \leq (\lambda+1)/\lambda\, Z^*$
- Form $D_1, \ldots, D_\lambda$ by deleting nodes in multiple sets
- $\sum_{j \in T-S} c_{e[j]} + \sum_i \sum_{j \in D_i} c_{e[j]} \leq 2c(\mathcal{T})$
- By symmetry, $E[\sum_{j \in T-S} c_{e[j]}] \leq 2c(\mathcal{T})/(\lambda+1)$
- Hence, $E[\sum_{j \in T-S} c_{e[j]}] \leq 2Z^* / \lambda \Rightarrow$ Stage II cost $\leq 2Z^*$!

$\Rightarrow$ Boosted Sampling is 4-approximation algorithm

2-Stage Stochastic Facility Location



Distribution over clients gives the set of clients to serve.

Stage I: Open some facilities in advance; pay cost f_i for facility i.

Stage I cost = $\sum_{(i \text{ opened})} f_i$.

Actual scenario A = { clients to serve}, materializes.

Stage II: Can open more facilities to serve clients in A; pay cost f_i^A to open facility i. Assign clients in A to facilities.

Stage II cost = $\sum_{i \text{ opened in scenario } A} f_i^A$ + (cost of serving clients in A).

Deterministic Facility Location

Minimize $\sum_i f_i y_i + \sum_{j,i} d_j c_{ij} x_{ij}$

subject to $\sum_i x_{ij} \geq 1 \quad \forall j$

$x_{ij} \leq y_i \quad \forall i, j$

$x_{ij}, y_i \geq 0 \quad \forall i, j$

y_i : indicates if facility i is open.

x_{ij} : indicates if client j is assigned to facility i.

d_j is the demand at client j

A Convex Program

p_A : probability of scenario $A \subseteq \mathcal{D}$.

y_i : indicates if facility i is opened in stage I.

$y_{A,i}$: indicates if facility i is opened in scenario A.

$x_{A,ij}$: whether client j is assigned to facility i in scenario A.

Minimize $h(y) = \sum_i f_i y_i + g(y)$ s.t. $y_i \geq 0$ for each i (SUFL-P)

where, $g(y) = \sum_{A \subseteq \mathcal{D}} p_A g_A(y)$

and $g_A(y) = \min. \sum_i F_i y_{A,i} + \sum_{j,i} c_{ij} x_{A,ij}$

s.t. $\sum_i x_{A,ij} \geq 1$ for each $j \in A$

$x_{A,ij} \leq y_i + y_{A,i}$ for each i,j

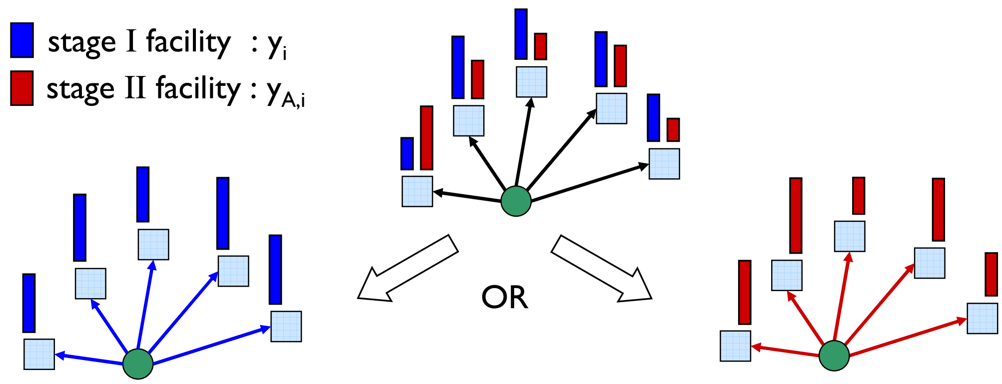
$x_{A,ij}, y_{A,i} \geq 0$ for each i,j.

Rounding (SUFL-P)

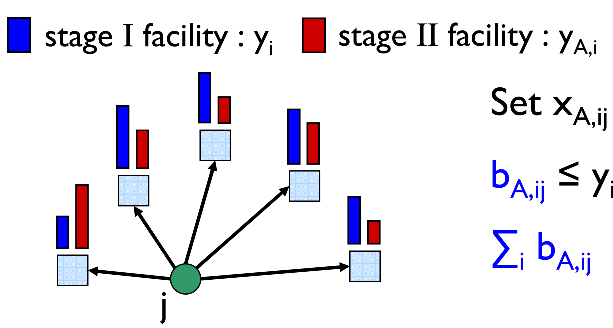
Let y : optimal solution with cost OPT.
(x_A, y_A) : optimal solution for scenario A.

Goal: Decouple stage I and the stage II scenarios.

Assign $j \in A$ exclusively to stage I facilities, or to stage II facilities.



Rounding (contd.)

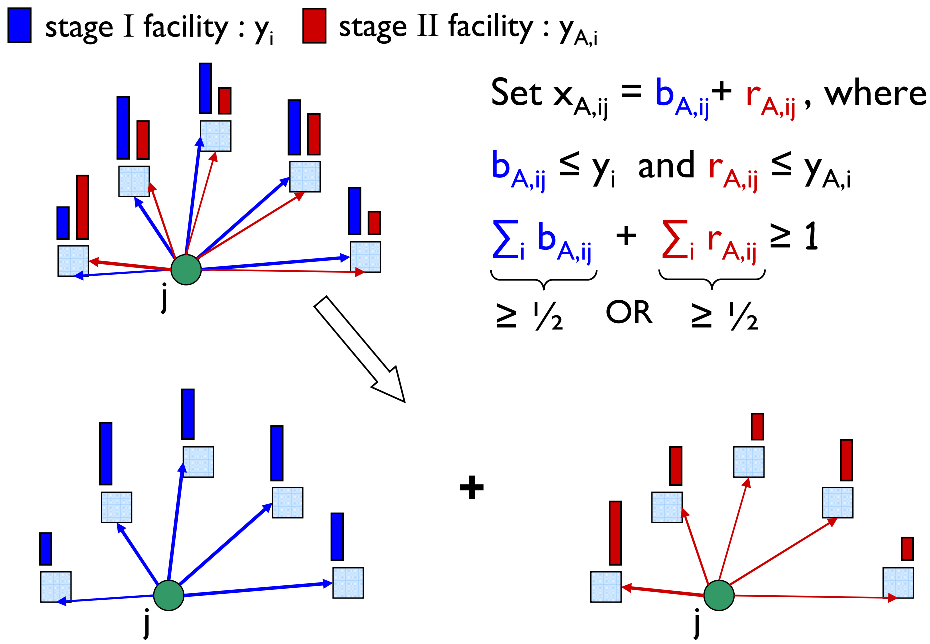


Set $x_{A,ij} = b_{A,ij} + r_{A,ij}$, where

$b_{A,ij} \le y_i$ and $r_{A,ij} \le y_{A,i}$

$\sum_i b_{A,ij} + \sum_i r_{A,ij} \ge 1$

Rounding (contd.)



Set $x_{A,ij} = b_{A,ij} + r_{A,ij}$, where

$b_{A,ij} \leq y_i$ and $r_{A,ij} \leq y_{A,i}$

$\underbrace{\sum_i b_{A,ij}}_{\geq 1/2} + \underbrace{\sum_i r_{A,ij}}_{\geq 1/2} \geq 1$

$\geq ½$ OR $\geq ½$

Rounding (contd.)

$\sum_i b_{A,ij} \geq ½ \Rightarrow$ $(2b_{A,ij})$ is a feasible assignment for j with facility variables $2y_i$.

$\sum_i r_{A,ij} \geq ½ \Rightarrow$ $(2r_{A,ij})$ is a feasible assignment for j with facility variables $2y_{A,i}$.

Have an α-approx. algorithm for UFL wrt. LP relaxation.

Stage I

- Solve UFL instance: facility set $\mathcal{F}$ with costs f_i, client set $D = \{(j,A) : \sum_i b_{A,ij} \geq ½\}$, (j,A) has demand p_A.
- $(\{2b_{A,ij}\}_{(j,A)\in D}, 2y)$ is a feasible fractional solution.
- Obtain integer solution: gives facilities to open in stage I.
- Takes care of client j in each scenario A where $\sum_i b_{A,ij} \geq ½$.

Rounding (contd.)

Stage II, scenario A

- Assign $j \in A$ such that $\sum_i b_{A,ij} \geq ½$ to stage I facility.
  $\Rightarrow$ Only need to assign remaining clients with $\sum_i r_{A,ij} \geq ½$.
- Solve UFL instance: facility set $\mathcal{F}$ with costs F_i , client set $D_A = \{ j \in A : \sum_i r_{A,ij} \geq ½ \}$.
- $(\{2r_{A,ij}\}_{j \in D_A}, 2y_A)$ is a feasible fractional solution.
- "Round" to get an integer solution
  – determines what other facilities to open in scenario A,
  – how to assign clients in D_A.

Shows a 2α integrality gap for stochastic UFL-LP. Modify slightly to get a 3.23-approximation algorithm for stochastic UFL.

Lecture #3

Stochastic Set Cover (SSC)

Universe $U = \{e_1, \ldots, e_n\}$, subsets $S_1, S_2, \ldots, S_m \subseteq U$, set S has weight ω_S.

Deterministic problem: Pick a minimum weight collection of sets that covers each element.

Stochastic version: Set of elements to be covered is given by a probability distribution.

- choose some sets initially paying ω_S for set S
- subset $A \subseteq U$ to be covered is revealed
- can pick additional sets paying W_S for set S.

Minimize (ω-cost of sets picked in stage I) +
$\mathbf{E}_{A \subseteq U}$ [W_S -cost of new sets picked for scenario A].

Stochastic Set Covering LP

p_A : probability of scenario $A \subseteq U$.
x_S : indicates if set S is picked in stage I.
$y_{A,S}$: indicates if set S is picked in scenario A.

Minimize $\sum_S \omega_S x_S + \sum_{A \subseteq U} p_A \sum_S W_S y_{A,S}$
subject to,

$$\sum_{S: e \in S} x_S + \sum_{S: e \in S} y_{A,S} \geq 1 \qquad \text{for each } A \subseteq U, e \in A$$
$$x_S, y_{A,S} \geq 0 \qquad \text{for each } S, A.$$

Exponential number of variables and exponential number of constraints.

Inflation factor ¸ = $\max_S W_S/\omega_S$

Sample Average Approximation

Sample Average Approximation (SAA) method:

- Sample initially N times from scenario distribution
- Solve 2-stage problem estimating p_A with frequency of occurrence of scenario A

How large should N be?

Kleywegt, Shapiro & Homem De-Mello 01: bound N by variance of a certain quantity – need not be polynomially bounded even for our class of programs.

SwamyS 05: show using ε-subgradients that for our class, N can be poly-bounded.

Nemirovskii & Shapiro: show that for stochastic set cover LP with non-scenario dependent costs, KSH01 gives polynomial bound on N for (preprocessing + SAA) algorithm. Later also without preprocessing.

Charikar, Chekuri, & Pal 05: give elegant "Chernoff"-based proof that an α-approximation for polynomial-scenario setting yields (1+ε)α-approximation for black box setting

A Compact Formulation

p_A : probability of scenario $A \subseteq U$.

x_S : indicates if set S is picked in stage I.

Minimize $h(x) = \sum_S \omega_S x_S + f(x)$ s.t. $x_S \geq 0$ for each S

where, $f(x) = \sum_{A \subseteq U} p_A f_A(x)$

and $f_A(x) = \min. \sum_S W_S y_{A,S}$

s.t. $\sum_{S: e \in S} y_{A,S} \geq 1 - \sum_{S: e \in S} x_S$ for each $e \in A$

$y_{A,S} \geq 0$ for each S.

Equivalent to earlier LP.

Each $f_A(x)$ is convex, so $f(x)$ and $h(x)$ are convex functions.

Sample Average Approximation

Sample Average Approximation (SAA) method:

- Sample N times from distribution
- Estimate p_A by q_A = frequency of occurrence of scenario A

(P) $\min_{x\in \mathcal{P}} \left(h(x) = \omega\cdot x + \sum_{A\subseteq U} p_A f_A(x)\right)$

(SAA-P) $\min_{x\in \mathcal{P}} \left(h'(x) = \omega\cdot x + \sum_{A\subseteq U} q_A f_A(x)\right)$

To show: With poly-bounded N, if $\bar{x}$ solves (SAA-P) then $h(\bar{x}) \approx OPT$.

Let $z_A \equiv$ optimal dual solution for scenario A at point $u \in \Re^m$.

$\Rightarrow d_u$ with $d_{u,S} = \omega_S - \sum_{A\subseteq U} q_A \sum_{e\in S} z_{A,e}$ is a subgradient of $h'(.)$ at u.

Lemma: With high probability, for "many" points u in $\mathcal{P}$,

d_u is a subgradient of $h'(.)$ at u,

d_u is an approximate subgradient of $h(.)$ at u.

Establishes "closeness" of $h(.)$ and $h'(.)$ and suffices to prove result.

Intuition: Can run ellipsoid on both (P) and (SAA-P) using the same vector d_u at feasible point u.

Sample Average Approximation

Sample Average Approximation (SAA) method:

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- Solve 2-stage problem estimating p_A with frequency of occurrence of scenario A

How large should N be?

Kleywegt, Shapiro & Homem De-Mello 01: bound N by variance of a certain quantity – need not be polynomially bounded even for our class of programs.

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Sample Average Approximation

(Charikar, Chekuri, and Pál)

Sample Average Approximation (SAA) method:

- Sample N times from distribution
- Estimate p_A by q_A = frequency of occurrence of scenario A

(P) $\min_{x\in\mathcal{P}}$ (h(x) = ω.x_+ $\sum_{A\subseteq U} p_A f_A(x)$) _

(SAA-P) $\min_{x\in\mathcal{P}}$ (h'(x) = ω.x + $\sum_{A\subseteq U} q_A f_A(x)$)

Prove just a weak version – let x' be a minimizer of h' – we want to show that for N polynomial, then x' is also of objective function value within a factor of 1+², with probability 1-∂

Chernoff Bound – Let X_k 2 [0,1], k=1,…,M, be ind. r.v.s & let X =$\sum_k X_k$. Then, for any ² > 0

Pr[|X-E[X]| > ²M] · 2 exp(-²² M)

We'll take N = c ¸² n log (1/∂) (1/²⁴)

Sample Average Approximation (SAA) method:

- Sample N times from distribution
- Estimate p_A by q_A = freq. of occurrence of scenario A

(P) OPT = $\min_{x\in\mathcal{P}}$ (h(x) = ω.x + $\sum_{A\subseteq U} p_A f_A(x)$ =ωx + f(x))

(SAA-P) $\min_{x\in\mathcal{P}}$ (h'(x) =_ω.x + $\sum_{A\subseteq U} q_A f_A(x)$_=ωx + f'(x))

Divide scenarios into high and low: say A is high if $f_A(0)$ ¸ ¸OPT/²

By def'n of ¸: for each x & A, $f_A(0)$ · ¸ωx + $f_A(x)$

Lemma: Let p be probability that A is high; then p · (1/¸)²/(1-²).

Proof: OPT = ωx* + $E_A[f_A(x^*)]$ for optimal x*.

¸ p $E_A[f_A(x^*)$ | A high] ¸ p $E_A[$ $f_A(0)$-¸ωx*| A high]

¸ p [¸ OPT/² - ¸ OPT]=p OPT ¸ (1-²)/² qed

Three Key Properties

$f(x) = \omega x + \sum_{A \mu U} p_A \, f_A(x)$

$f_{hi}(x) = \sum_{A\ high} p_A \, f_A(x)$ & $f_{lo}(x) = f(x) - f_{hi}(x) - \omega x$

and analogous for f'

- For each x, $|f_{lo}(x) - f'_{lo}(x)| \cdot {}^2OPT$ w.h.p.
- For each x, $f'_{hi}(0) - f'_{hi}(x) \cdot 2^2\omega x$ w.h.p.
- For each x, $f_{hi}(0) - f_{hi}(x) \cdot 2^2\omega x$.

Three Key Properties

$f(x) = \omega x + \sum_{A \mu U} p_A \, f_A(x)$ & f' replace p by q

$f_{hi}(x) = \sum_{A\ high} p_A \, f_A(x)$ & $f_{lo}(x) = f(x) - f_{hi}(x) - \omega x$

and analogous for f'

- For each x, $|f_{lo}(x) - f'_{lo}(x)| \cdot {}^2OPT$ w.h.p.
- For each x, $f'_{hi}(0) - f'_{hi}(x) \cdot 2^2\omega x$ w.h.p.
- For each x, $f_{hi}(0) - f_{hi}(x) \cdot 2^2\omega x$.

For SAA minimizer x': (by above + $f_{hi}(x) \cdot f_{hi}(0)$)

$$f(x') - f'(x') \cdot {}^2OPT + 2^2\omega x' + f_{hi}(0) - f'_{hi}(0)$$

$$+ \quad f'(x^*) - f(x^*) \cdot {}^2OPT + 2^2\omega x^* + f'_{hi}(0) - f_{hi}(0)\ )$$

$$f(x') - 2^2\omega x' \cdot f(x^*) + 2^2\omega x^* + 2^2OPT\ )$$

$$(1-2^2)f(x') \cdot (1+4^2)OPT \quad !!$$

Three Key Properties

- For each x, $|f_{lo}(x) – f'_{lo}(x)| \cdot {}^{2}OPT$ w.h.p.
- For each x, $f'_{hi}(0) – f'_{hi}(x) \cdot 2^{2}\omega x$ w.h.p.
- For each x, $f_{hi}(0) – f_{hi}(x) \cdot 2^{2}\omega x$.

Lemma. With probability $1-\partial$, the fraction of high scenarios is at most $2^{2}/_{¸}$ (by Chernoff)

This yields 2nd and 3rd properties directly.

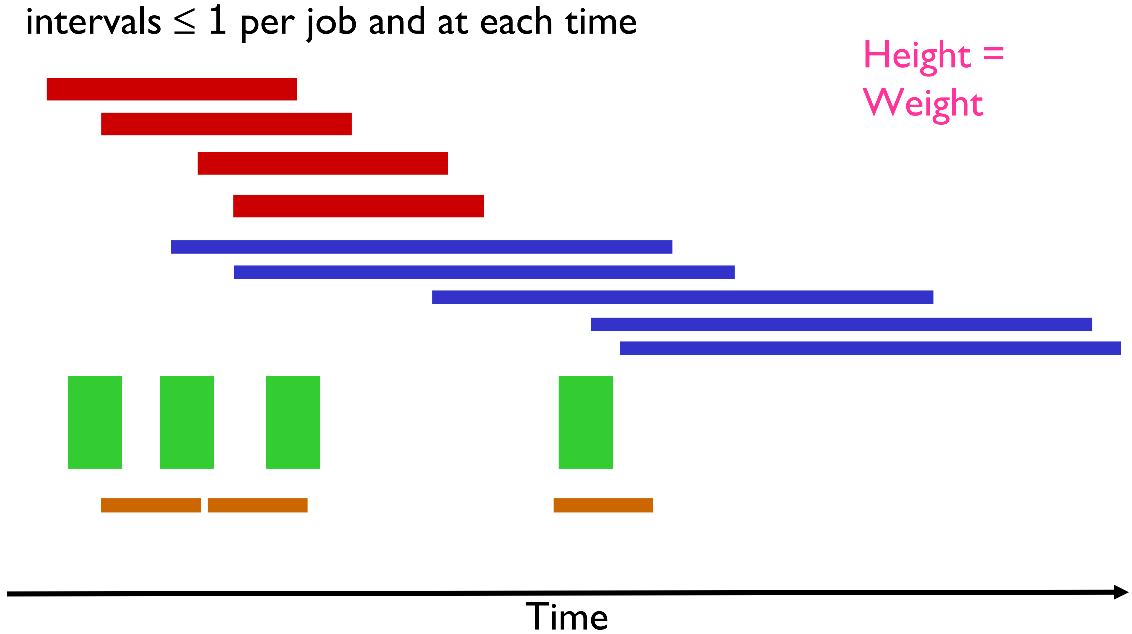
For 1st, view $f'_{lo}(x)$ as the mean of N independent random variable F_i that is $f_A(x)$ if A is low, but 0 otherwise

Apply Chernoff bound to $(1/N)\sum_i F_i/[¸\, OPT/^{2}]$ to get 1st property

Maximum-weight on-time set

Jobs N = {1,2,…,n} - job j has set of allowed time intervals $S_j = \{[s_{1j},e_{1j}),\ldots,[s_{kj},e_{kj})\}$ with corresponding weights w_{ij}

Deterministic problem: Pick a maximum-weight collection of intervals ≤ 1 per job and at each time



Maximum-weight on-time set

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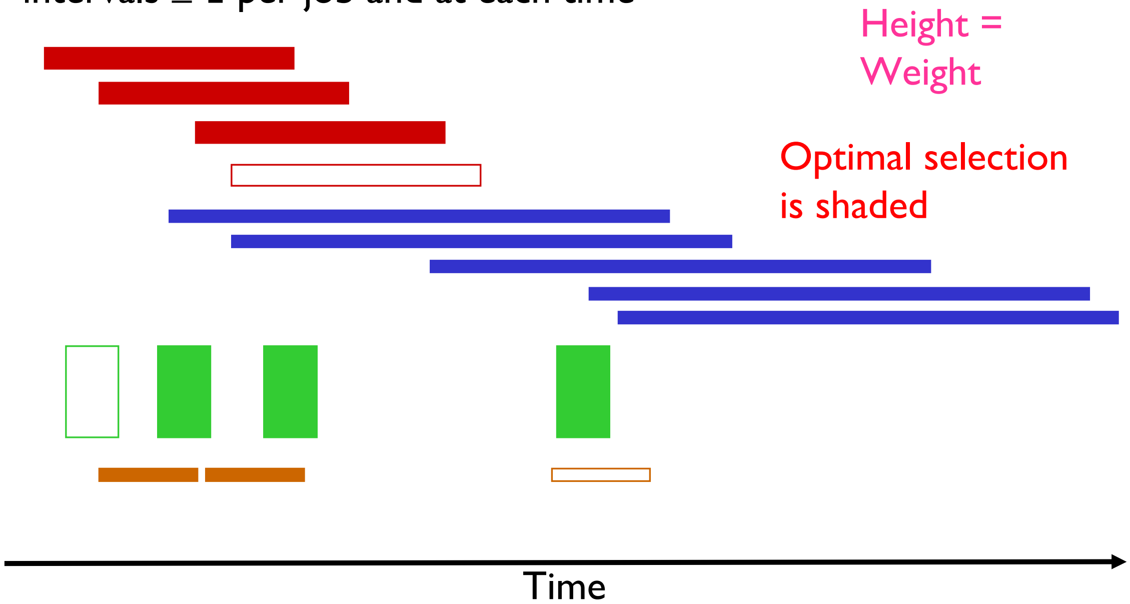

Maximum-weight on-time set

Jobs $N = \{1,2,\ldots,n\}$ - job j has set of allowed time intervals
$S_j = \{[s_{1j},e_{1j}),\ldots,[s_{kj},e_{kj})\}$ with corresponding weights w_{ij}

Deterministic problem: Pick a maximum-weight collection of intervals ≤ 1 per job and at each time

Linear Programming Relaxation

Let T_t be the set of intervals (for all jobs) containing time t : (i,j)

x_{ij} : indicates whether $[s_{ij},e_{ij})$ selected for job j

Maximize $\sum_{i,j} w_{ij}\, x_{ij}$

Subject to	$\sum_i x_{ij} \leq 1,$	for each j=1,…,n
	$\sum_{(i,j) \in T_t} x_{ij} \leq 1$	for each t
	$x_{ij} \geq 0$	for each i,j

Theorem [Bar-Noy, Bar-Yehuda, Freund, Naor, & Schieber]
Primal-dual 2-approximation algorithm for max-weight schedule

2-Stage Stochastic Variant

Scenario $A \subseteq N$ of active jobs occurs with probability p_A

Stage I: Choose set $D \subseteq N$ of jobs to defer to subcontractor and receive small weight ω_j for each $j \in D$

Stage II: Given realized scenario A, make selection T_A where $(i,j) \in T_A \Rightarrow j \in A\text{-}D$ and has weight W_{ij}

Goal: Maximize the total expected weight scheduled (where expectation is with respect to active subset probabilities)

A Primal-Dual Theorem

Theorem: [S & Sozio] We can (adapt the 2-approximation algorithm for deterministic setting to) obtain a 2-approximation algorithm for stochastic maximum-weight on-time scheduling.

Note: it is trivial to obtain a 4-approximation algorithm (flip a coin and either decide to either put all of your eggs in Stage I or Stage II) and almost as simple to obtain a 3-approximation algorithm

We focus first on the polynomial-scenario model

Maximum-weight on-time set

Jobs $N = \{1,2,\ldots,n\}$ - job j has set of allowed time intervals

$S_j = \{[s_{1j},e_{1j}),\ldots,[s_{kj},e_{kj})\}$ with corresponding weights w_{ij}

Deterministic problem: Pick a maximum-weight collection of intervals ≤ 1 per job and at each time

Linear Programming Relaxation

Let T_t be the set of intervals (for all jobs) containing time t : (i,j)

x_{ij} : indicates whether $[s_{ij},e_{ij})$ selected for job j

Maximize $\sum_{i,j} w_{ij} x_{ij}$

Subject to

$\sum_i x_{ij} \leq 1$	for each j=1,…,n
$\sum_{(i,j) \in T_t} x_{ij} \leq 1$	for each t
$x_{ij} \geq 0$	for each i,j

Theorem [Bar-Noy, Bar-Yehuda, Freund, Naor, & Schieber]
Primal-dual 2-approximation algorithm for max-weight schedule

Dual Linear Program

Let T_t be the set of intervals (for all jobs) containing time t

Minimize $\sum_j u_j + \sum_t v_t$
Subject to

$$u_j + \sum_{t:\,(i,j) \in T_t} v_t \geq w_{ij} \text{ for each } (i,j)$$

$$u_j,\ v_t \geq 0$$

The primal-dual algorithm has two phases:

•first it constucts a feasible dual solution, while building a stack of possible pairs (i,j) to be selected;

•next it pops the stack, selecting any pair that doesn't conflict with those already selected;

•amortization shows dual cost is at most twice the value of the primal.

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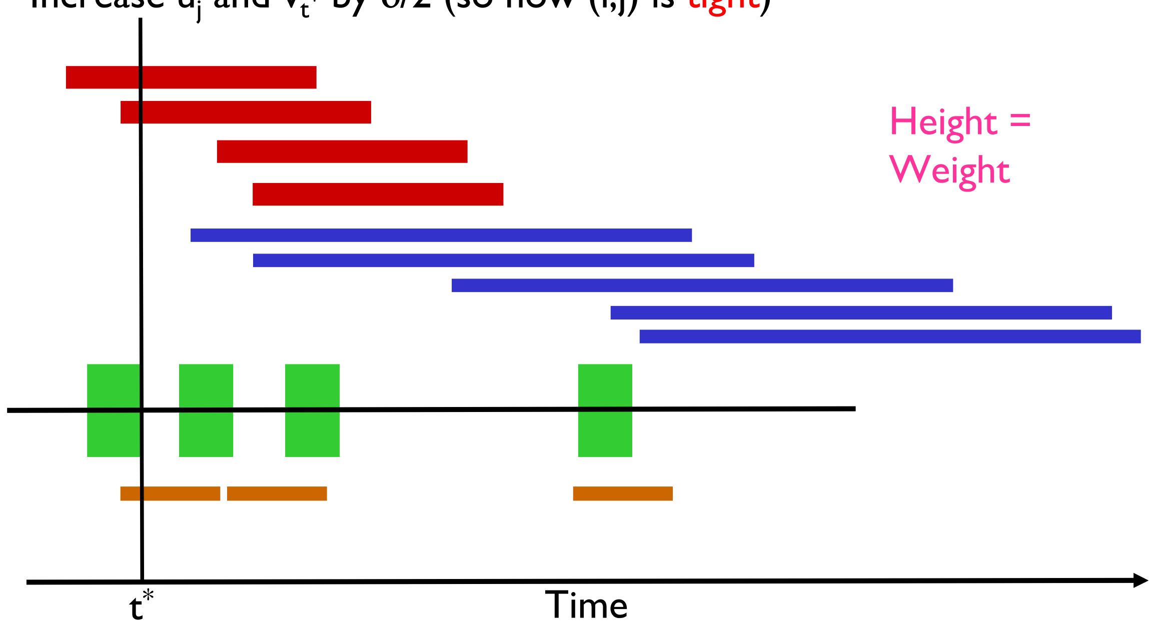
The Primal-Dual Algorithm of Bar-Noy et al.

- Pick the uncovered interval (i,j) with the earliest ending point t^*
- Compute its deficit $\delta = w_{ij} - u_j - \sum_{t: (i,j) \in T_t} v_t$
- Increase u_j and v_{t^*} by $\delta/2$ (so now (i,j) is tight)



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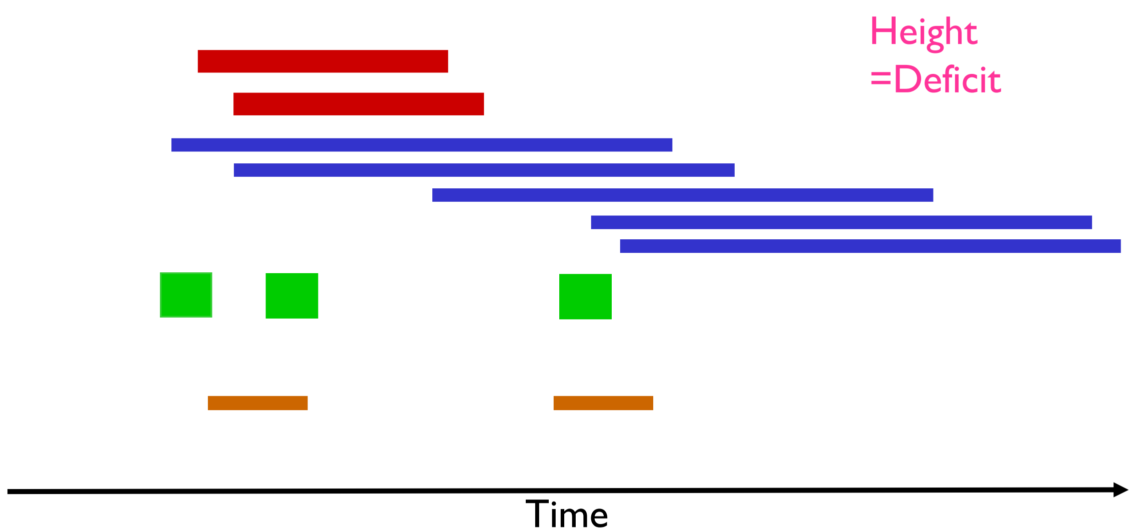

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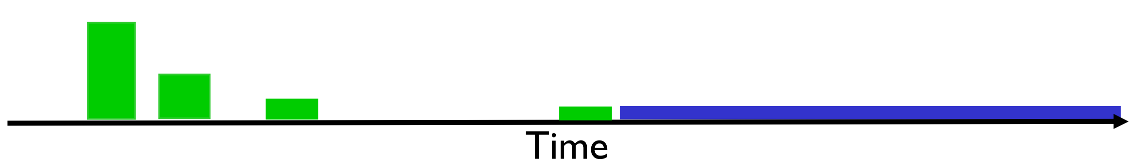

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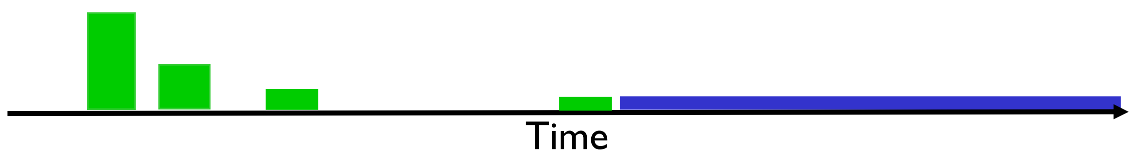
•Analysis: every selected interval is tight; every iteration adds δ to dual objective and contributes at least $\delta/2$ to "paying for" selected (i,j)'s; hence, dual objective is at most twice amount paid!!



Linear Programming Relaxation for 2-Stage Problem

Let T_t be the set of intervals (for all jobs) containing time t

x_j : indicates whether job j is deferred in stage I

$y_{ij}(S)$: indicates whether $[s_{ij}, e_{ij})$ selected for job j in stage II for scenario S

Maximize $\sum_j \omega_j x_j + \sum_{i,j,S} p_S W_{ij}\, y_{ij}(S)$

Subject to $x_j + \sum_i y_{ij}(S) \leq 1,$ for each j,S

$\sum_{(i,j) \in T_t} y_{ij}(S) \leq 1,$ for each t,S

$x_j,\ y_{ij}(S) \geq 0$ for each i,j,S

DUAL Minimize $\sum_{j,S} u_j(S) + \sum_{t,S} v_t(S)$

Subject to $\sum_S u_j(S) \geq \omega_j$ for each j

$u_j(S) + \sum_{t:\,(i,j) \in T_t} v_t(S) \geq p(S)\, W_{ij}$ for each (i,j), S

$u_j(S),\ v_t(S) \geq 0$

A Simple 2-Stage Algorithm

For each scenario $A \subseteq N$ with probability $p_A > 0$

run the deterministic algorithm with job set A where weight of job j for $[s_{ij}, e_{ij})$ is $p_A W_{ij}$

let $u_j(A)$ denote the dual values constucted by the algorithm

Stage I: Let D be the set of jobs j for which

$$\omega_j > \Sigma_A \; u_j(A)$$

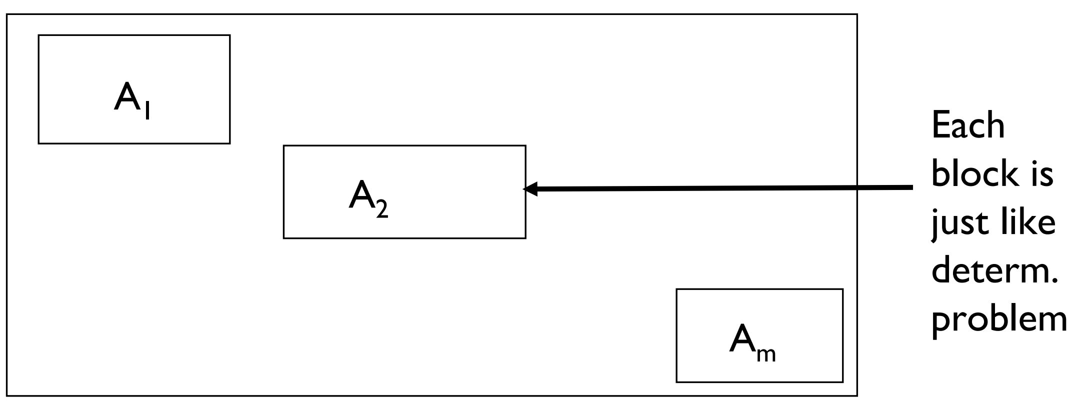
Stage II: Given realized scenario A,

recompute first phase of algorithm (to get duals)

but in second phase never select (i,j) for $j \in D$

Main Idea of Analysis

What is 2-stage dual? Block-structured by scenario A with additional linking constraints:



$$\Sigma_A \; u_j(S) \qquad \geq \omega_j \; \forall \; j$$

So we can adapt the scenario-by-scenario constuction as building a feasible dual solution for the 2-stage linear relaxation

What about black box model?

Just use sampling to estimate the deferral rule! - use M samples

For each sampled scenario $A \subseteq N$ run deterministic algorithm with job set A where weight of job j for $[s_{ij}, e_{ij})$ is W_{ij} to obtain dual values $u_j(A)$ – let A_k be k^{th} sample

Stage I: Let D be the set of jobs j for which

$$(1+\varepsilon)\ \omega_j > (1/M) \sum_k u_j(A_k)$$

Stage II: Given realized scenario A, compute T_A and then recompute first phase of algorithm (to get duals) but in second phase never select (i,j) for $j \in D$

Number of samples needed is polynomial in n, $1/\varepsilon$, and $\lambda = \max_j W_j/\omega_j$

Similar to "sample average approximation" results of [Swamy & S, Shapiro & Nemirovski, and Charikar, Chekuri, & Pál]

Some Additional Details

Previously used profits equal to $p_A W_{ij}$ for i^{th} interval of job j in scenario A – what now?

Ignore p_A – call rescaled duals $u_j^*(A)$ where

$$u_j(A) = p_A\ u_j^*(A)$$

Had used $r = \sum_A p_A\ u_j(A)$ as threshold

Now use $\mathbf{r}^* = \sum_K (1/M)\ u_j^*(A_k)$ instead

Use Chernoff bounds to prove $\mathbf{r}^*$ ¼ r w/high prob.

Chernoff – Let X_k 2 [0,1] , k=1,…,M be ind. r.v.s & let $X = \sum_k X_k$. Then, for any ² > 0

Pr[|X-E[X]| > ² M] · 2 exp(-²² M)

What are the [0,1] random variables?

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Applying the Chernoff Bound

Let $X_k = u_j^*(A_k) /(¸\ \omega_j)$

Why is X_k 2 [0,1]?

There is some i such that

$u_j^*(A_k)$ · W_{ij} (perhaps i that became tight)

and so $u_j^*(A_k)$ · W_{ij} · ¸ ω_j

Now take M= £((¸²/²²) log (n/°)) to get

Pr[|X-E[X]| > ²M/¸] ·exp(-²²M/ ¸²) ,

Pr[| **r*** - r | > ² ω_j] · °/n

And apply "union bound" to get failure for all jobs j occurs with probability at most °

Another 2-Stage Stochastic Variant

Scenario $A \subseteq N$ of active jobs occurs with probability p_A

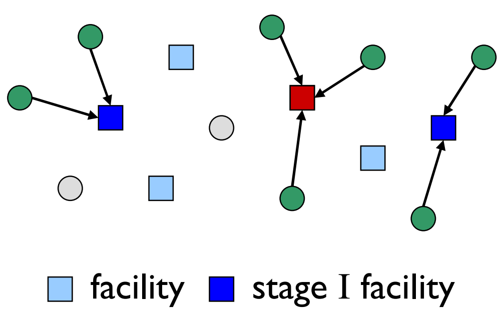
Stage I: Choose set $C \subseteq N$ of jobs j to commit to service and get weight ω_j

Stage II: Given realized scenario A, make selection T_A where $\exists\ (i,j) \in T_A$ for each $j \in C$ plus some additional ones

Goal: Maximize the total expected weight scheduled

Bad News: There is an approximation-preserving reduction from the deterministic maximum independent set problem, and hence no "reasonable" performance guarantee can be proved (unless P=NP).

2-Stage Stochastic Facility Location



Distribution over clients gives the set of clients to serve.

Stage I: Open some facilities in advance; pay cost f_i for facility i.

Stage I cost = $\sum_{(i \text{ opened})} f_i$.

Actual scenario A = { ● clients to serve}, materializes.

Stage II: Can open more facilities to serve clients in A; pay cost f_i^A to open facility i. Assign clients in A to facilities.

Stage II cost = $\sum_{i \text{ opened in scenario } A} f_i^A$ + (cost of serving clients in A).

Several Ways to Skin The Cat Facility Location Yet Again

- Can apply LP-rounding approach as done for set covering [S & Swamy]
- Can apply the boosted sampling approach if the second stage costs are proportional
- In polynomial scenario setting can adapt primal-dual algorithm of Jain & Vazirani for deterministic version to get 3-approximation algorithm [Mahdian]
- Can then apply result Sample Average Approximation result of [Charikar, Chekuri, & Pál] to extend to "black box" model

Discrete Stochastic Optimization and Approximation Algorithms

- Area of emerging importance
- Rich source of algorithmic questions
- Can one prove a strong result for approximate stochastic dynamic programming? [Levi Roundy & S] [Halman, Klabjan, Mostagir, Orlin & Simchi-Levi]
- When is sampling information good enough to derive near-optimal solutions?
- Reconsider some well-studied problems but now in "black box" model, not just specific distributions
- Expectation is not enough

Thank You.