Cost Sharing and Approximation Algorithms

Exercise 2

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Problems 1 and 2 deal with the Steiner tree problem: We are given an undirected graph $G = (V,E)$ with non-negative edge costs $c : E \rightarrow \mathbb{R}^+$, a set of terminals $R = \{t_1, \ldots, t_k\} \subseteq V$ and a designated root node $r \in V$. For a given subset $S \subseteq R$ of terminals, a Steiner tree on $S$ is a minimum cost tree in $G$ that spans all nodes in $S \cup \{r\}$. We use $\text{opt}(S)$ to refer to its cost.

**Problem 1.** In the cost sharing variant of the Steiner tree problem, the set of players corresponds to the set of terminal nodes, i.e., $U = R$. Every player wants to connect her terminal $t_i$ to the root node $r$. The cost $C(S)$ to connect all players in $S \subseteq U$ with $r$ is defined as the cost $\text{opt}(S)$ of a Steiner tree on $S$.

Develop a 2-budget balanced and weakly group-strategyproof cost sharing mechanism for the Steiner tree cost sharing game.

**Problem 2.** Give a 2-approximate and 2-strict algorithm for the Steiner tree problem.

Hints for Problems 1 and 2: Convince yourself about the following:

- We can assume without loss of generality that $G$ is a complete graph and that the edge costs satisfy the triangle inequality.
- We obtain a 2-approximate Steiner tree for $S \subseteq R$ by computing a minimum spanning tree on $S \cup \{r\}$.

**Problem 3.** Suppose we are given a set of players $U$ and a cost function $C : 2^U \rightarrow \mathbb{R}^+$. A cost allocation $(x_i)_{i \in N}$ assigns a non-negative cost share $x_i$ to every player $i \in U$. The cost allocation $(x_i)_{i \in N}$ is said to be in the $\alpha$-core ($\alpha \geq 1$) if:

1. $\frac{1}{\alpha} C(U) \leq \sum_{i \in U} x_i \leq C(U)$
2. $\sum_{i \in S} x_i \leq C(S)$ for every $S \subseteq U$.

(a) Show that every cross-monotonic and $\beta$-budget balanced cost sharing function $\xi$ gives rise to a cost allocation in the $\alpha$-core with $\alpha = \beta$.

(b) Show that if the cost function $C$ is such that $C(U) \geq \beta \sum_{i \in U} C(\{i\})$ for some $\beta > 1$ then there is no cost allocation in the $\alpha$-core for $\alpha < \beta$. 

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