Randomized Mechanism Design: Approximation and Online Algorithms

Part 2: Combinatorial Auctions

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The combinatorial auction problem

A set $M = \{1, \ldots, m\}$ shall be allocated to n bidders with private valuations for bundles of items

Definitions:

- feasible allocations: $A = \{(S_1, \ldots, S_n) \subseteq M^n \,|\, S_i \cap S_j = \emptyset, i \neq j\}$
- valuation functions: $v_i : 2^M \to \mathbb{R}_{\geq 0}$, $i \in [n]$
- objective: maximize social welfare $\sum_{i=1}^{n} v_i(S_i)$

Assumptions:

- free disposal: $S \subseteq T \Rightarrow v_i(S) \leq v_i(T)$
- normalization: $v_i(\emptyset) = 0$

Overview

0: Introduction

1: Approximation algorithms

– single-minded bidders –

– multi-dimensional bidders –

2: Online algorithms

– overselling algorithm –

– oblivious randomized rounding –

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Single-minded bidders

- Bidders are called *single-minded* if, for every bidder i, there exists a bundle $S_i^* \subseteq M$ and a value $v_i^* \in \mathbb{R}_{\geq 0}$ such that

$$v_i(T) = \begin{cases} v_i^* & \text{if } T \supseteq S_i^* \\ 0 & \text{otherwise} \end{cases}$$

- Bids correspond to tuples (S_i^*, v_i^i).
- Given the output of a mechanism, bidder i is called *winning* if it is assigned a bundle $T \supseteq S_i^*$.
- An output is called *exact*, if every bidder i is assigned S_i^* (rather than some superset).
- A mechanism producing only exact outputs is called *exact*.

Computational hardness

Proposition

The allocation problem among single-minded bidders is NP-hard.

Proof: Reduction from independent set.

- Consider a graph $G = (V, E)$. Each node is represented by a bidder. Each edge is represented by a good.
- For bidder i, set $S_i^* = \{e \in E | i \in e\}$ and $v_i^* = 1$.
- This way, winning bidders correspond to nodes in an independent set. □

Indeed, the reduction implies

Proposition

Approximating the optimal allocation among single-minded bidders to within a factor of $m^{1/2-\epsilon}$, for any $\epsilon > 0$, is NP-hard.

Incentive compatibility for single-minded bidders

Characterization of truthfulness

An exact mechanism for single minded bidders in which losers pay 0 is truthful if and only if it satisfies the following two properties:

- **Monotonicity:** A bidder who wins with bid (S_i^*, v_i^*) keeps winning for any $v_i' > v_i^*$ and for any $S_i' \subset S_i^*$ (for any fixed setting of the other bids).
- **Critical Payment:** A winning bidder pays the minimum value needed for winning: The infimum of all values v_i' such that (S_i^*, v_i') wins.

Incentive compatible mechanism for single-minded bidders

Greedy allocation

- Reorder the bids such that $\frac{v_1^*}{\sqrt{|S_1^*|}} \geq \frac{v_2^*}{\sqrt{|S_2^*|}} \geq \cdots \geq \frac{v_n^*}{\sqrt{|S_n^*|}}$.
- Initialize the set of winning bidders to $W = \emptyset$.
- For $i = 1 \ldots n$ do: If $S_i^* \cap \bigcup_{j \in W} S_j^* = \emptyset$ then add i to W.

The Greedy allocation is monotone. Combining it with critical payment gives a truthful mechanism.

Approximation factor of the Greedy algorithm

Theorem [Lehmann et. al, 2002]

The Greedy mechanism guarantees a $\sqrt{m}$-approximation of the optimal social welfare.

Proof:

- For $i \in W$, let $OPT_i = \{j \in OPT, j \geq i | S_i^* \cap S_j^* \neq \emptyset\}$.
- As $v_j^* \leq \sqrt{|S_j^*|} \cdot v_i^* / \sqrt{|S_i^*|}$, for $j \in OPT_i$, we obtain

$$\sum_{j \in OPT_i} v_j^* \leq \frac{v_i^*}{\sqrt{|S_i^*|}} \sum_{j \in OPT_i} \sqrt{|S_j^*|}$$

- We will show that $\sum_{j \in OPT_i} \sqrt{|S_j^*|} \leq \sqrt{|S_i^*|}\sqrt{m}$, which gives

$$v(OPT) \leq \sum_{i \in W} \sum_{j \in OPT_i} v_j^* \leq \sum_{i \in W} v_i^* \sqrt{m} = \sqrt{m} \cdot v(GREEDY) \quad .$$

Approximation factor of the Greedy algorithm

Claim

$$\sum_{j \in OPT_i} \sqrt{|S_j^*|} \leq \sqrt{|S_i^*|}\sqrt{m}$$

- By the Cauchy-Schwarz inequality

$$\sum_{j \in OPT_i} \sqrt{|S_j^*|} \leq \sqrt{|OPT_i|}\sqrt{\sum_{j \in OPT_i} |S_j^*|}.$$

- Now $|OPT_i| \leq |S_i^*|$ since every S_j^*, for $j \in OPT_i$, intersects S_i^* and these intersections are disjoint. (Why?)
- Furthermore, $\sum_{j \in OPT_i} |S_j^*| \leq m$ since OPT_i is an allocation.

□

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Problem description

ILP description of the problem

$$\begin{aligned} \text{Maximize} \quad & \sum_{(i,S)} x_{i,S} v_i(S) & \\ \text{subject to} \quad & \sum_{S} x_{i,S} \leq 1 & \text{for each bidder } i \\ & \sum_{(i,S)|j \in S} x_{i,S} \leq 1 & \text{for each item } j \\ & x_{i,S} \geq 0 & \end{aligned}$$

The LP-ralaxation of this problem can be solved efficiently using

Demand oracles:

Given a price p_j, for each item j, the *demand oracle for bidder i* answers queries of the following kind:

What is the utility-maximizing bundle?

Incentive compatibility for multi-dimensional bidders

Characterization of truthfulness

A mechanism is truthful if and only if it satisfies the following two properties for every i:

i) For every bundle $T \subseteq M$, there exists a price $q_T^{(i)}(v_{-i})$.

That is, for all v_i with $f_i(v_i, v_{-i}) = T$, $p(v_i, v_{-i}) = q_T^{(i)}(v_{-i})$.

ii) The social choice function maximizes the utility for player i.

That is, for every bidder i,

$$f(v) = \operatorname*{argmax}_{(S_1,\ldots,S_n)\in A^{(i)}(v_{-i})} \left(v_i(S_i) - q_{S_i}^{(i)}(v_{-i})\right)$$

with $A^{(i)}(v_{-i}) \subseteq A$ being a non-empty subset of allocations.

Examples: VCG, Fixed Price Auctions, Iterative Auctions

A universally truthful auction mechanism

[Dobzinski, Nisan, Schapira 2006]

1. Partition bidders into three sets SEC-PRICE, FIXED, STAT with probability $1 - \epsilon$, $\epsilon/2$, and $\epsilon/2$, respectively.
2. Calculate optimal fractional solution opt^*_{STAT} of the bidders in STAT.
3. Perfom a second price auction for selling a full bundle to a bidder in SEC-PRICE with a reserve price $r = v(opt^*_{STAT})/\sqrt{m}$.
4. If the second price auction was not successful then:
Perform a fixed price auction selling items at a fixed price $p = \epsilon v(\epsilon opt^*_{STAT})/8m$, considering bidders in some fixed order.

Analyzing the approximation ratio

Bidder i is called *t-dominant* if $v_i(M) \geq v(opt)/t$.

Lemma

Suppose that there is a $\sqrt{m}$-dominant bidder and $r \leq {}^{v(opt)}/_{\sqrt{m}}$. Then the mechanism provides a $\sqrt{m}$-approximation with probability at least $1 - \epsilon$.

Lemma

Suppose there is no $\sqrt{m}$-dominant bidder. Then, with probability at least $1 - {}^{16}/_{\epsilon\sqrt{m}}$., both $v(opt_{STAT})$ and $v(opt_{FIXED})$ are lower-bounded by $v(opt) \cdot {}^{\epsilon}/_{4}$.

An analogous statement holds wrt opt^*, opt^*_{STAT}, and opt^*_{FIXED}.

Analyzing the approximation ratio

Analysis of fixed price auction

Suppose that the following conditions hold:

- There is no $\sqrt{m}$-dominant bidder.
- The item price p satisfies: $\frac{\epsilon^2 v(opt^*)}{32m} \leq p \leq \frac{\epsilon v(opt^*)}{8m}$.
- $v(opt^*_{FIXED}) \geq v(opt^*) \cdot \epsilon/4$.

We will show that the revenue of the fixed-price auction is $\Omega(\epsilon^3 v(opt^*_{FIXED})/\sqrt{m})$.

This gives

Theorem [Dobzinski et. al, 2010]

The mechanism provides an approximation ratio of $O(\sqrt{m}/\epsilon^3)$ with probability at least $1 - \epsilon$.

Analysis of fixed price auction

Let $\{y_{i,S}\}$ be the values of the variables in opt^*_{FIXED}.

Let $\mathcal{T}$ be the set of pairs (i, S) with $y_{i,S} > 0$ and $v_i(S) \geq p \cdot |S|$.

Let $opt^*_{FIXED|\mathcal{T}} = \{y_{i,S}\}_{(i,S)\in\mathcal{T}}$.

Claim

$$v(opt^*_{FIXED|\mathcal{T}}) = \sum_{(i,S)\in\mathcal{T}} y_{i,S} v_i(S) \geq \frac{1}{2} \cdot v(opt^*_{FIXED}).$$

Proof:

Define $\bar{\mathcal{T}}$ to be the complement of $\mathcal{T}$. It holds

$$\sum_{(i,S)\in\bar{\mathcal{T}}} y_{i,S} \cdot v_i(S) \leq \sum_{(i,S)\in\bar{\mathcal{T}}} y_{i,S} \cdot |S| \cdot p \leq m \cdot p$$

$$\leq m \cdot \frac{\epsilon v(opt^*)}{8m} \leq \frac{\epsilon v(opt^*_{FIXED})}{2} \quad .$$

□

Analysis of fixed price auction

It remains to show $v(FP) = \Omega(v(opt^*_{FIXED|\mathcal{T}}))$, where FP denotes the allocation of the fixed price auction.

We consider bidders in the order of the fixed price auction and study the following

dynamic process:

Whenever the fixed price auction chooses a bundle S_i for a bidder i, we remove the following bundles from $\mathcal{T}$:

1. (i, S) for any bundle S
2. (j, S) for any bidder j and any bundle S with $S \cap S_i \neq \emptyset$

At the end of the process the set $\mathcal{T}$ is empty!

Analysis of fixed price auction

When adding S_i to FP, the set $\mathcal{T}$ loses the following values

1. $$\sum_{(i,S)\in\mathcal{T}} y_{i,S} \cdot v_i(S) \le \sum_{(i,S)\in\mathcal{T}} y_{i,S} \cdot v_i(M) \le v_i(M) \le \frac{v(opt^*)}{\sqrt{m}}$$
2. $$\sum_{(i,S)\in\mathcal{T}|j\in S} y_{i,S} \cdot v_i(S) \le \frac{v(opt^*)}{\sqrt{m}}, \text{ for every } j \in S_i$$

That is, for each item that we add to FP, the set $\mathcal{T}$ loses a value of at most $2 \cdot \frac{v(opt^*)}{\sqrt{m}}$.

On the other hand, FP achieves revenue $p \ge \epsilon^2 \cdot \frac{v(opt^*)}{32m}$, for each of the picked items. □

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Online mechanisms – model and approach

We assume that there are n bidders with arbitrary valuations.

The n bidders arrive one by one in random order.

The bidder arriving at time i, $1 \leq i \leq n$, is called the *ith bidder*.

The iterative pricing approach

When the i-th bidder arrives the mechanism calls the demand oracle with prices p_e^i that only depend on vauations of bidders $1, \ldots, i-1$ but not on the valuations of bidders $i, \ldots, n$.

By the direct characterization, this approach yields incentive compatible mechanisms.

Online mechanisms – competitive ratio

What do we achieve?

- Suppose each items is available with multiplicity $b \geq 1$.
  Competitive ratio: $O(m^{1/(b+1)} \log(bm))$.
- For $b = \log m$ this gives competitive ratio $O(\log m)$.
- Suppose bundles have size at most d.
  Competitive ratio: $O(d^{1/b} \log(bm))$.
- Suppose valuations are submodular or XOS.
  Competitive ratio: $O(\log m)$.

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Analytic trick: Violate constraints

Overselling MPU algorithm [inspired by Bartal, Gonen, Nisan 2003]

For each good $e \in U$ do $p_e^1 := p_0$.

For each bidder $i = 1, 2, \dots, n$ do

  Set $S_i := \text{Oracle}_i(U_i, p^i)$.

  Update for each good $e \in S_i$: $p_e^{i+1} := p_e^i \cdot 2^{1/b}$.

Suppose L is a lower bound of $v(opt)$ such that at most one bidder exceeds L. We set $p_0 = L/4bm$.

For the time being, assume that $U_i = M$.

$\text{Oracle}_i(U_i, p^i)$ returns the utility-maximal bundle for bidder i for prices p^i restricted to items in $U_i \subseteq M$.

How many copies per item are sold?

Lemma 1

At most sb copies of each item are sold, where $s = \log(4bm) + \frac{2}{b}$.

Proof:

- Suppose $\lceil sb - 2 \rceil \geq b \log(4bm)$ copies of item e have been sold after some step.
- Then the price of e is larger than $p_0 \cdot 2^{\log(4bm)} \geq L$.
- After this step, only one further copiy of e might be given to that bidder whose maximum valuation exceeds L.
- Hence, at most $\lceil sb - 1 \rceil \leq sb$ copies of e are assigned, which proves the lemma. $\square$

Lower bounding social welfare achieved by the algorithm

Let p_e^* denote the final prices (after the algorithm stopped).

Lemma 2

$v(S) \geq b \sum_{e \in U} p_e^* - bmp_0$.

Proof:

As bidders are individually rational, $v_i(S_i) \geq \sum_{e \in S_i} p_e^i$. Thus

$$v(S) \geq \sum_{i=1}^{n} \sum_{e \in S_i} p_e^i = \sum_{i=1}^{n} \sum_{e \in S_i} p_0 r^{\ell_e^i} = p_0 \sum_{e \in U} \sum_{k=0}^{\ell_e^*-1} r^k = p_0 \sum_{e \in U} \frac{r^{\ell_e^*} - 1}{r - 1}$$

where $r = 2^{1/b}$, ℓ_e^i is the number of copies of e sold before bidder i, and ℓ_e^* is the number of copies sold at the end of the execution.

Applying $p_e^* = p_0 r^{\ell_e^*}$ and $1/(r-1) = 1/(2^{1/b} - 1) \geq b$ gives the lemma. □

Lemma 3

$v(S) \geq v(opt) - b\sum_{e \in M} p_e^*$, provided $U_1 = \cdots U_n = M$.

Proof:

Consider any feasible allocation $T = (T_1, \ldots, T_n)$.

As the algorithm uses a utility-maximizing demand oracle, we have

$$v_i(S_i) - \sum_{e \in S_i} p_e^i \geq v_i(T_i) - \sum_{e \in T_i} p_e^i,$$

which implies

$$v_i(S_i) \geq v_i(T_i) - \sum_{e \in T_i} p_e^i.$$

As $p_e^* \geq p_e^i$, for every i and e, we obtain

$$v_i(S_i) \geq v_i(T_i) - \sum_{e \in T_i} p_e^*. \qquad (*)$$

Lower bounding social welfare achieved by the algorithm

Summing over all bidders gives

$$v(S) = \sum_{i=1}^{n} v_i(S_i) \geq \sum_{i=1}^{n} v_i(T_i) - \sum_{i=1}^{n} \sum_{e \in T_i} p_e^* \geq v(T) - b \sum_{e \in M} p_e^*$$

because T is feasible so that each item is given to at most b sets.

Taking for T_i to be the bundle assigned to bidder i in an optimal solution gives

$$v(S) \geq v(opt) - b \sum_{e \in U} p_e^* .$$

□

Lower bounding social welfare achieved by the algorithm

Lemma 2

$v(S) \geq b \sum_{e \in U} p_e^* - bmp_0$.

Lemma 3

$v(S) \geq v(opt) - b \sum_{e \in U} p_e^*$, provided $U_1 = \cdots U_n = M$.

Substituting Lemma 2 into Lemma 3 gives

$$v(S) \geq v(opt) - v(S) - bmp_0 \geq v(opt) - v(S) - \frac{1}{4} v(opt)$$

as $p_0 = L/4bm \leq v(opt)/4bm$.

This gives $2v(S) \geq \frac{3}{4} v(opt)$ and, hence, $v(S) \geq \frac{3}{8} v(opt)$.

Properties of the overselling MPU algorithm

The algorithm is $\frac{3}{8}$-competitive with respect to the optimal offline social welfare.

However, its output is not feasible as it oversells items by a factor of $O(\log bm)$.

Is the algorithm incentive compatible?

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Algorithmic trick: Use randomization to ensure feasibility

MPU algorithm with oblivious randomized rounding

For each good $e \in U$ do $p_e^1 := p_0$, $b_e^1 := b$.

For each bidder $i = 1, 2, \ldots, n$ do

Set $S_i := \text{Oracle}_i(U_i, p^i)$, for $U_i = \{e \in U \mid b_e^i > 0\}$.

Update for each good $e \in S_i$: $p_e^{i+1} := p_e^i \cdot 2^{1/b}$.

With probability q set $R_i := S_i$ else $R_i := \emptyset$.

Update for each good $e \in R_i$: $b_e^{i+1} := b_e^i - 1$.

Lemma 4

Suppose the probability $q > 0$ is chosen sufficiently small such that, for any $1 \leq i \leq n$, and any bundle $T \subseteq U$,

$$\underbrace{\mathbf{E}\left[v_i(T \cap U_i)\right] \geq \frac{1}{2} v_i(T)}_{\text{expected value assumption}} \quad .$$

Then $\mathbf{E}\left[v(S)\right] \geq \frac{1}{8} v(opt)$ and $\mathbf{E}\left[v(R)\right] \geq \frac{q}{8} v(opt)$.

Proof:

Consider any feasible allocation $T_1, \ldots, T_n$.

The set S_i is chosen by $\text{Oracle}_i(U_i, p^i)$ so that

$$v_i(S_i) \geq v_i(T_i \cap U_i) - \sum_{e \in T_i \cap U_i} p_e^i ,$$

for any outcome of the algorithm's random coin flips.

Lower bounding social welfare achieved by the algorithm

This implies

$$\mathbf{E}\left[v_i(S_i)\right] \geq \mathbf{E}\left[v_i(T_i \cap U_i)\right] - \sum_{e \in T_i \cap U_i} \mathbf{E}\left[p_e^i\right] .$$

Applying the expected value assumption, we obtain

$$\mathbf{E}\left[v_i(S_i)\right] \geq \frac{1}{2} v_i(T_i) - \sum_{e \in T_i} \mathbf{E}\left[p_e^i\right] .$$

Observe that this equation is similar to equation (*) in the proof of Lemma 3 so that the rest of the analysis proceeds analogous to the analysis for the overselling MPU algorithm. □

Lower bounding social welfare achieved by the algorithm

Lemma 5

The expected value assumption holds for

$$q = \frac{1}{2\mathrm{e}d^{1/b}\left(\log(4bm) + \frac{2}{b}\right)} \;,$$

where b denotes the multiplicity and d the maximum bundle size.

This implies

Theorem [Krysta, V., 2012]

The algorithm is $O(d^{1/b}\log(bm))$-competitive.

Proof of Lemma 5:

By Lemma 1, item $e \in U$ is contained in at most $\ell := b \cdot \log(4bm) + 2$ of the provisional bundles $S_1, \ldots, S_{i-1}$.

Each of these ℓ bundles is turned into a final bundle with probability $q = b/(2\mathrm{e}d^{1/b}\ell)$.

Observe that $e \notin U_i$ if at least b of the ℓ bundles became final.

The probability that $e \notin U_i$ is thus

$$\binom{\ell}{b} \cdot q^b \leq \left(\frac{\mathrm{e}\ell}{b}\right)^b \cdot \left(\frac{b}{2\mathrm{e}d^{1/b}\ell}\right)^b = \frac{1}{2d} \ .$$

By the union bound, we have $\mathbf{Pr}\left[\exists e \in T : e \notin U_i\right] \leq |T| \cdot \frac{1}{2d} \leq \frac{1}{2}$.

Thus, $\mathbf{E}\left[v_i(T \cap U_i)\right] \geq v_i(T) \cdot \mathbf{Pr}\left[\neg\exists e \in T : e \notin U_i\right] \geq \frac{1}{2} v_i(T)$. $\square$

Submodular and XOS valuations

Submodular:
$v_i(S \cup T) \leq v_i(S) + v_i(T) - v_i(S \cap T)$, for every S, T

Subadditive (a.k.a. complement free):
$v_i(S \cup T) \leq v_i(S) + v_i(T)$, for every S, T

Fractional-subadditive (a.k.a. XOS):
$v_i(S) \leq \sum_{K \subseteq S} \alpha_K v_i(K)$ for every *fractional cover* α_K, i.e.,

- $0 \leq \alpha_K \leq 1$, for all $K \subseteq S$, and
- $\sum_{i | j \in K} \alpha_K \geq 1$, for every item $j \in S$

Submodular $\subseteq$ Fractional-Subadditive $\subseteq$ Subadditive

Lemma 6

If valuation functions are fractional-subadditive then the expected value assumption holds for

$$q = \frac{1}{2(\log(4\mu m) + 2)} \enspace .$$

This implies

Theorem [Krysta, V., 2012]

The algorithm is $O(\log(m)$-competitive for XOS valuations.

Proof of Lemma 6:

Any item $e \in U$ is contained in at most $\ell := b \cdot \log(4bm) + 2$ of the provisional bundles $S_1, \ldots, S_{i-1}$. Each of these ℓ bundles is turned into a final bundle with probability $q = 1/(2\ell)$.

$$\mathbf{Pr}\left[e \notin U_i\right] = \mathbf{Pr}\left[\text{one of the } \ell \text{ bundles becomes final}\right] \leq \frac{1}{2} \ .$$

Now fix T arbitrarily. For any given subset $K \subseteq T$, let $\alpha(K)$ denote the probability that $T \cap U_i = K$. For any $e \in T$,

$$\sum_{T \supseteq K \ni e} \alpha(K) = \mathbf{Pr}\left[e \in U_i\right] \geq \frac{1}{2} \ .$$

That is, α is a fractional half-cover of T. By fractional subadditivity,

$$\mathbf{E}\left[v_i(T \cap U_i)\right] = \sum_{K \subseteq T} \alpha(K) v_i(K) \geq \frac{1}{2} v_i(T) \ .$$

□

Recommended Reading

- Chapter 9, 11, and 12 in *"Algorithmic Game Theory," Nisan N., Roughgarden T., Tardos E., Vazirani V. (Eds.)*, 2007.
- D. Lehmann, L. O'Callaghan, and Y. Shoham. Truth revelation in approximately efficient combinatorial auctions. J. ACM, 49(5), 2002.
- S. Dobzinski, N. Nisan, and M. Schapira. Truthful randomized mechanisms for combinatorial auctions. J. Comput. Syst. Sci. 78(1): 15-25, 2012.
- S. Dobzinski. Two randomized mechanisms for combinatorial auctions. APPROX 2007.
- R. Lavi and C. Swamy. Truthful and near-optimal mechanism design via linear programming. J. ACM 58(6), 2011.

Recommended Reading

- U. Feige. On maximizing welfare when utility functions are subadditive. *SIAM J. Comput.*, 39(1), 2009. (see also the STOC 2006 version which contains some addtional content)
- P. Krysta and B. Vöcking, Online Mechanism Design (Randomized Rounding on the Fly). ICALP 2012.
- Y. Bartal, R. Gonen, and N. Nisan. Incentive compatible multi unit combinatorial auctions. TARK 2003.
- B. Awerbuch, Y. Azar, A. Meyerson. Reducing truth-telling online mechanisms to online optimization. STOC 2003.