EDP Exercises (Parts 1 and 2)

Problem 1. The goal of this problem is to obtain a tight analysis of the greedy algorithm for the undirected edge-disjoint paths problem.

(a) Let $G(V, E)$ be a simple unit capacity undirected graph and let $\mathcal{T}$ be the collection of all source-sink pairs such that the shortest distance in G between any source and its sink is at least ℓ. Then show that the value of the maximum (fractional) multicommodity flow for the pairs in $\mathcal{T}$ is bounded by $O(n^2/\ell^2)$.

(b) Use the preceding result to conclude that the greedy algorithm gives an $O(n^{2/3})$-approximation to EDP in undirected graphs.

(c) Show that there exists an infinite family of (acyclic) directed and undirected instances on which the approximation ratio achieved by the greedy algorithm is $\Omega(n^{2/3})$.

Problem 2. In this exercise is to show that the fractional well-linked decomposition can be used to recover an integral well-linked decomposition.

(a) Let G be a connected graph with a weight function π defined on its vertices such that $0 \leq \pi(v) \leq 1$ for all $v \in V$. Then we can find edge-disjoint trees $T_1, T_2, ..., T_q$ containing all the vertices such that for each T_i, we have $1 \leq \pi(T_i) \leq 3$, and the weight of each vertex contributes to exactly one tree. Assume that the total weight of all the vertices is greater than 3 at the start.

   (Hint: Start with a spanning tree of G and cluster.)

(b) Given an input instance G, X, M where X is π-cut well-linked, we can recover G, X', M' such that X' is $\frac{1}{2}$-cut well-linked, $|X'| = \Omega(\pi(X))$, and $M' \subseteq M$ is a matching defined over X'.

(c) Given an input instance G, X, M where X is π-flow well-linked, we can recover G, X', M' such that X' is $\frac{1}{2}$-flow well-linked, $|X'| = \Omega(\pi(X))$, and $M' \subseteq M$ is a matching defined over X'.

Problem 3. We say that a set A is routable to a set B if there are $|A|$ edge-disjoint paths from A to B in G such that each node in $A \cup B$ is an end-point of at most one path. Paths are allowed to be of 0 length (for nodes in $A \cap B$). Also note that if A can be routed to B then $|A| \leq |B|$.

Show that if X is a well-linked set in G then for any $A, B \subseteq X$ such that $|A| \leq |B|$, the set A is routable to B in G.

Problem 4. In this exercise we will analyze the behavior of the randomized rounding scheme when applied to the multicommodity flow relaxation for solving EDP.

(a) Show that given a feasible instance of undirected/directed EDP (all pairs can be routed with congestion 1), the randomized rounding technique can be used to recover an integral solution with congestion $O(\log n / \log \log n)$.

(b) Show that given any feasible instance of EDP, one can construct a $\Theta(1/\log n)$-integral solution such that congestion on any edge is $O(1)$.

(c) Suppose that you are given an instance of undirected/directed EDP with a feasible multicommodity flow solution (all pairs can be routed with congestion 1) where all flow paths have length at most L. Show that there exists an integral solution whose congestion is bounded by $O(\log L + \log \log n)$.

(Hint: Use LLL.)

Problem 5. In this exercise, we will show that the integrality gap of the multicommodity flow relaxation for EDP with no congestion is bounded by $O(\sqrt{n})$. Let F be the total flow routed by the multicommodity flow relaxation.

(a) Show that if $F/2$ units of flow is routed on paths of length greater than $\sqrt{n}$, there must exist a vertex v such that $\Omega(F/\sqrt{n})$ units of flow passes through v.

(b) Use the above fact to show that if $F/2$ units of flow is routed on paths of length greater than $\sqrt{n}$, you can recover an integral solution of value $\Omega(F/\sqrt{n})$.

(Hint: Use the clustering scheme from problem 2(a).)

(c) Conclude that you can always integrally route $\Omega(F/\sqrt{n})$ pairs.