ADFOCS 2015

Exercises on the s-t Path TSP Problem

August 17-21, 2015

1. Consider the following standard linear programming relaxation of the traveling salesman problem:

$$\text{Min} \quad \sum_{e \in E} c_e x_e$$

subject to:

$$\begin{aligned} x(\delta(v)) &= 2, && \forall v \in V, \\ x(\delta(S)) &\geq 2, && \forall S \subset V, S \neq \emptyset, \\ 0 \leq x_e &\leq 1, && \forall e \in E. \end{aligned}$$

Let x^* be an optimal solution to the LP, and let OPT_{LP} be the optimal value of the LP.

(a) Show that $\frac{n-1}{n} x^*$ is feasible for the spanning tree polytope.

(b) Show that Christofides' algorithm returns a solution of cost at most $\frac{3}{2} OPT_{LP}$.

2. Recall the LP relaxation for the s-t TSP path problem:

$$\text{Min} \quad \sum_{e \in E} c_e x_e$$

subject to:

$$x(\delta(v)) = \begin{cases} 1, & v = s, t, \\ 2, & v \neq s, t, \end{cases}$$

$$x(\delta(S)) \geq \begin{cases} 1, & |S \cap \{s, t\}| = 1, \\ 2, & |S \cap \{s, t\}| \neq 1, \end{cases}$$

$$0 \leq x_e \leq 1, \qquad \forall e \in E.$$

Let x^* be an optimal solution to the LP, and let OPT_{LP} be the optimal value of the LP.

In this problem, we will show that the Best-of-Many Christofides' algorithm is a $\frac{3}{2}$-approximation algorithm if $x_e^* \in \{0, \frac{1}{2}, 1\}$ for all $e \in E$. Let $F_1, \ldots, F_k$ be the spanning trees in the convex combination given by x^*, and let $T_1, \ldots, T_k$ be the sets of vertices whose parity needs fixing in the trees $F_1, \ldots, F_k$ (respectively). A set S is odd for a tree F_i if $|S \cap T_i|$ is odd.

(a) Prove that for any tree F_i, $x^*(\delta(S))$ is integral for any odd set S.

(b) Prove that for any tree F_i, $x^*(\delta(S)) \geq 2$ for any odd set S.

(c) Prove that Best-of-Many Christofides' algorithm returns a solution of cost at most $\frac{3}{2}OPT_{LP}$.

3. Consider a solution x to the LP relaxation in Problem 1 that is a *fractional 2-matching*: $x_e \in \{0, 1/2, 1\}$ for all $e \in E$, and all the edges with $x_e = 1/2$ form vertex-disjoint cycles with odd numbers of edges. A *2-matching* is a set of edges such that each vertex has degree two, and each connected component has at least 3 vertices in it (and is thus a collection of cycles, each of which has at least 3 vertices in it). A *graphical 2-matching* is a multiset of edges such that each vertex has even degree, and each connected component has at least 3 vertices in it. In this exercise, we will show that we can find a 2-matching of cost at most 4/3 times the value of the LP solution.

   To do this, consider a graph G' obtained from the LP solution by any path of edges e with $x_e = 1$ and replacing the path with a single edge, and including any edge e with $x_e = 1/2$. The resulting graph is cubic (each vertex has degree exactly 3) and is 2-edge-connected (argue to yourself that this must be true). Let the cost c'_e of any edge e in the graph G' be the cost of the path (if e corresponds to a path in the original graph) or the negative of the cost of the edge e (if the edge e had $x_e = 1/2$) in the original graph. Compute a minimum-cost perfect matching M. We now construct a set of edges F. If $e \in M$ corresponded to a path in the original graph, then add two copies of each edge in the path to F. If $e \notin M$ and e corresponded to a path in the original graph, then include one copy of each edge in the path in F. If for edge e we had $x_e = 1/2$ and $e \notin M$, then add this edge to F (if $e \in M$ then we do not add this edge to F).

   (a) Argue that the resulting set of edges F must be a graphical 2-matching.

   (b) Prove that the cost of the edges in F must be at most $\frac{4}{3}\sum_{e\in E} c_e x_e$.

   (c) Prove that there is a 2-matching of cost at most the cost of the edges in F.

4. In the *prize-collecting* traveling salesman problem, we are given a complete graph $G = (V, E)$, costs $c_e \geq 0$ on the edges $e \in E$ that obey the triangle inequality, a root vertex $r \in V$, and penalties $\pi_i \geq 0$ for all $i \in V$. The goal is to find a set of vertices S with $r \in S$, and a tour T on S that minimizes $\sum_{e\in T} c_e + \sum_{i\notin S} \pi_i$; that is, we minimize the total cost of the tour on S plus the penalties of the vertices not in S.

   In the *prize-collecting Steiner tree problem*, we are given a graph $G = (V, E)$, costs $c_e \geq 0$ on the edges $e \in E$, a root vertex $r \in V$, and penalties $\pi_i \geq 0$ for all $i \in V$. The goal is to find a set of vertices S with $r \in S$, and a tree T spanning S that minimizes $\sum_{e\in T} c_e + \sum_{i\notin S} \pi_i$; that is, we minimize the total cost of the tree on S plus the penalties of the vertices not in S.

Consider the following linear program.

$$\text{Min} \quad \sum_{e \in E} c_e x_e + \sum_{i \in V} \pi_i (1 - y_i)$$

subject to

$$\begin{aligned}
\sum_{e \in \delta(S)} x_e &\geq y_i, & \forall S \subseteq V - r, S \neq \emptyset, \forall i \in S, \\
y_r &= 1, & \\
y_i &\geq 0, & \forall i \in V, \\
x_e &\geq 0, & \forall e \in E.
\end{aligned}$$

We assert that the LP can be solved in polynomial time (using an algorithm known as the ellipsoid method). Let (x^*, y^*) be an optimal solution. Consider the following algorithm for the prize-collecting Steiner tree problem. We let $S = \{i \in V : y_i^* \geq \alpha\}$ for some α. Find a minimum-cost spanning tree T on S. It is possible to show that $\sum_{e \in T} c_e \leq \frac{2}{\alpha} \sum_{e \in E} c_e x_e^*$.

(a) Argue that the linear program is a relaxation of the prize-collecting Steiner tree problem.

(b) If you are familiar with the ellipsoid method, argue that the linear program can be solved in polynomial time.

(c) Show that there is a value for α such that the algorithm is a 3-approximation algorithm for the prize-collecting Steiner tree problem.

(d) Use the 3-approximation algorithm for the prize-collecting Steiner tree problem to devise an approximation algorithm for the prize-collecting traveling salesman problem. How good a performance guarantee can you get?