s-t path TSP

David P. Williamson
Cornell University

August 17-21, 2015
ADFOCS
**Traveling Salesman Problem (TSP)**

**Input:**
- A complete, undirected graph $G = (V, E)$;
- Edge costs $c(i, j) \geq 0$ for all $e = (i, j) \in E$.

**Goal:** Find the min-cost tour that visits each city exactly once.

Costs are *symmetric* ($c(i, j) = c(j, i)$) and obey the *triangle inequality* ($c(i, k) \leq c(i, j) + c(j, k)$).

*Asymmetric* TSP (ATSP) input has complete directed graph, and $c(i, j)$ may not equal $c(j, i)$.
The traveling salesman problem

From Bill Cook, tour of 647 US colleges (www.math.uwaterloo.ca/tsp/college)
The traveling salesman problem

From Bill Cook, tour of 647 US colleges
(www.math.uwaterloo.ca/tsp/college)
Definition

An $\alpha$-approximation algorithm is a polynomial-time algorithm that returns a solution of cost at most $\alpha$ times the cost of an optimal solution.

Long known: A $\frac{3}{2}$-approximation algorithm due to Christofides (1976). No better approximation algorithm yet known.
Christofides’ algorithm

Compute minimum spanning tree (MST) $F$ on $G$, then compute a minimum-cost perfect matching $M$ on odd-degree vertices of $T$. “Shortcut” Eulerian traversal in resulting Eulerian graph of $F \cup M$. 
Christofides’ algorithm

Compute minimum spanning tree (MST) $F$ on $G$, then compute a minimum-cost perfect matching $M$ on odd-degree vertices of $T$. “Shortcut” Eulerian traversal in resulting Eulerian graph of $F \cup M$. 
Christofides’ algorithm

Compute minimum spanning tree (MST) $F$ on $G$, then compute a minimum-cost perfect matching $M$ on odd-degree vertices of $T$. “Shortcut” Eulerian traversal in resulting Eulerian graph of $F \cup M$. 
Christofides’ algorithm

Compute minimum spanning tree (MST) $F$ on $G$, then compute a minimum-cost perfect matching $M$ on odd-degree vertices of $T$. “Shortcut” Eulerian traversal in resulting Eulerian graph of $F \cup M$. 
The s-t path TSP:
Usual TSP input plus $s, t \in V$, find a min-cost path from $s$ to $t$ visiting all other nodes in between (an s-t Hamiltonian path).

Hoogeveen (1991) shows that the natural variant of Christofides’ algorithm gives a $\frac{5}{3}$-approximation algorithm.
The $s$-$t$ path TSP:
Usual TSP input plus $s, t \in V$, find a min-cost path from $s$ to $t$
visiting all other nodes in between (an $s$-$t$ Hamiltonian path).

Hoogeveen (1991) shows that the natural variant of Christofides’
algorithm gives a $\frac{5}{3}$-approximation algorithm.

What is the natural variant for the $s$-$t$ path TSP?
Eulerian path

There is an Eulerian path that starts at $s$, ends at $t$, and visits every edge exactly once iff $s$ and $t$ have odd-degree and all other vertices have even degree.
Eulerian path

There is an Eulerian path that starts at $s$, ends at $t$, and visits every edge exactly once iff $s$ and $t$ have odd-degree and all other vertices have even degree.
Hoogeveen’s algorithm

Let $F$ be the min-cost spanning tree. Let $T$ be the set of vertices whose *parity needs changing*: $s$ iff $s$ has even degree in $F$, $t$ iff $t$ has even degree in $F$, and $v \neq s, t$ iff $v$ has odd degree. Then find a minimum-cost perfect matching $M$ on the vertices in $T$. Find Eulerian path on $F \cup M$; shortcut to an $s$-$t$ Hamiltonian path.
Hoogeveen’s algorithm

Let $F$ be the min-cost spanning tree. Let $T$ be the set of vertices whose *parity needs changing*: $s$ iff $s$ has even degree in $F$, $t$ iff $t$ has even degree in $F$, and $v \neq s, t$ iff $v$ has odd degree. Then find a minimum-cost perfect matching $M$ on the vertices in $T$. Find Eulerian path on $F \cup M$; shortcut to an $s$-$t$ Hamiltonian path.
Hoogeveen’s algorithm

Let $F$ be the min-cost spanning tree. Let $T$ be the set of vertices whose *parity needs changing*: $s$ iff $s$ has even degree in $F$, $t$ iff $t$ has even degree in $F$, and $v \neq s, t$ iff $v$ has odd degree. Then find a minimum-cost perfect matching $M$ on the vertices in $T$. Find Eulerian path on $F \cup M$; shortcut to an $s$-$t$ Hamiltonian path.
Hoogeveen’s algorithm

Let $F$ be the min-cost spanning tree. Let $T$ be the set of vertices whose *parity needs changing*: $s$ iff $s$ has even degree in $F$, $t$ iff $t$ has even degree in $F$, and $v \neq s, t$ iff $v$ has odd degree. Then find a minimum-cost perfect matching $M$ on the vertices in $T$. Find Eulerian path on $F \cup M$; shortcut to an $s$-$t$ Hamiltonian path.
Rather than a minimum-cost perfect matching on $T$, will construct a minimum-cost \textit{T-join}: a set of edges that has odd degree at every vertex in $T$, even degree at every vertex not in $T$. 

![Diagram of T-join](attachment:image.png)
Rather than a minimum-cost perfect matching on $T$, we will construct a minimum-cost $T$-join: a set of edges that has odd degree at every vertex in $T$, even degree at every vertex not in $T$. 
Rather than a minimum-cost perfect matching on $T$, will construct a minimum-cost $T$-join: a set of edges that has odd degree at every vertex in $T$, even degree at every vertex not in $T$. 
Rather than a minimum-cost perfect matching on $T$, will construct a minimum-cost $T$-join: a set of edges that has odd degree at every vertex in $T$, even degree at every vertex not in $T$. 
Let $F$ be the min-cost spanning tree. Let $T$ be the set of vertices whose parity needs changing. Then find a minimum-cost $T$-join $J$. Find Eulerian path on $F \cup J$; shortcut to an $s$-$t$ Hamiltonian path.

Theorem Hoogeveen’s algorithm is a $5/3$-approximation algorithm.
Hoogeveen’s algorithm

Let \( F \) be the min-cost spanning tree. Let \( T \) be the set of vertices whose parity needs changing. Then find a minimum-cost \( T \)-join \( J \). Find Eulerian path on \( F \cup J \); shortcut to an \( s-t \) Hamiltonian path.

Theorem

Hoogeveen’s algorithm is a \( \frac{5}{3} \)-approximation algorithm.
Proof of theorem

Let $F$ be edges in MST, $c(F) = \sum_{e \in F} c_e$.

Let $O$ be edges in optimal soln, $OPT = c(O)$.

Clearly $c(F) \leq OPT$ since $O$ is a spanning tree.

Let $T$ be vertices in $F$ whose parity needs changing.

Idea: Construct 3 $T$-joins of total cost $c(F) + OPT$.

Then $\text{MST} + \text{min-cost T-join} \leq c(F) + \frac{1}{3} (c(F) + OPT) 
\leq OPT + \frac{2}{3} OPT = \frac{5}{3} OPT$.

Let $R$ be edges on $s$-$t$ path in MST $F$.

Color edges of $O$ green or blue: start at $s$, color blue until first node in $T$, then switch colors as each node in $T$ reached. Gives $G$ (green), $B$ (blue).
F-R a T-join: \( F \cup (F - R) \) has even degree at every node except s, t

G a T-join: pairs up nodes in T.

B is not a T-join: \( F \cup B \) has even degree at all nodes but then \( B \cup R \) is a T-join.

\[ c(F - R) + c(G) + c(B \cup R) = c(F) + c(O). \]
Proof of theorem
Proof of theorem
Proof of theorem
Proof of theorem
Proof of theorem
Proof of theorem
Proof of theorem
Tight Example

The analysis is tight. Consider the graph TSP instance below: cost $c_e$ for $e = (i, j)$ is number of edges in shortest $i$-$j$ path in graph.
The analysis is tight. Consider the graph TSP instance below: cost $c_e$ for $e = (i, j)$ is number of edges in shortest $i$-$j$ path in graph.
Tight Example

The analysis is tight. Consider the graph TSP instance below: cost \( c_e \) for \( e = (i, j) \) is number of edges in shortest \( i-j \) path in graph.
The analysis is tight. Consider the graph TSP instance below: cost $c_e$ for $e = (i, j)$ is number of edges in shortest $i$-$j$ path in graph.
Improvements

No improvement on Hoogeveen’s algorithm for s-t path TSP, until just the last few years.

Hoogeveen (1991) $\frac{5}{3}$
Improvements

No improvement on Hoogeveen’s algorithm for s-t path TSP, until just the last few years.

Hoogeveen (1991) $\frac{5}{3}$

An, Kleinberg, Shmoys (2012) $\frac{1+\sqrt{5}}{2} \approx 1.618$
No improvement on Hoogeveen’s algorithm for \( s-t \) path TSP, until just the last few years.

- Hoogeveen (1991) \( \frac{5}{3} \)

- An, Kleinberg, Shmoys (2012) \( \frac{1 + \sqrt{5}}{2} \approx 1.618 \)

- Sebő (2013) \( \frac{8}{5} = 1.6 \)
**Improvements**

No improvement on Hoogeveen’s algorithm for \(s-t\) path TSP, until just the last few years.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Year</th>
<th>Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hoogeveen</td>
<td>1991</td>
<td>(\frac{5}{3})</td>
</tr>
<tr>
<td>An, Kleinberg, Shmoys</td>
<td>2012</td>
<td>(\frac{1+\sqrt{5}}{2} \approx 1.618)</td>
</tr>
<tr>
<td>Sebő</td>
<td>2013</td>
<td>(\frac{8}{5} = 1.6)</td>
</tr>
<tr>
<td>Vygen</td>
<td>2015</td>
<td>1.599</td>
</tr>
</tbody>
</table>

**Goal:** Understand the An et al. and Sebő algorithm and analysis.
A Linear Programming Relaxation

Min $\sum_{e \in E} c_e x_e$

subject to:

$x(\delta(v)) = \begin{cases} 1 & v = s, t \\ 2 & v \neq s, t \end{cases}$

$x(\delta(S)) \geq \begin{cases} 1 & |s \cap S \cap t| = 1 \\ 2 & |s \cap S \cap t| \neq 1 \end{cases}$

$0 \leq x_e \leq 1, \quad \forall e \in E,$

where $\delta(S)$ is the set of edges with exactly one endpoint in $S$, and $x(E') \equiv \sum_{e \in E'} x_e$. 
A Linear Programming Relaxation

Minimize \( \sum_{e \in E} c_e x_e \)

subject to:
\[
x(\delta(v)) = \begin{cases} 
1, & v = s, t, \\
2, & v \neq s, t,
\end{cases}
\]

\[
x(\delta(S)) \geq \begin{cases} 
1, & |S \cap \{s, t\}| = 1, \\
2, & |S \cap \{s, t\}| \neq 1,
\end{cases}
\]

\[0 \leq x_e \leq 1, \quad \forall e \in E,
\]

where \( \delta(S) \) is the set of edges with exactly one endpoint in \( S \), and
\[x(E') \equiv \sum_{e \in E'} x_e.\]
LP relaxation
LP relaxation
LP relaxation
LP relaxation
LP relaxation
The spanning tree polytope

The spanning tree polytope (convex hull of all spanning trees) is defined by the following inequalities:

\[
\sum_{e \in E} x_e = |V| - 1,
\]

\[
x(E(S)) \leq |S| - 1, \quad \forall |S| \subseteq V, |S| \geq 2,
\]

\[
\sum_{e \in E(S)} x_e \geq 0, \quad \forall e \in E,
\]

where \(E(S)\) is the set of all edges with both endpoints in \(S\).
The LP relaxation and spanning trees

Lemma

Any solution \( x \) feasible for the \( s-t \) path TSP LP relaxation is in the spanning tree polytope.
\[ x(E) = \sum_{e \in E} x_e = \frac{1}{2} \sum_{v \in V} x(\delta(v)) = \frac{1}{2} (|V| - 2) \cdot 2 + 2 = |V| - 1 \]

\[ x(E(S)) = \frac{1}{2} \left( \sum_{v \in S} x(\delta(v)) - x(\delta(S)) \right) \]

- If \(|S \cap E_s, t_3| = 1\)
  \[ x(E(S)) \leq \frac{1}{2} \left( 1 + 2(|S| - 1) - 1 \right) = |S| - 1. \]
- If \(|S \cap E_s, t_3 = \emptyset|\)
- If \(|S \cap E_s, t_3 = \{s, t_3|\} \)
Proof

\[ x(δ(v)) = \begin{cases} 
1, & v = s, t, \\
2, & v \neq s, t,
\end{cases} \]

\[ x(δ(S)) \geq \begin{cases} 
1, & |S \cap \{s, t\}| = 1, \\
2, & |S \cap \{s, t\}| \neq 1,
\end{cases} \]

\[ 0 \leq x_e \leq 1, \quad \forall e \in E. \]

\[ x(E) = |V| - 1, \]

\[ x(E(S)) \leq |S| - 1, \quad \forall |S| \subseteq V, |S| \geq 2, \]

\[ x_e \geq 0, \quad \forall e \in E. \]
Let $OPT_{LP}$ be the value of an optimal solution $x^*$ to the LP relaxation.

**Theorem (An, Kleinberg, Shmoys (2012))**

*Hoogeveen’s algorithm returns a solution of cost at most $\frac{5}{3} OPT_{LP}$.***
An extremely useful lemma

Let $F$ be a spanning tree, and let $T$ be the vertices whose parity needs fixing in $F$.

**Definition**

$S$ is an **odd set** if $|S \cap T|$ is odd.

**Lemma**

Let $S$ be an odd set. If $|S \cap \{s, t\}| = 1$, then $|F \cap \delta(S)|$ is even. If $|S \cap \{s, t\}| \neq 1$, then $|F \cap \delta(S)|$ is odd.
Lemma

Let $S$ be an odd set. If $|S \cap \{s, t\}| = 1$, then $|F \cap \delta(S)|$ is even. If $|S \cap \{s, t\}| \neq 1$, then $|F \cap \delta(S)|$ is odd.
Lemma

Let $S$ be an odd set. If $|S \cap \{s, t\}| = 1$, then $|F \cap \delta(S)|$ is even. If $|S \cap \{s, t\}| \neq 1$, then $|F \cap \delta(S)|$ is odd.
 Lemma

Let $S$ be an odd set. If $|S \cap \{s, t\}| = 1$, then $|F \cap \delta(S)|$ is even. If $|S \cap \{s, t\}| \neq 1$, then $|F \cap \delta(S)|$ is odd.
Lemma

Let $S$ be an odd set. If $|S \cap \{s, t\}| = 1$, then $|F \cap \delta(S)|$ is even. If $|S \cap \{s, t\}| \neq 1$, then $|F \cap \delta(S)|$ is odd.
Lemma

Let $S$ be an odd set. If $|S \cap \{s, t\}| = 1$, then $|F \cap \delta(S)|$ is even. If $|S \cap \{s, t\}| \neq 1$, then $|F \cap \delta(S)|$ is odd.
Proof of lemma

\[
\sum_{v \in S} \deg_F(v) = 2|E(S) \cap F| + |\delta(S) \cap F|
\]
Proof of Lemma

If $|s_n|^2 = 1$, spec $S \subseteq S$. set $T$ iff $\deg(S)$ even.

$S$ odd $\Rightarrow$ even $\#$ of odd deg. vertices in $S$.

$|s_n|^2$ odd

$\sum \deg(v) - 2|E(S) \cap F| = |\partial(S) \cap F|

\text{even even}

\text{In fact } |\partial(S) \cap F| \geq 2

$|s_n|^2, t_3| = 1$ & $S$ odd $\Rightarrow$ odd $\#$ odd deg. verts in $S$

$\sum \deg(v) - 2|E(S) \cap F| = |\partial(S) \cap F|

\text{odd even odd}$
The solution to the following linear program is the minimum-cost $T$-join for costs $c \geq 0$:

$$\text{Min} \quad \sum_{e \in E} c_e x_e$$

subject to:

$$x(\delta(S)) \geq 1, \quad \forall S \subseteq V, |S \cap T| \text{ odd}$$

$$x_e \geq 0, \quad \forall e \in E.$$
The solution to the following linear program is the minimum-cost $T$-join for costs $c \geq 0$:

\[
\begin{align*}
\text{Min} & \quad \sum_{e \in E} c_e x_e \\
\text{subject to:} & \quad x(\delta(S)) \geq 1, \quad \forall S \subseteq V, |S \cap T| \text{ odd} \\
& \quad x_e \geq 0, \quad \forall e \in E.
\end{align*}
\]

\[
\sum_{v \in S} \deg_J(v) = 2|E(S) \cap J| + |\delta(S) \cap J|
\]
Proof of theorem

Theorem (An, Kleinberg, Shmoys (2012))

Hoogeveen’s algorithm returns a solution of cost at most $\frac{5}{3} \text{OPT}_L$.

Lemma

Let $S$ be an odd set. If $|S \cap \{s, t\}| = 1$, then $|F \cap \delta(S)|$ is even. If $|S \cap \{s, t\}| \neq 1$, then $|F \cap \delta(S)|$ is odd.

Min $\sum_{e \in E} c_e x_e$

$x(\delta(S)) \geq 1, \quad \forall S \subseteq V, |S \cap T| \text{ odd}$

$x_e \geq 0, \quad \forall e \in E.$
Proof: Let $x^*$ be an opt. soln to LP relaxation. 

Cost of MST $\leq \sum_{ee} c_e x_e^* = \text{OPT}_{LP}$. 

since $x^*$ is feasible for spanning tree polytope. 

Let $X_F \in \{0,1\}^{E_1}$ s.t. $X_F(e) = \begin{cases} 1 & \text{if } ee \in F \\ 0 & \text{o.w.} \end{cases}$ 

Claim: $y = \frac{1}{3} X_F + \frac{1}{3} x^*$ feasible for T-join LP. 

Then $c(F \cup J) = c(F) + c(J) \leq \text{OPT}_{LP} + \frac{1}{3} c(F) + \frac{1}{3} \text{OPT}_{LP}$ \[ \leq \frac{5}{3} \text{OPT}_{LP} \]
\[ y = \frac{1}{3} X_{F} + \frac{1}{3} x^{*} \text{ feas. for T-join LP.} \]

Need to show that if \(|S_{N}| \text{ odd, then } y(\delta(S)) \geq 1.\]

If \(|S_{N} \cap S, t \cap 1| \neq 1\), then
\[ y(\delta(S)) = \frac{1}{3} |F \cap \delta(S)| + \frac{1}{3} x^{*}(\delta(S)) \geq \frac{1}{3} + \frac{2}{3} = 1 \]

If \(|S_{N} \cap S, t \cap 1| = 1\), then
\[ y(\delta(S)) = \frac{1}{3} |F \cap \delta(S)| + \frac{1}{3} x^{*}(\delta(S)) \geq \frac{2}{3} + \frac{1}{3} = 1 \]

\[ \square \]
Convex combination

Let $x^*$ be an optimal LP solution. Let $\chi_F$ be the characteristic vector of a set of edges $F$, so that

$$\chi_F(e) = \begin{cases} 1 & e \in F \\ 0 & e \notin F \end{cases}$$

Since $x^*$ is in the spanning tree polytope, can write $x^*$ as a convex combination of spanning trees $F_1, \ldots, F_k$: 

$$x^* = \sum_{i=1}^{k} \lambda_i \chi_{F_i},$$

such that $\sum_{i=1}^{k} \lambda_i = 1$, $\lambda_i \geq 0$. 
An, Kleinberg, Shmoys (2012) propose the Best-of-Many Christofides’ algorithm: given optimal LP solution $x^*$, compute convex combination of spanning trees

$$x^* = \sum_{i=1}^{k} \lambda_i x_{F_i}.$$ 

For each spanning tree $F_i$, let $T_i$ be the set of vertices whose parity needs fixing, let $J_i$ be the minimum-cost $T_i$-join. Find $s$-$t$ Hamiltonian path by shortcutting $F_i \cup J_i$. Return the shortest path found over all $i$. 
Best-of-Many Christofides’ Algorithm

\[ x^* = \sum_{i=1}^{k} \lambda_i \chi_{F_i}. \]

For each spanning tree \( F_i \), let \( T_i \) be the set of vertices whose parity needs fixing, \( J_i \) be the minimum-cost \( T_i \)-join. Find \( s-t \) Hamiltonian path by shortcutting \( F_i \cup J_i \). Return the shortest path found over all \( i \).

**Theorem**

The Best-of-Many Christofides’ algorithm returns a solution of cost at most \( \frac{5}{3} \text{OPT}_{LP} \).