ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Exercises on Symmetric TSP

These problems are taken from various sources. Problems marked * are more difficult but also more fun :).

Also, please do the exercises in any order. There are a lot of them so you will probably not have the time to do them all during the exercise session.

1 Show that any graph has an even number of odd-degree vertices.

2 For a 2-edge connected cubic graph, show that $x_e = 1/3$ for all $e \in E$ is a feasible solution to Edmond's perfect matching polytope.

Recall that Edmond's perfect matching polytope is given by

$$\sum_{e \in \delta(v)} x_e = 1 \qquad \text{for all } v \in V,$$
$$\sum_{e \in \delta(S)} x_e \geq 1 \qquad \text{for all } S \subseteq V : |S| \text{ odd},$$
$$x \geq 0,$$

where $\delta(S)$ denotes the set of edges with exactly one endpoint in S, i.e., those edges that crosses the cut $(S, \bar{S})$.

3 Can you find an n-vertex instance for which there is a tour of length n but Christofides' algorithm returns a solution of length close to $3n/2$?

This shows that the analysis of Christofides' algorithm is tight.

4 Consider a cubic 2-edge-connected graph $G = (V, E)$ with nonnegative edge weights $w : E \to \mathbb{R}_+$. Show that there is a tour of weight at most $\frac{4}{3}\sum_{e \in E} w(e)$.

5 (*) It is well-known that the vertices of a 2-edge connected graph can be ordered $v_1, v_2, \ldots, v_n$ such that for $i = 2, \ldots, n-1$, v_i has a neighbor v_k to the left (i.e., $k < i$) and a neighbor v_ℓ to the right (i.e., $\ell > i$). This is called an st-numbering of the graph. Can you use the st-numbering to give an alternative definition of the set R of edges we wish to remove to prove that any cubic 2-edge connected graph has a connected Eulerian graph with at most $4n/3 - 2/3$ edges?

6 (**) Generalize the proof to show that any 2-edge connected subcubic graph has a connected Eulerian graph with at most $4n/3 - 2/3$ edges.

7 (**) Consider a cubic 2-edge-connected graph $G = (V, E)$ with nonnegative edge weights $w : E \to \mathbb{R}_+$. Suppose that an optimal solution x^* to the Held-Karp relaxation has the following form:

- x^* is half-integral, i.e., the edges can be partitioned into two sets $E_{1/2} = \{e \in E : x^*_e = 1/2\}$ and $E_1 = \{e \in E : x^*_e = 1\}$.
- The edges $E_{1/2}$ form disjoint triangles.

Show that in this case, there exists a tour of value at most $\frac{4}{3}\sum_{e \in E} x^*_e w(e)$.