ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Exercises on Asymmetric TSP

These problems are taken from various sources. Problems marked * are more difficult but also more fun :).

Also, please do the exercises in any order although it might be a good idea to do 3 before 4.

1 We say that an ATSP instance is β-balanced if $c_{ij}/\beta \leq c_{ji} \leq \beta c_{ij}$ for each pair i, j of vertices/cities.

Design an $O(\beta)$-approximation algorithm for β-balanced instances.

2 (*) Consider the following algorithm for ATSP:

- Find a directed cycle C in G minimizing $\frac{c(E(C))}{|C|}$ and add the edges of C, i.e., $E(C)$, to the solution.
- Remove all but one of the vertices of C from G and proceed recursively until G is reduced to a single vertex.

Prove that this algorithm is a $O(\ln n)$-approximation algorithm for ATSP.

3 Give a polynomial time algorithm for finding a min cost cycle cover.
Hint: use a bipartite graph in some way.

4 (*) Show that there exists a cycle cover of no higher cost than the optimal value to the LP-relaxation of ATSP.

Hint: use that $\{x \in \mathbb{R}^E_{\geq 0} : x(\delta(v)) = 1 \text{ for } v \in V\}$ defines the perfect matching polytope for bipartite graphs. Can you prove that?

5 Prove the following:

"Given a solution x to the LP relaxation of ATSP, z defined by

$$z_{\{i,j\}} = \frac{n-1}{n}(x_{ij} + x_{ji})$$

is a feasible solution to the MST polytope."

6 Suppose that we relax the Held-Karp relaxation by instead of demanding that the in- and out-degree of each vertex is *exactly* one, we require that the in- and out-degree of each vertex is *at least* one. We thus obtain the following relaxation:

$$\begin{aligned} \text{minimize} \quad & \sum_{e \in E} x_e w(e) \\ \sum_{e \in \delta^+(v)} x_e \geq 1 \quad & \text{for all } v \in V, \\ \sum_{e \in \delta^-(v)} x_e \geq 1 \quad & \text{for all } v \in V, \\ \sum_{e \in \delta^+(S)} x_e \geq 1 \quad & \text{for all } \emptyset \subset S \subset V, \\ x \geq 0. & \end{aligned}$$

Show that this relaxation has an unbounded integrality gap.

7 Give an edge-weighted undirected graph $G = (V, E, w)$ satisfying: the number of cuts of value at most αc is roughly $\binom{n}{2\alpha}$, where c is the value of a min-cut.

(This shows that Karger's result is basically tight.)