Complexity of Matrix Multiplication and Bilinear Problems

[Lecture 1 – Exercises]

Exercise 1
Let q be a positive integer. Consider the following three tensors:

$$\begin{aligned} T_1 &= \sum_{i=1}^{q} x \otimes y_i \otimes z_i, \\ T_2 &= \sum_{i=1}^{q} x_i \otimes y \otimes z_i, \\ T_3 &= \sum_{i=1}^{q} x_i \otimes y_i \otimes z. \end{aligned}$$

Each of these three tensors is isomorphic to the tensor of some matrix multiplication. For each $i \in \{1, 2, 3\}$, identity the value of m_i, n_i and p_i such that $T_i \cong \langle m_i, n_i, p_i \rangle$.

Exercise 2
For any positive integer r, let $\langle r \rangle$ denote the tensor

$$\langle r \rangle = \sum_{i=1}^{r} x_i \otimes y_i \otimes z_i.$$

What does this tensor represent?

Exercise 3
Let $t \in \mathbb{F}^u \otimes \mathbb{F}^v \otimes \mathbb{F}^w$ and $t' \in \mathbb{F}^{u'} \otimes \mathbb{F}^{v'} \otimes \mathbb{F}^{w'}$ be two tensors. We say that t' is a restriction of t, and write $t' \leq t$, if there exist three linear maps

$$\begin{aligned} \alpha &\colon \mathbb{F}^u \to \mathbb{F}^{u'} \\ \beta &\colon \mathbb{F}^v \to \mathbb{F}^{v'} \\ \gamma &\colon \mathbb{F}^w \to \mathbb{F}^{w'} \end{aligned}$$

such that $(\alpha \otimes \beta \otimes \gamma)t = t'$.

(i) Check that for any tensor t the rank of t is the smallest integer r such that $t \leq \langle r \rangle$.

(ii) Check that $R(t') \leq R(t)$ holds for any two tensors t, t' such that $t' \leq t$.

Exercise 4
Let $t \in \mathbb{F}^u \otimes \mathbb{F}^v \otimes \mathbb{F}^w$ and $t' \in \mathbb{F}^{u'} \otimes \mathbb{F}^{v'} \otimes \mathbb{F}^{w'}$ be two tensors. We say that t' is a degeneration of t, and write $t' \trianglelefteq t$, if there exist three linear maps

$$\begin{aligned} \alpha &\colon \mathbb{F}[\lambda]^u \to \mathbb{F}[\lambda]^{u'} \\ \beta &\colon \mathbb{F}[\lambda]^v \to \mathbb{F}[\lambda]^{v'} \\ \gamma &\colon \mathbb{F}[\lambda]^w \to \mathbb{F}[\lambda]^{w'} \end{aligned}$$

and a nonnegative integer c such that such that $(\alpha \otimes \beta \otimes \gamma)t = \lambda^c t' + \lambda^{c+1} t''$ for some tensor $t'' \in \mathbb{F}[\lambda]^{u'} \otimes \mathbb{F}[\lambda]^{v'} \otimes \mathbb{F}[\lambda]^{w'}$.

(i) Show that $t' \leq t$ implies $t' \trianglelefteq t$.

(ii) Check that for any tensor t the border rank of t is the smallest integer r such that $t \trianglelefteq \langle r \rangle$.

(iii) Check that $\underline{R}(t') \leq \underline{R}(t)$ holds for any two tensors t, t' such that $t' \trianglelefteq t$.

Exercise 5
Consider the computation of the product of two matrices A and B of the following form:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & 0 \\ b_{31} & 0 \end{pmatrix}.$$

(i) Write the tensor corresponding to this computational task.

(ii) Show that the border rank of this tensor is at most 5.
Hint: you can start by expanding the expression

$$\begin{aligned}
&(a_{11} + \lambda^2 a_{13}) \otimes b_{31} \otimes (c_{11} - \lambda c_{12}) \\
&+ (a_{11} + \lambda^2 a_{22}) \otimes (b_{21} - \lambda b_{12}) \otimes c_{21} \\
&+ (a_{11} + \lambda^2 a_{23}) \otimes (b_{31} + \lambda b_{12}) \otimes (c_{21} + \lambda c_{12}) \\
&- a_{11} \otimes (b_{21} + b_{31}) \otimes (c_{11} + c_{21})
\end{aligned}$$

and see what you obtain.

Exercise 6
Let $n = 2\ell + 1$ be an odd integer.

(i) Verify that the size of the set

$$\big\{(i,j,k) \in \{-\ell, ..., \ell\} \times \{-\ell, ..., \ell\} \times \{-\ell, ..., \ell\} \mid i + j + k = 0\big\}$$

is at least $\frac{3n^2}{4}$.

(ii) Show that

$$\left\langle \left\lceil \frac{3n^2}{4} \right\rceil \right\rangle \trianglelefteq \langle n, n, n \rangle.$$

Remark: The same results hold for even n as well.