Complexity of Matrix Multiplication and Bilinear Problems

François Le Gall

Graduate School of Informatics
Kyoto University

ADFOCS17 - Lecture 1
22 August 2017

Algebraic Complexity Theory

- ✓ Algebraic complexity theory: study of computation using algebraic models

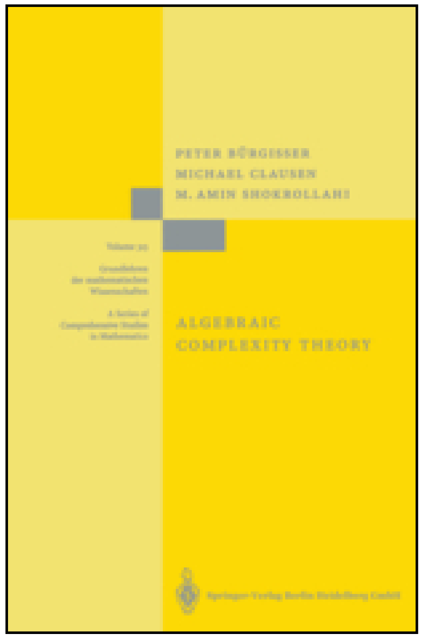
- ✓ Main Achievements:
  - ➢ lower bounds on the complexity (in algebraic models of computation) of concrete problems
  - ➢ powerful techniques to construct fast algorithms for computational problems with an algebraic structure

- ✓ Several subareas:
  - ➢ high degree algebraic complexity: study of high-degree polynomials
  - ➢ low degree algebraic complexity: linear forms, bilinear forms,…

in particular matrix multiplication

the main concepts in low degree algebraic complexity theory have been introduced for the study of the complexity of matrix multiplication

Some General References



Algebraic Complexity Theory

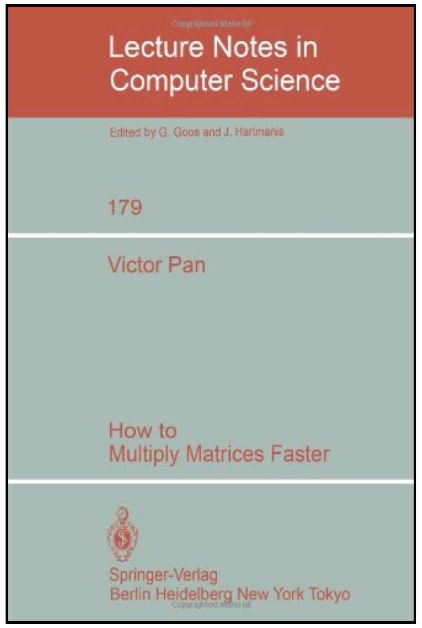
Bürgisser, Clausen and Shokrollahi

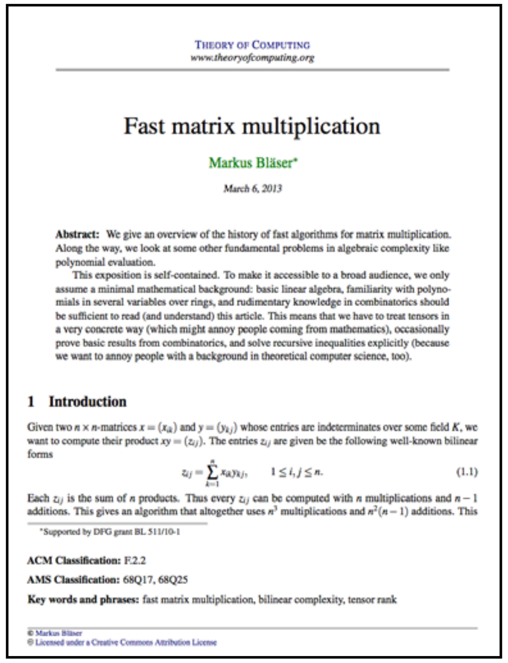
(Springer, 1997)

How to Multiply Matrices Faster

Pan

(Springer, 1984)





THEORY OF COMPUTING
www.theoryofcomputing.org

Fast matrix multiplication

Markus Bläser*

March 6, 2013

Abstract: We give an overview of the history of fast algorithms for matrix multiplication. Along the way, we look at some other fundamental problems in algebraic complexity like polynomial evaluation.

This exposition is self-contained. To make it accessible to a broad audience, we only assume a minimal mathematical background: basic linear algebra, familiarity with polynomials in several variables over rings, and rudimentary knowledge in combinatorics should be sufficient to read (and understand) this article. This means that we have to treat tensors in a very concrete way (which might annoy people coming from mathematics), occasionally prove basic results from combinatorics, and solve recursive inequalities explicitly (because we want to annoy people with a background in theoretical computer science, too).

1 Introduction

Given two $n \times n$-matrices $x = (x_{ik})$ and $y = (y_{kj})$ whose entries are indeterminates over some field K, we want to compute their product $xy = (z_{ij})$. The entries z_{ij} are given be the following well-known bilinear forms

$$z_{ij} = \sum_{k=1}^{n} x_{ik} y_{kj}, \qquad 1 \leq i, j \leq n. \tag{1.1}$$

Each z_{ij} is the sum of n products. Thus every z_{ij} can be computed with n multiplications and $n-1$ additions. This gives an algorithm that altogether uses n^3 multiplications and $n^2(n-1)$ additions. This

*Supported by DFG grant BL 511/10-1

ACM Classification: F.2.2

AMS Classification: 68Q17, 68Q25

Key words and phrases: fast matrix multiplication, bilinear complexity, tensor rank

© Markus Bläser
© Licensed under a Creative Commons Attribution License

Fast Matrix Multiplication

Bläser

(Theory of Computing Library, Graduate Survey 5, 2013)

Matrix Multiplication

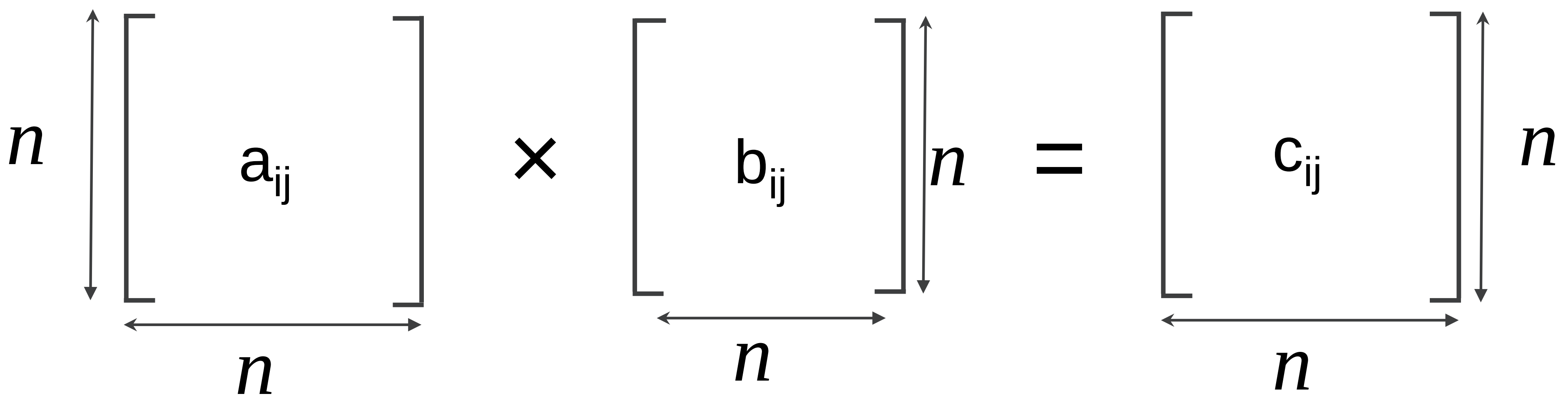
- ✓ This is one of the most fundamental problems in mathematics and computer science

- ✓ Many problems in linear algebra have the same complexity as matrix multiplication:
  - ➢ inverting a matrix
  - ➢ solving a system of linear equations
  - ➢ computing a system of linear equations
  - ➢ computing the determinant
  - ➢ …

- ✓ In several areas of theoretical computer science, the best known algorithms use matrix multiplication:
  - ➢ computing the transitivity closure of a graph
  - ➢ computing the all-pairs shortest paths in graphs
  - ➢ detecting directed cycles in a graph
  - ➢ exact algorithms for MAX-2SAT
  - ➢ …

Matrix Multiplication: Trivial Algorithm

Compute the product of two n x n matrices A and B over a field $\mathbb{F}$



n multiplications and (*n*-1) additions

$$c_{ij} = \boxed{\sum_{k=1}^{n} a_{ik}b_{kj}} \qquad \text{for all } 1 \leq i \leq n \text{ and } 1 \leq j \leq n$$

Trivial algorithm: $n^2(2n\text{-}1)=O(n^3)$ arithmetic operations

We can do better

Overview of the Lectures

✓ Fundamental techniques for fast matrix multiplication (1969~1987)

- Basics of bilinear complexity theory: exponent of matrix multiplication, Strassen’s algorithm, bilinear algorithms
- First technique: tensor rank and recursion
- Second technique: border rank
- Third technique: the asymptotic sum inequality
- Fourth technique: the laser method

Lecture 1

✓ Recent progress on matrix multiplication (1987~)

- Laser method on powers of tensors → currently fastest known algorithm for matrix multiplication
- Other approaches
- Lower bounds
- Rectangular matrix multiplication

Lecture 2

✓ Applications of matrix multiplications, open problems

Lecture 3

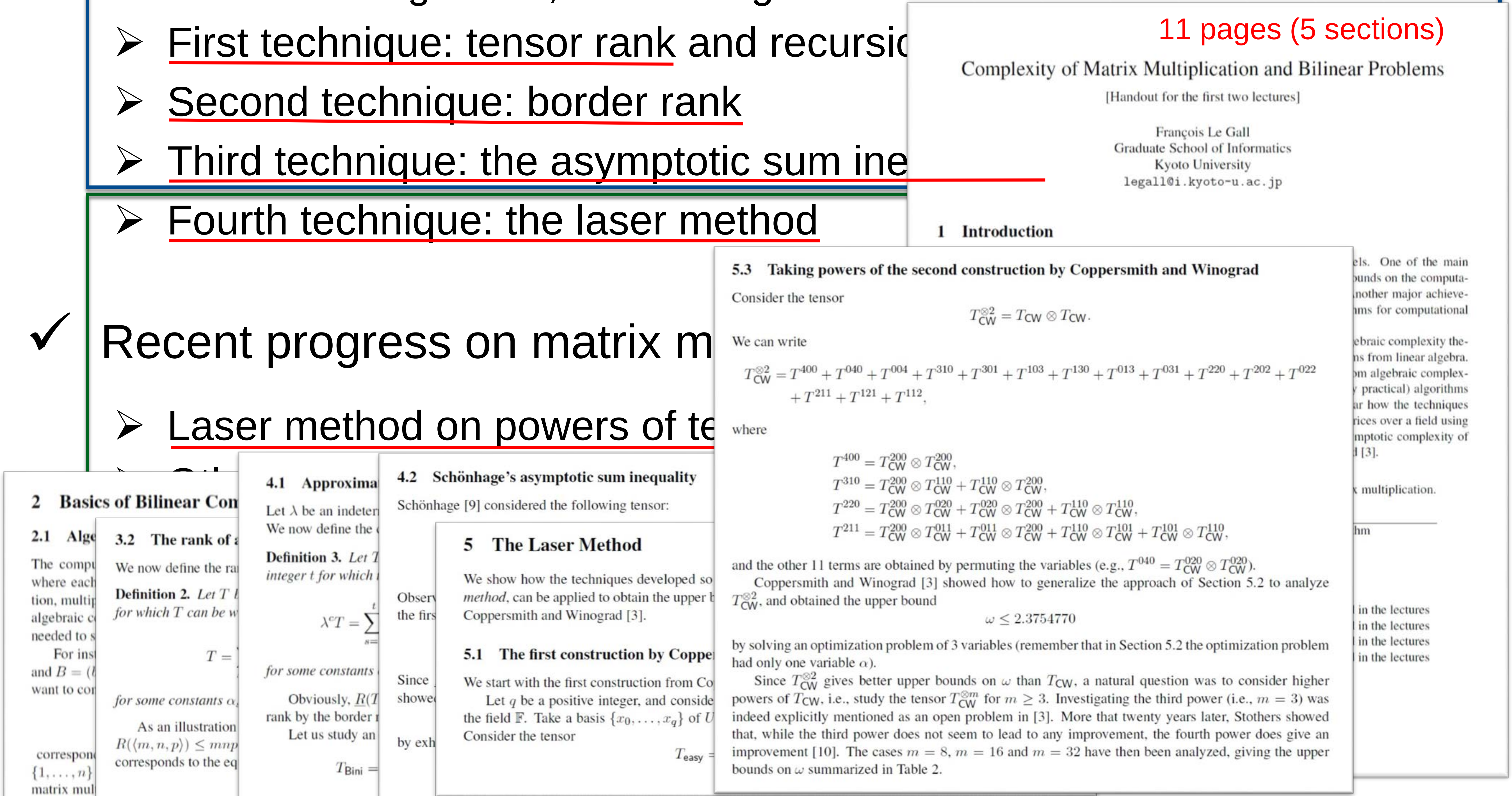
Handout for the First Part

✓ Fundamental techniques for fast matrix multiplication (1969~1987)

- Basics of bilinear complexity theory: exponent of matrix multiplication, Strassen's algorithm, bilinear algorithms
- First technique: tensor rank and recursio
- Second technique: border rank
- Third technique: the asymptotic sum ine
- Fourth technique: the laser method

✓ Recent progress on matrix m

- Laser method on powers of te
- Oth



11 pages (5 sections)

Complexity of Matrix Multiplication and Bilinear Problems

[Handout for the first two lectures]

François Le Gall
Graduate School of Informatics
Kyoto University
legall@i.kyoto-u.ac.jp

1 Introduction

els. One of the main
ounds on the computa-
nother major achieve-
hms for computational

ebraic complexity the-
ns from linear algebra.
om algebraic complex-
y practical) algorithms
ar how the techniques
rices over a field using
mptotic complexity of
d [3].

x multiplication.

hm

in the lectures
in the lectures
in the lectures
in the lectures

5.3 Taking powers of the second construction by Coppersmith and Winograd

Consider the tensor

$$T_{\mathsf{CW}}^{\otimes 2} = T_{\mathsf{CW}} \otimes T_{\mathsf{CW}}.$$

We can write

$$T_{\mathsf{CW}}^{\otimes 2} = T^{400} + T^{040} + T^{004} + T^{310} + T^{301} + T^{103} + T^{130} + T^{013} + T^{031} + T^{220} + T^{202} + T^{022} + T^{211} + T^{121} + T^{112},$$

where

$$\begin{aligned} T^{400} &= T_{\mathsf{CW}}^{200} \otimes T_{\mathsf{CW}}^{200}, \\ T^{310} &= T_{\mathsf{CW}}^{200} \otimes T_{\mathsf{CW}}^{110} + T_{\mathsf{CW}}^{110} \otimes T_{\mathsf{CW}}^{200}, \\ T^{220} &= T_{\mathsf{CW}}^{200} \otimes T_{\mathsf{CW}}^{020} + T_{\mathsf{CW}}^{020} \otimes T_{\mathsf{CW}}^{200} + T_{\mathsf{CW}}^{110} \otimes T_{\mathsf{CW}}^{110}, \\ T^{211} &= T_{\mathsf{CW}}^{200} \otimes T_{\mathsf{CW}}^{011} + T_{\mathsf{CW}}^{011} \otimes T_{\mathsf{CW}}^{200} + T_{\mathsf{CW}}^{110} \otimes T_{\mathsf{CW}}^{101} + T_{\mathsf{CW}}^{101} \otimes T_{\mathsf{CW}}^{110}, \end{aligned}$$

and the other 11 terms are obtained by permuting the variables (e.g., $T^{040} = T_{\mathsf{CW}}^{020} \otimes T_{\mathsf{CW}}^{020}$).

Coppersmith and Winograd [3] showed how to generalize the approach of Section 5.2 to analyze $T_{\mathsf{CW}}^{\otimes 2}$, and obtained the upper bound

$$\omega \leq 2.3754770$$

by solving an optimization problem of 3 variables (remember that in Section 5.2 the optimization problem had only one variable α).

Since $T_{\mathsf{CW}}^{\otimes 2}$ gives better upper bounds on ω than T_{CW}, a natural question was to consider higher powers of T_{CW}, i.e., study the tensor $T_{\mathsf{CW}}^{\otimes m}$ for $m \geq 3$. Investigating the third power (i.e., $m = 3$) was indeed explicitly mentioned as an open problem in [3]. More that twenty years later, Stothers showed that, while the third power does not seem to lead to any improvement, the fourth power does give an improvement [10]. The cases $m = 8$, $m = 16$ and $m = 32$ have then been analyzed, giving the upper bounds on ω summarized in Table 2.

5 The Laser Method

We show how the techniques developed so
method, can be applied to obtain the upper b
Coppersmith and Winograd [3].

5.1 The first construction by Coppe

We start with the first construction from Co
Let q be a positive integer, and conside
the field $\mathbb{F}$. Take a basis $\{x_0, \ldots, x_q\}$ of U
Consider the tensor

$T_{\mathsf{easy}} =$

4.2 Schönhage's asymptotic sum inequality

Schönhage [9] considered the following tensor:

Obser
the firs

Since
showe

by exh

4.1 Approxima

Let λ be an indeter
We now define the

Definition 3. *Let T
integer t for which*

$\lambda^t T = \sum_{s=}^{t}$

for some constants

Obviously, $\underline{R}(T$
rank by the border
Let us study an

$T_{\mathsf{Bini}} =$

2 Basics of Bilinear Con

2.1 Alge

The comp
where each
tion, multi
algebraic c
needed to s
For ins
and $B = ($
want to co

correspon
$\{1, \ldots, n\}$
matrix mul

3.2 The rank of a

We now define the ra

Definition 2. *Let T
for which T can be w*

$T =$

for some constants α

As an illustration
$R(\langle m, n, p \rangle) \leq mnp$
corresponds to the eq

Overview of the Lectures

✓ Fundamental techniques for fast matrix multiplication (1969~1987)

- Basics of bilinear complexity theory: exponent of matrix multiplication, Strassen's algorithm, bilinear algorithms
- First technique: tensor rank and recursion
- Second technique: border rank
- Third technique: the asymptotic sum inequality
- Fourth technique: the laser method

Lecture 1

✓ Recent progress on matrix multiplication (1987~)

- Laser method on powers of tensors
- Other approaches
- Lower bounds
- Rectangular matrix multiplication

Lecture 2

✓ Applications of matrix multiplications, open problems

Lecture 3

Algebraic Model of Computation

Compute the product of two n x n matrices A and B over a field $\mathbb{F}$

Model #1: algebraic circuits

- ✓ gates: $+, -, \times, \div$ (operations on two elements of the field)
- ✓ inputs: a_{ij}, b_{ij} ($2n^2$ inputs)
- ✓ output: $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$

Model #2: straight-line programs (sequence of instructions)

$C(n) =$ size of the shortest straight-line program computing the product

Informally: minimal number of arithmetic operations needed to compute the product

Straightforward algorithm:

$C(n) \leq n^2(2n-1)$ using the formulas $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$

for instance $C(2) \leq 12$ (8 multiplications and 4 additions)

The Exponent of Matrix Multiplication

Compute the product of two n x n matrices A and B over a field $\mathbb{F}$

$C(n) =$ size of the shortest straight-line program computing the product

Exponent of matrix multiplication

$$\omega = \inf \{ \alpha \mid C(n) = O(n^{\alpha}) \}$$

n^2 entries need to be computed

Obviously, $2 \leq \omega \leq 3$

equivalently:

$$\omega = \inf \{ \alpha \mid C(n) \leq n^{\alpha} \text{ for all large enough } n \}$$

Straightforward algorithm:

$C(n) \leq n^2(2n-1)$ using the formulas $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$

$$C(n) = O(n^3)$$

The Exponent of Matrix Multiplication

Compute the product of two n x n matrices A and B over a field $\mathbb{F}$

$C(n) =$ size of the shortest straight-line program computing the product

Exponent of matrix multiplication

$$\omega = \inf\{\, \alpha \mid C(n) = O(n^{\alpha}) \,\}$$

Obviously, $2 \leq \omega \leq 3$

equivalently:

$$\omega = \inf\{\, \alpha \mid C(n) \leq n^{\alpha} \text{ for all large enough } n \}$$

Two remarks:

- ✓ this is an **inf** and not a **min** since the exponent may be achieved by an algorithm with complexity of the form "O($n^{\omega+\varepsilon}$) for any ε>0"
- ✓ ω may depend on the field (but can depend only on its characteristic)

History of the main improvements on the exponent of square matrix multiplication

Upper bound	Year	Authors
$\omega \leq 3$		
$\omega < 2.81$	1969	Strassen
$\omega < 2.79$	1979	Pan
$\omega < 2.78$	1979	Bini, Capovani, Romani and Lotti
$\omega < 2.55$	1981	Schönhage
$\omega < 2.53$	1981	Pan
$\omega < 2.52$	1982	Romani
$\omega < 2.50$	1982	Coppersmith and Winograd
$\omega < 2.48$	1986	Strassen
$\omega < 2.376$	1987	Coppersmith and Winograd
$\omega < 2.374$	2010	Stothers
$\omega < 2.3729$	2012	Vassilevska Williams
$\omega < 2.3728639$	2014	LG

What is ω? $\omega = 2$?

History of the main improvements on the exponent of square matrix multiplication

Upper bound	Year	Authors	
$\omega \leq 3$			
$\omega < 2.81$	1969	Strassen	Rank of a tensor
$\omega < 2.79$	1979	Pan	
$\omega < 2.78$	1979	Bini, Capovani, Romani and Lotti	Border rank of a tensor
$\omega < 2.55$	1981	Schönhage	Asymptotic sum inequality
$\omega < 2.53$	1981	Pan	
$\omega < 2.52$	1982	Romani	
$\omega < 2.50$	1982	Coppersmith and Winograd	
$\omega < 2.48$	1986	Strassen	Laser method
$\omega < 2.376$	1987	Coppersmith and Winograd	
$\omega < 2.374$	2010	Stothers	
$\omega < 2.3729$	2012	Vassilevska Williams	
$\omega < 2.3728639$	2014	LG	

History of the main improvements on the exponent of square matrix multiplication

Remark: the recent algorithms are not practical

$O(n^{2.55})$, but with a large constant in the big-O notation

Upper b		
$\omega \leq 3$		
$\omega < 2.81$	1969	Strassen
$\omega < 2.79$	1979	Pan
$\omega < 2.78$	1979	Bini, Capovani, Romani and Lotti
$\omega < 2.55$	1981	Schönhage
$\omega < 2.53$	1981	Pan
$\omega < 2.52$	1982	Romani
$\omega < 2.50$	1982	Coppersmith and Winograd
$\omega < 2.48$	1986	Strassen
$\omega < 2.376$	1987	Coppersmith and Winograd
$\omega < 2.374$	2010	Stothers
$\omega < 2.3729$	2012	Vassilevska Williams
$\omega < 2.3728639$	2014	LG

Asymptotic sum inequality

Laser method

The Exponent of Matrix Multiplication

Compute the product of two n x n matrices A and B over a field F

$C(n) =$ size of the shortest straight-line program computing the product

Exponent of matrix multiplication

$$\omega = \inf \{ \alpha \mid C(n) = O(n^{\alpha}) \}$$

In 1969, Strassen gave the first sub-cubic time algorithm for matrix multiplication

Complexity: $O(n^{2.81})$ arithmetic operations

$$C(n) = O(n^{2.81})$$

➡ $\omega \leq 2.81$

Strassen's algorithm (for the product of two 2×2 matrices)

Goal: compute the product of $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ by $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

1. Compute:

$$\begin{aligned}
&\underline{m_1 = a_{11} * (b_{12} - b_{22})},\\
&\underline{m_2 = (a_{11} + a_{12}) * b_{22}},\\
&m_3 = (a_{21} + a_{22}) * b_{11},\\
&m_4 = a_{22} * (b_{21} - b_{11}),\\
&m_5 = (a_{11} + a_{22}) * (b_{11} + b_{22}),\\
&m_6 = (a_{12} - a_{22}) * (b_{21} + b_{22}),\\
&m_7 = (a_{11} - a_{21}) * (b_{11} + b_{12}).
\end{aligned}$$

2. Output:

$$\begin{aligned}
-m_2 + m_4 + m_5 + m_6 &= c_{11},\\
\underline{m_1 + m_2 = c_{12}}&,\\
m_3 + m_4 &= c_{21},\\
m_1 - m_3 + m_5 - m_7 &= c_{22}.
\end{aligned}$$

entries of the output matrix

7 multiplications

18 additions/substractions

$C(2) \leq 25$

worse than the trivial algorithm
(8 multiplications and 4 additions)

Strassen's algorithm (for the product of two $2^k \times 2^k$ matrices)

Goal: compute the product of $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ by $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$

A_{ij}, B_{ij}: matrices of size $2^{k-1} \times 2^{k-1}$

1. Compute:

$$M_1 = A_{11} * (B_{12} - B_{22}),$$
$$\vdots$$
$$M_7 = (A_{11} - A_{21}) * (B_{11} + B_{12}).$$

2. Output:

$$-M_2 + M_4 + M_5 + M_6 = C_{11},$$
$$\vdots$$
$$M_1 - M_3 + M_5 - M_7 = C_{22}.$$

7 multiplications of two $2^{k-1} \times 2^{k-1}$ matrices

- done recursively using Strassen's algorithm

18 additions/substractions of two $2^{k-1} \times 2^{k-1}$ matrices

- $2^{2(k-1)}$ scalar operations for each

Strassen’s algorithm (for the product of two $2^k \times 2^k$ matrices)

Goal: compute the product of $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ by $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$

Observation: the complexity of Strassen’s algorithm is dominated by the number of (scalar) multiplications

-1

Complexity of this algorithm

$$T(2^k) = 7 \times T(2^{k-1}) + 18 \times 2^{2(k-1)}$$

$$= O(7^k)$$

$$= O\left(\left(2^k\right)^{\log_2 7}\right)$$

Conclusion: $C(2^k) = O((2^k)^{\log_2 7})$

$\omega \leq \log_2 7 = 2.807\ldots$

[Strassen 69]

Remember:

7 multiplications of two $2^{k-1} \times 2^{k-1}$ matrices

18 ad

rices

- $2^{2(k-1)}$ scalar operations for each

Exponent of matrix multiplication

$$\omega = \inf \{ \alpha \mid C(n) = O(n^\alpha) \}$$

Bilinear Algorithms

A bilinear algorithm for matrix multiplication is an algebraic algorithm of the form:

t is the bilinear complexity of the algorithm

1. Compute $$\begin{aligned} m_1 &= (\text{linear combination of the } a_{ij}\text{'s}) * (\text{linear combination of the } b_{ij}\text{'s}) \\ &\vdots \\ m_t &= (\text{linear combination of the } a_{ij}\text{'s}) * (\text{linear combination of the } b_{ij}\text{'s}) \end{aligned}$$
2. Each entry c_{ij} is computed by taking a linear combination of $m_1, \ldots, m_t$

i.e., we do not allow products of the form $a_{ij} * a_{i'j'}$ or $b_{ij} * b_{i'j'}$

$C^{\text{bil}}(n) =$ bilinear complexity of the best bilinear algorithm computing the product of two n × n matrices

Bilinear Algorithms

By generalizing Strassen's recursive argument we obtain:

Proposition 1

Let m and t be any positive integers. Suppose that there exists a bilinear algorithm that computes the product of two $m \times m$ matrices with bilinear complexity t. Then

$$\omega \leq \log_m(t).$$

In short: $\mathcal{C}^{bil}(m) \leq t \Longrightarrow \omega \leq \log_m(t)$

Example (Strassen's bound): $\mathcal{C}^{bil}(2) \leq 7 \Longrightarrow \omega \leq \log_2(7)$

Proof: $\mathcal{C}^{bil}(m) \leq t \Longrightarrow \mathcal{C}^{bil}(m^k) \leq t^k$ for any $k \geq 1$

"recursion"

and $t^k = (m^k)^{\log_m(t)}$

$$\Longrightarrow C(m^k) = O(tk)$$

"complexity dominated by the number of multiplications"

Exponent of matrix multiplication

$$\omega = \inf\{\, \alpha \mid C(n) = O(n^\alpha) \,\}$$

→

Corollary

$$\omega = \inf\{\, \alpha \mid C^{\mathrm{bil}}(n) = O(n^\alpha) \,\}$$

Overview of the Lectures

✓ Fundamental techniques for fast matrix multiplication (1969~1987)

- Basics of bilinear complexity theory: exponent of matrix multiplication, Strassen’s algorithm, bilinear algorithms
- First technique: tensor rank and recursion
- Second technique: border rank
- Third technique: the asymptotic sum inequality
- Fourth technique: the laser method

Lecture 1

✓ Recent progress on matrix multiplication (1987~)

- Laser method on powers of tensors
- Other approaches
- Lower bounds
- Rectangular matrix multiplication

Lecture 2

✓ Applications of matrix multiplications, open problems

Lecture 3

The tensor of matrix multiplication

Definition 1

The tensor corresponding to the multiplication of an $m \times n$ matrix by an $n \times p$ matrix is

$$\sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} a_{ik} \otimes b_{kj} \otimes c_{ij}.$$

The tensor of matrix multiplication

Definition 1

The tensor corresponding to the multiplication of an $m \times n$ matrix by an $n \times p$ matrix is

$$\langle m, n, p \rangle = \sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} a_{ik} \otimes b_{kj} \otimes c_{ij}.$$

The tensor of matrix multiplication

Definition 1

The tensor corresponding to the multiplication of an $m \times n$ matrix by an $n \times p$ matrix is

$$\langle m, n, p \rangle = \sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} a_{ik} \otimes b_{kj} \otimes c_{ij}.$$

intuitive interpretation:

- this is a formal sum

The tensor of matrix multiplication

Definition 1

The tensor corresponding to the multiplication of an $m \times n$ matrix by an $n \times p$ matrix is

$$\langle m, n, p \rangle = \sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} a_{ik} \otimes b_{kj} \otimes c_{ij}.$$

intuitive interpretation:

- this is a formal sum
- when the a_{ik} and the b_{kj} are replaced by the corresponding entries of matrices, the coefficient of c_{ij} becomes $\sum_{k=1}^{n} a_{ik} b_{kj}$

The tensor of matrix multiplication

Definition 1

The tensor corresponding to the multiplication of an $m \times n$ matrix by an $n \times p$ matrix is

$$\langle m, n, p\rangle = \sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} a_{ik} \otimes b_{kj} \otimes c_{ij}.$$

intuitive interpretation:

- this is a formal sum
- when the a_{ik} and the b_{kj} are replaced by the corresponding entries of matrices, the coefficient of c_{ij} becomes $\sum_{k=1}^{n} a_{ik} b_{kj}$

why this is useful:

- one object instead of mp objects

The tensor of matrix multiplication

Definition 1

The tensor corresponding to the multiplication of an $m \times n$ matrix by an $n \times p$ matrix is

$$\langle m, n, p \rangle = \sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} a_{ik} \otimes b_{kj} \otimes c_{ij}.$$

intuitive interpretation:

- this is a formal sum
- when the a_{ik} and the b_{kj} are replaced by the corresponding entries of matrices, the coefficient of c_{ij} becomes $\sum_{k=1}^{n} a_{ik} b_{kj}$

why this is useful:

- one object instead of mp objects
- shows the symmetries between the two input matrices and the output matrix (see later)

The tensor of matrix multiplication

Definition 1

The tensor corresponding to the multiplication of an $m \times n$ matrix by an $n \times p$ matrix is

$$\langle m, n, p \rangle = \sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} a_{ik} \otimes b_{kj} \otimes c_{ij}.$$

The tensor of matrix multiplication

Definition 1

The tensor corresponding to the multiplication of an $m \times n$ matrix by an $n \times p$ matrix is

$$\langle m, n, p \rangle = \sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} a_{ik} \otimes b_{kj} \otimes c_{ij}.$$

Rank (slightly informal definition):

$R(\langle m, n, p \rangle) =$ minimal t such that $\langle m, n, p \rangle$ can be written as the sum of t terms of the form

(lin. comb. of a_{ij}) $\otimes$ (lin. comb. of b_{ij}) $\otimes$ (lin. comb. of c_{ij}).

The tensor of matrix multiplication

Definition 1

The tensor corresponding to the multiplication of an $m \times n$ matrix by an $n \times p$ matrix is

$$\langle m,n,p\rangle = \sum_{i=1}^{m}\sum_{j=1}^{p}\sum_{k=1}^{n} \underbrace{\boxed{a_{ik} \otimes b_{kj} \otimes c_{ij}}}_{\text{one term}}.$$

$$R(\langle m,n,p\rangle) \leq mnp$$

Rank (slightly informal definition):

$R(\langle m,n,p\rangle)$ = minimal t such that $\langle m,n,p\rangle$ can be written as the sum of t terms of the form

(lin. comb. of a_{ij}) $\otimes$ (lin. comb. of b_{ij}) $\otimes$ (lin. comb. of c_{ij}).

The tensor of matrix multiplication

Definition 1

The tensor corresponding to the multiplication of an $m \times n$ matrix by an $n \times p$ matrix is

$$\langle m,n,p\rangle = \sum_{i=1}^{m}\sum_{j=1}^{p}\sum_{k=1}^{n} \boxed{a_{ik}\otimes b_{kj}\otimes c_{ij}}.$$

one term

$$R(\langle m,n,p\rangle) \le mnp$$

Rank (slightly informal definition):

$R(\langle m,n,p\rangle) =$ minimal t such that $\langle m,n,p\rangle$ can be written as the sum of t terms of the form (lin. comb. of a_{ij}) $\otimes$ (lin. comb. of b_{ij}) $\otimes$ (lin. comb. of c_{ij}).

$$\begin{aligned}
\langle 2,2,2\rangle = {} & a_{11}\otimes(b_{12}-b_{22})\otimes(c_{12}+c_{22})\\
& +(a_{11}+a_{12})\otimes b_{22}\otimes(-c_{11}+c_{12})\\
& +(a_{21}+a_{22})\otimes b_{11}\otimes(c_{21}-c_{22})\\
& +a_{22}\otimes(b_{21}-b_{11})\otimes(c_{11}+c_{21})\\
& +(a_{11}+a_{22})\otimes(b_{11}+b_{22})\otimes(c_{11}+c_{22})\\
& +(a_{12}-a_{22})\otimes(b_{21}+b_{22})\otimes c_{11}\\
& +(a_{11}-a_{21})\otimes(b_{11}+b_{12})\otimes(-c_{22})
\end{aligned}$$

Strassen's algorithm gives

$$R(\langle 2,2,2\rangle) \le 7$$

The tensor of matrix multiplication

1. Compute:

$$\begin{aligned} m_1 &= a_{11} * (b_{12} - b_{22}), \\ m_2 &= (a_{11} + a_{12}) * b_{22}, \\ m_3 &= (a_{21} + a_{22}) * b_{11}, \\ m_4 &= a_{22} * (b_{21} - b_{11}), \\ m_5 &= (a_{11} + a_{22}) * (b_{11} + b_{22}), \\ m_6 &= (a_{12} - a_{22}) * (b_{21} + b_{22}), \\ m_7 &= (a_{11} - a_{21}) * (b_{11} + b_{12}). \end{aligned}$$

2. Output:

$$\begin{aligned} -m_2 + m_4 + m_5 + m_6 &= c_{11}, \\ m_1 + m_2 &= c_{12}, \\ m_3 + m_4 &= c_{21}, \\ m_1 - m_3 + m_5 - m_7 &= c_{22}. \end{aligned}$$

of an $m \times n$ matrix by

$${}_{ik} \otimes b_{kj} \otimes c_{ij}.$$

one term

$$R(\langle m, n, p \rangle) \leq mnp$$

e written

$b_{ij}) \otimes$ (lin. comb. of c_{ij}).

$$\begin{aligned} \langle 2,2,2 \rangle = & a_{11} \otimes (b_{12} - b_{22}) \otimes (c_{12} + c_{22}) \\ & + (a_{11} + a_{12}) \otimes b_{22} \otimes (-c_{11} + c_{12}) \\ & + (a_{21} + a_{22}) \otimes b_{11} \otimes (c_{21} - c_{22}) \\ & + a_{22} \otimes (b_{21} - b_{11}) \otimes (c_{11} + c_{21}) \\ & + (a_{11} + a_{22}) \otimes (b_{11} + b_{22}) \otimes (c_{11} + c_{22}) \\ & + (a_{12} - a_{22}) \otimes (b_{21} + b_{22}) \otimes c_{11} \\ & + (a_{11} - a_{21}) \otimes (b_{11} + b_{12}) \otimes (-c_{22}) \end{aligned}$$

Strassen's algorithm gives

$$R(\langle 2, 2, 2 \rangle) \leq 7$$

The tensor of matrix multiplication

1. Compute:

$$\begin{aligned} m_1 &= \underline{a_{11} * (b_{12} - b_{22})}, \\ m_2 &= \underline{(a_{11} + a_{12}) * b_{22}}, \\ m_3 &= (a_{21} + a_{22}) * b_{11}, \\ m_4 &= a_{22} * (b_{21} - b_{11}), \\ m_5 &= (a_{11} + a_{22}) * (b_{11} + b_{22}), \\ m_6 &= (a_{12} - a_{22}) * (b_{21} + b_{22}), \\ m_7 &= (a_{11} - a_{21}) * (b_{11} + b_{12}). \end{aligned}$$

2. Output:

$$\begin{aligned} -m_2 + m_4 + m_5 + m_6 &= c_{11}, \\ \boxed{m_1 + m_2 = c_{12},} & \\ m_3 + m_4 &= c_{21}, \\ m_1 - m_3 + m_5 - m_7 &= c_{22}. \end{aligned}$$

of an $m \times n$ matrix by

$\boxed{{}_{ik} \otimes b_{kj} \otimes c_{ij}}.$

one term

$$R(\langle m, n, p\rangle) \leq mnp$$

e written

$b_{ij}) \otimes$ (lin. comb. of c_{ij}).

$$\begin{aligned} \langle 2,2,2\rangle = & \underline{a_{11} \otimes (b_{12} - b_{22})} \otimes (\boxed{c_{12}} + c_{22}) \\ & + \underline{(a_{11} + a_{12}) \otimes b_{22}} \otimes (-c_{11} + \boxed{c_{12}}) \\ & + (a_{21} + a_{22}) \otimes b_{11} \otimes (c_{21} - c_{22}) \\ & + a_{22} \otimes (b_{21} - b_{11}) \otimes (c_{11} + c_{21}) \\ & + (a_{11} + a_{22}) \otimes (b_{11} + b_{22}) \otimes (c_{11} + c_{22}) \\ & + (a_{12} - a_{22}) \otimes (b_{21} + b_{22}) \otimes c_{11} \\ & + (a_{11} - a_{21}) \otimes (b_{11} + b_{12}) \otimes (-c_{22}) \end{aligned}$$

Strassen's algorithm gives

$$R(\langle 2,2,2\rangle) \leq 7$$

The tensor of matrix multiplication

1. Compute:

$$
\begin{aligned}
m_1 &= \underline{a_{11} * (b_{12} - b_{22})}, \\
m_2 &= \underline{(a_{11} + a_{12}) * b_{22}}, \\
m_3 &= (a_{21} + a_{22}) * b_{11}, \\
m_4 &= a_{22} * (b_{21} - b_{11}), \\
m_5 &= (a_{11} + a_{22}) * (b_{11} + b_{22}), \\
m_6 &= (a_{12} - a_{22}) * (b_{21} + b_{22}), \\
m_7 &= (a_{11} - a_{21}) * (b_{11} + b_{12}).
\end{aligned}
$$

2. Output:

$$
\begin{aligned}
-m_2 + m_4 + m_5 + m_6 &= c_{11}, \\
\boxed{m_1 + m_2 = c_{12},}& \\
m_3 + m_4 &= c_{21}, \\
m_1 - m_3 + m_5 - m_7 &= c_{22}.
\end{aligned}
$$

of an $m \times n$ matrix by

${}_{ik} \otimes b_{kj} \otimes c_{ij}$.

one term

$R(\langle m, n, p \rangle) \leq mnp$

e written

b_{ij}) ⊗ (lin. comb. of c_{ij}).

$$
\begin{aligned}
\langle 2,2,2 \rangle = &\underline{a_{11} \otimes (b_{12} - b_{22})} \otimes (\boxed{c_{12}} + c_{22}) \\
&+ \underline{(a_{11} + a_{12}) \otimes b_{22}} \otimes (-c_{11} + \boxed{c_{12}}) \\
&+ (a_{21} + a_{22}) \otimes b_{11} \otimes (c_{21} - c_{22}) \\
&+ a_{22} \otimes (b_{21} - b_{11}) \otimes (c_{11} + c_{21}) \\
&+ (a_{11} + a_{22}) \otimes (b_{11} + b_{22}) \otimes (c_{11} + c_{22}) \\
&+ (a_{12} - a_{22}) \otimes (b_{21} + b_{22}) \otimes c_{11} \\
&+ (a_{11} - a_{21}) \otimes (b_{11} + b_{12}) \otimes (-c_{22})
\end{aligned}
$$

Strassen's algorithm gives

$R(\langle 2, 2, 2 \rangle) \leq 7$

rank = bilinear complexity

The tensor of matrix multiplication

Definition 1

The tensor corresponding to the multiplication of an $m \times n$ matrix by an $n \times p$ matrix is

$$\langle m, n, p \rangle = \sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} \underbrace{\boxed{a_{ik} \otimes b_{kj} \otimes c_{ij}}}_{\text{one term}}.$$

$$R(\langle m, n, p \rangle) \leq mnp$$

Rank (slightly informal definition):

$R(\langle m, n, p \rangle) =$ minimal t such that $\langle m, n, p \rangle$ can be written as the sum of t terms of the form (lin. comb. of a_{ij}) $\otimes$ (lin. comb. of b_{ij}) $\otimes$ (lin. comb. of c_{ij}).

$$\begin{aligned}
\langle 2,2,2 \rangle = & a_{11} \otimes (b_{12} - b_{22}) \otimes (c_{12} + c_{22}) \\
& + (a_{11} + a_{12}) \otimes b_{22} \otimes (-c_{11} + c_{12}) \\
& + (a_{21} + a_{22}) \otimes b_{11} \otimes (c_{21} - c_{22}) \\
& + a_{22} \otimes (b_{21} - b_{11}) \otimes (c_{11} + c_{21}) \\
& + (a_{11} + a_{22}) \otimes (b_{11} + b_{22}) \otimes (c_{11} + c_{22}) \\
& + (a_{12} - a_{22}) \otimes (b_{21} + b_{22}) \otimes c_{11} \\
& + (a_{11} - a_{21}) \otimes (b_{11} + b_{12}) \otimes (-c_{22})
\end{aligned}$$

Strassen's algorithm gives

$$R(\langle 2, 2, 2 \rangle) \leq 7$$

rank = bilinear complexity

The tensor of matrix multiplication

Definition 1

The tensor corresponding to the multiplication of an $m \times n$ matrix by an $n \times p$ matrix is

$$\langle m,n,p\rangle = \sum_{i=1}^{m}\sum_{j=1}^{p}\sum_{k=1}^{n} \underbrace{\boxed{a_{ik}\otimes b_{kj}\otimes c_{ij}}}_{\text{one term}}.$$

Rank (slightly informal definition):

$$R(\langle m,n,p\rangle) \le mnp$$

$R(\langle m,n,p\rangle)$ = minimal t such that $\langle m,n,p\rangle$ can be written as the sum of t terms of the form (lin. comb. of a_{ij}) $\otimes$ (lin. comb. of b_{ij}) $\otimes$ (lin. comb. of c_{ij}).

$$\begin{aligned}\langle 2,2,2\rangle = & a_{11}\otimes(b_{12}-b_{22})\otimes(c_{12}+c_{22})\\ & +(a_{11}+a_{12})\otimes b_{22}\otimes(-c_{11}+c_{12})\\ & +(a_{21}+a_{22})\otimes b_{11}\otimes(c_{21}-c_{22})\\ & +a_{22}\otimes(b_{21}-b_{11})\otimes(c_{11}+c_{21})\\ & +(a_{11}+a_{22})\otimes(b_{11}+b_{22})\otimes(c_{11}+c_{22})\end{aligned}$$

$$\omega = \inf\left\{\alpha \mid R(\langle n,n,n\rangle) = O(n^{\alpha})\right\}$$

Strassen's algorithm gives

$$R(\langle 2,2,2\rangle) \le 7$$

rank = bilinear complexity

Properties of this tensor

Definition 1

The tensor corresponding to the multiplication of an $m \times n$ matrix by an $n \times p$ matrix is

$$\langle m, n, p \rangle = \sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} a_{ik} \otimes b_{kj} \otimes c_{ij}.$$

Property (Equation (3.2))

$$\langle m, n, p \rangle \otimes \langle m', n', p' \rangle \cong \langle mm', nn', pp' \rangle$$

Properties of this tensor

Definition 1

The tensor corresponding to the multiplication of an $m \times n$ matrix by an $n \times p$ matrix is

$$\langle m, n, p \rangle = \sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} a_{ik} \otimes b_{kj} \otimes c_{ij}.$$

Property (Equation (3.2))

$$\langle m, n, p \rangle \otimes \langle m', n', p' \rangle \cong \langle mm', nn', pp' \rangle$$

Proof:

$$\langle m, n, p \rangle \otimes \langle m', n', p' \rangle = \sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} \sum_{i'=1}^{m'} \sum_{j'=1}^{p'} \sum_{k'=1}^{n'} (a_{ik} \otimes a'_{i'k'}) \otimes (b_{kj} \otimes b'_{k'j'}) \otimes (c_{ij} \otimes c'_{i'j'})$$

Properties of this tensor

Definition 1

The tensor corresponding to the multiplication of an $m \times n$ matrix by an $n \times p$ matrix is

$$\langle m, n, p \rangle = \sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} a_{ik} \otimes b_{kj} \otimes c_{ij}.$$

Property (Equation (3.2))

$$\langle m, n, p \rangle \otimes \langle m', n', p' \rangle \cong \langle mm', nn', pp' \rangle$$

Proof:

$$\langle m, n, p \rangle \otimes \langle m', n', p' \rangle = \sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} \sum_{i'=1}^{m'} \sum_{j'=1}^{p'} \sum_{k'=1}^{n'} \underbrace{(a_{ik} \otimes a'_{i'k'})}_{a_{ii'kk'}} \otimes \underbrace{(b_{kj} \otimes b'_{k'j'})}_{b_{kk'jj'}} \otimes \underbrace{(c_{ij} \otimes c'_{i'j'})}_{c_{ii'jj'}}$$

Properties of this tensor

Definition 1

The tensor corresponding to the multiplication of an $m \times n$ matrix by an $n \times p$ matrix is

$$\langle m,n,p\rangle = \sum_{i=1}^{m}\sum_{j=1}^{p}\sum_{k=1}^{n} a_{ik}\otimes b_{kj}\otimes c_{ij}.$$

Property (Equation (3.2))

$$\langle m,n,p\rangle \otimes \langle m',n',p'\rangle \cong \langle mm',nn',pp'\rangle$$

Proof:

$$\langle m,n,p\rangle \otimes \langle m',n',p'\rangle = \sum_{i=1}^{m}\sum_{j=1}^{p}\sum_{k=1}^{n}\sum_{i'=1}^{m'}\sum_{j'=1}^{p'}\sum_{k'=1}^{n'} \underbrace{(a_{ik}\otimes a'_{i'k'})}_{a_{ii'kk'}} \otimes \underbrace{(b_{kj}\otimes b'_{k'j'})}_{b_{kk'jj'}} \otimes \underbrace{(c_{ij}\otimes c'_{i'j'})}_{c_{ii'jj'}}$$

Intuitive explanation of $\langle n,n,n\rangle \otimes \langle n,n,n\rangle \cong \langle n^2,n^2,n^2\rangle$:

Properties of this tensor

Definition 1

The tensor corresponding to the multiplication of an $m \times n$ matrix by an $n \times p$ matrix is

$$\langle m, n, p \rangle = \sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} a_{ik} \otimes b_{kj} \otimes c_{ij}.$$

Property (Equation (3.2))

$$\langle m, n, p \rangle \otimes \langle m', n', p' \rangle \cong \langle mm', nn', pp' \rangle$$

Proof:

$$\langle m, n, p \rangle \otimes \langle m', n', p' \rangle = \sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} \sum_{i'=1}^{m'} \sum_{j'=1}^{p'} \sum_{k'=1}^{n'} \underbrace{(a_{ik} \otimes a'_{i'k'})}_{a_{ii'kk'}} \otimes \underbrace{(b_{kj} \otimes b'_{k'j'})}_{b_{kk'jj'}} \otimes \underbrace{(c_{ij} \otimes c'_{i'j'})}_{c_{ii'jj'}}$$

Intuitive explanation of $\langle n, n, n \rangle \otimes \langle n, n, n \rangle \cong \langle n^2, n^2, n^2 \rangle$:

$\langle n, n, n \rangle$: product of two $n \times n$ matrices, each entry being an element in $\mathbb{F}$

Properties of this tensor

Definition 1

The tensor corresponding to the multiplication of an $m \times n$ matrix by an $n \times p$ matrix is

$$\langle m, n, p \rangle = \sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} a_{ik} \otimes b_{kj} \otimes c_{ij}.$$

Property (Equation (3.2))

$$\langle m, n, p \rangle \otimes \langle m', n', p' \rangle \cong \langle mm', nn', pp' \rangle$$

Proof:

$$\langle m, n, p \rangle \otimes \langle m', n', p' \rangle = \sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} \sum_{i'=1}^{m'} \sum_{j'=1}^{p'} \sum_{k'=1}^{n'} \underbrace{(a_{ik} \otimes a'_{i'k'})}_{a_{ii'kk'}} \otimes \underbrace{(b_{kj} \otimes b'_{k'j'})}_{b_{kk'jj'}} \otimes \underbrace{(c_{ij} \otimes c'_{i'j'})}_{c_{ii'jj'}}$$

Intuitive explanation of $\langle n, n, n \rangle \otimes \langle n, n, n \rangle \cong \langle n^2, n^2, n^2 \rangle$:

$\langle n, n, n \rangle$: product of two $n \times n$ matrices, each entry being an element in $\mathbb{F}$

$\langle n, n, n \rangle \otimes \langle n, n, n \rangle$: product of two $n \times n$ matrices, each entry being an $n \times n$ matrix

Properties of this tensor

Definition 1

The tensor corresponding to the multiplication of an $m \times n$ matrix by an $n \times p$ matrix is

$$\langle m,n,p\rangle = \sum_{i=1}^{m}\sum_{j=1}^{p}\sum_{k=1}^{n} a_{ik}\otimes b_{kj}\otimes c_{ij}.$$

Property (Equation (3.2))

$$\langle m,n,p\rangle \otimes \langle m',n',p'\rangle \cong \langle mm',nn',pp'\rangle$$

Proof:

$$\langle m,n,p\rangle \otimes \langle m',n',p'\rangle = \sum_{i=1}^{m}\sum_{j=1}^{p}\sum_{k=1}^{n}\sum_{i'=1}^{m'}\sum_{j'=1}^{p'}\sum_{k'=1}^{n'} \underbrace{(a_{ik}\otimes a'_{i'k'})}_{a_{ii'kk'}} \otimes \underbrace{(b_{kj}\otimes b'_{k'j'})}_{b_{kk'jj'}} \otimes \underbrace{(c_{ij}\otimes c'_{i'j'})}_{c_{ii'jj'}}$$

Intuitive explanation of $\langle n,n,n\rangle \otimes \langle n,n,n\rangle \cong \langle n^2,n^2,n^2\rangle$:

$\langle n,n,n\rangle$: product of two $n \times n$ matrices, each entry being an element in $\mathbb{F}$

$\langle n,n,n\rangle \otimes \langle n,n,n\rangle$: product of two $n \times n$ matrices, each entry being an $n \times n$ matrix

$=$ product of two $n^2 \times n^2$ matrices over $\mathbb{F}$

Properties of this tensor

Definition 1

The tensor corresponding to the multiplication of an $m \times n$ matrix by an $n \times p$ matrix is

$$\langle m, n, p \rangle = \sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} a_{ik} \otimes b_{kj} \otimes c_{ij}.$$

Property (Equation (3.2))

$$\langle m, n, p \rangle \otimes \langle m', n', p' \rangle \cong \langle mm', nn', pp' \rangle$$

Proof:

$$\langle m, n, p \rangle \otimes \langle m', n', p' \rangle = \sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} \sum_{i'=1}^{m'} \sum_{j'=1}^{p'} \sum_{k'=1}^{n'} \underbrace{(a_{ik} \otimes a'_{i'k'})}_{a_{ii'kk'}} \otimes \underbrace{(b_{kj} \otimes b'_{k'j'})}_{b_{kk'jj'}} \otimes \underbrace{(c_{ij} \otimes c'_{i'j'})}_{c_{ii'jj'}}$$

Property: submultiplicativity of the rank (Equation (3.3), special case)

$$R(\langle mm', nn', pp' \rangle) \leq R(\langle m, n, p \rangle) \times R(\langle m', n', p' \rangle)$$

Properties of this tensor

Definition 1

The tensor corresponding to the multiplication of an $m \times n$ matrix by an $n \times p$ matrix is

$$\langle m, n, p \rangle = \sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} a_{ik} \otimes b_{kj} \otimes c_{ij}.$$

Property (Equation (3.4))

$$R(\langle m, n, p \rangle) = R(\langle m, p, n \rangle) = \cdots = R(\langle p, n, m \rangle)$$

(by permuting the variables, which preserves the rank)

Properties of this tensor

Definition 1

The tensor corresponding to the multiplication of an $m \times n$ matrix by an $n \times p$ matrix is

$$\langle m, n, p \rangle = \sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} a_{ik} \otimes b_{kj} \otimes c_{ij}.$$

Property (Equation (3.4))

$$R(\langle m, n, p \rangle) = R(\langle m, p, n \rangle) = \cdots = R(\langle p, n, m \rangle)$$

(by permuting the variables, which preserves the rank)

Consequence: $R(\langle n, n, n^2 \rangle) = R(\langle n, n^2, n \rangle)$

Properties of this tensor

Definition 1

The tensor corresponding to the multiplication of an $m \times n$ matrix by an $n \times p$ matrix is

$$\langle m, n, p \rangle = \sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} a_{ik} \otimes b_{kj} \otimes c_{ij}.$$

Property (Equation (3.4))

$$R(\langle m, n, p \rangle) = R(\langle m, p, n \rangle) = \cdots = R(\langle p, n, m \rangle)$$

(by permuting the variables, which preserves the rank)

Consequence: $R(\langle n, n, n^2 \rangle) = R(\langle n, n^2, n \rangle)$

$n \times n$ matrix by $n \times n^2$ matrix

$n \times n^2$ matrix by $n^2 \times n$ matrix

Properties of this tensor

Definition 1

The tensor corresponding to the multiplication of an $m \times n$ matrix by an $n \times p$ matrix is

$$\langle m, n, p \rangle = \sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} a_{ik} \otimes b_{kj} \otimes c_{ij}.$$

Property (Equation (3.4))

$$R(\langle m, n, p \rangle) = R(\langle m, p, n \rangle) = \cdots = R(\langle p, n, m \rangle)$$

(by permuting the variables, which preserves the rank)

Consequence: $R(\langle n, n, n^2 \rangle) = R(\langle n, n^2, n \rangle)$

$n \times n$ matrix by $n \times n^2$ matrix

$n \times n^2$ matrix by $n^2 \times n$ matrix

same (bilinear) complexity!

The first Inequality

Remember:

Proposition 1

Let m and t be any positive integers. Suppose that there exists a bilinear algorithm that computes the product of two $m \times m$ matrices with bilinear complexity t. Then

$$\omega \leq \log_m(t).$$

In short: $\mathcal{C}^{bil}(m) \leq t \Longrightarrow \omega \leq \log_m(t)$

The first Inequality

Remember:

Proposition 1

Let m and t be any positive integers. Suppose that there exists a bilinear algorithm that computes the product of two $m \times m$ matrices with bilinear complexity t. Then

$$\omega \leq \log_m(t).$$

In short: $\mathcal{C}^{bil}(m) \leq t \Longrightarrow \omega \leq \log_m(t)$ or $\mathcal{C}^{bil}(m) \leq t \Longrightarrow m^{\omega} \leq t$

The first Inequality

Remember:

Proposition 1

Let m and t be any positive integers. Suppose that there exists a bilinear algorithm that computes the product of two $m \times m$ matrices with bilinear complexity t. Then

$$\omega \leq \log_m(t).$$

In short: $\mathcal{C}^{bil}(m) \leq t \Longrightarrow \omega \leq \log_m(t)$ or $\mathcal{C}^{bil}(m) \leq t \Longrightarrow m^\omega \leq t$

In our new terminology: $R(\langle m, m, m \rangle) \leq t \Longrightarrow m^\omega \leq t$

The first Inequality

Remember:

Proposition 1

Let m and t be any positive integers. Suppose that there exists a bilinear algorithm that computes the product of two $m \times m$ matrices with bilinear complexity t. Then

$$\omega \leq \log_m(t).$$

In short: $\mathcal{C}^{bil}(m) \leq t \Longrightarrow \omega \leq \log_m(t)$ or $\mathcal{C}^{bil}(m) \leq t \Longrightarrow m^{\omega} \leq t$

In our new terminology: $R(\langle m, m, m\rangle) \leq t \Longrightarrow m^{\omega} \leq t$

Theorem 1

$$R(\langle m, n, p\rangle) \leq t \Longrightarrow (mnp)^{\omega/3} \leq t$$

The first Inequality

Remember:

Proposition 1

Let m and t be any positive integers. Suppose that there exists a bilinear algorithm that computes the product of two $m \times m$ matrices with bilinear complexity t. Then

$$\omega \leq \log_m(t).$$

In short: $\mathcal{C}^{bil}(m) \leq t \Longrightarrow \omega \leq \log_m(t)$ or $\mathcal{C}^{bil}(m) \leq t \Longrightarrow m^\omega \leq t$

In our new terminology: $R(\langle m, m, m \rangle) \leq t \Longrightarrow m^\omega \leq t$

Theorem 1

$$R(\langle m, n, p \rangle) \leq t \Longrightarrow (mnp)^{\omega/3} \leq t$$

<u>Proof</u>: consider $T = \langle m, n, p \rangle \otimes \langle n, p, m \rangle \otimes \langle p, m, n \rangle$

The first Inequality

Remember:

Proposition 1

Let m and t be any positive integers. Suppose that there exists a bilinear algorithm that computes the product of two $m \times m$ matrices with bilinear complexity t. Then

$$\omega \leq \log_m(t).$$

In short: $\mathcal{C}^{bil}(m) \leq t \Longrightarrow \omega \leq \log_m(t)$ or $\mathcal{C}^{bil}(m) \leq t \Longrightarrow m^{\omega} \leq t$

In our new terminology: $R(\langle m, m, m\rangle) \leq t \Longrightarrow m^{\omega} \leq t$

Theorem 1

$$R(\langle m, n, p\rangle) \leq t \Longrightarrow (mnp)^{\omega/3} \leq t$$

Proof: consider $T = \langle m, n, p\rangle \otimes \langle n, p, m\rangle \otimes \langle p, m, n\rangle$

$$T \cong \langle mnp, mnp, mnp\rangle$$

The first Inequality

Remember:

Proposition 1

Let m and t be any positive integers. Suppose that there exists a bilinear algorithm that computes the product of two $m \times m$ matrices with bilinear complexity t. Then

$$\omega \leq \log_m(t).$$

In short: $\mathcal{C}^{bil}(m) \leq t \Longrightarrow \omega \leq \log_m(t)$ or $\mathcal{C}^{bil}(m) \leq t \Longrightarrow m^\omega \leq t$

In our new terminology: $R(\langle m, m, m \rangle) \leq t \Longrightarrow m^\omega \leq t$

Theorem 1

$$R(\langle m, n, p \rangle) \leq t \Longrightarrow (mnp)^{\omega/3} \leq t$$

Proof: consider $T = \langle m, n, p \rangle \otimes \langle n, p, m \rangle \otimes \langle p, m, n \rangle$

$$T \cong \langle mnp, mnp, mnp \rangle$$

$$R(T) \leq R(\langle m, n, p \rangle) \times R(\langle n, p, m \rangle) \times R(\langle p, m, n \rangle)$$

The first Inequality

Remember:

Proposition 1

Let m and t be any positive integers. Suppose that there exists a bilinear algorithm that computes the product of two $m \times m$ matrices with bilinear complexity t. Then

$$\omega \leq \log_m(t).$$

In short: $\mathcal{C}^{bil}(m) \leq t \Longrightarrow \omega \leq \log_m(t)$ or $\mathcal{C}^{bil}(m) \leq t \Longrightarrow m^\omega \leq t$

In our new terminology: $R(\langle m, m, m\rangle) \leq t \Longrightarrow m^\omega \leq t$

Theorem 1

$$R(\langle m, n, p\rangle) \leq t \Longrightarrow (mnp)^{\omega/3} \leq t$$

Proof: consider $T = \langle m, n, p\rangle \otimes \langle n, p, m\rangle \otimes \langle p, m, n\rangle$

$$T \cong \langle mnp, mnp, mnp\rangle$$

$$R(T) \leq R(\langle m, n, p\rangle) \times R(\langle n, p, m\rangle) \times R(\langle p, m, n\rangle) = R(\langle m, n, p\rangle)^3$$

The first Inequality

Remember:

Proposition 1

Let m and t be any positive integers. Suppose that there exists a bilinear algorithm that computes the product of two $m \times m$ matrices with bilinear complexity t. Then

$$\omega \leq \log_m(t).$$

In short: $\mathcal{C}^{bil}(m) \leq t \Longrightarrow \omega \leq \log_m(t)$ or $\mathcal{C}^{bil}(m) \leq t \Longrightarrow m^\omega \leq t$

In our new terminology: $R(\langle m, m, m \rangle) \leq t \Longrightarrow m^\omega \leq t$

Theorem 1

$$R(\langle m, n, p \rangle) \leq t \Longrightarrow (mnp)^{\omega/3} \leq t$$

Proof: consider $T = \langle m, n, p \rangle \otimes \langle n, p, m \rangle \otimes \langle p, m, n \rangle$

$$T \cong \langle mnp, mnp, mnp \rangle$$

$$R(T) \leq R(\langle m, n, p \rangle) \times R(\langle n, p, m \rangle) \times R(\langle p, m, n \rangle) = R(\langle m, n, p \rangle)^3 \leq t^3$$

The first Inequality

Remember:

Proposition 1

Let m and t be any positive integers. Suppose that there exists a bilinear algorithm that computes the product of two $m \times m$ matrices with bilinear complexity t. Then

$$\omega \leq \log_m(t).$$

In short: $\mathcal{C}^{bil}(m) \leq t \Longrightarrow \omega \leq \log_m(t)$ or $\mathcal{C}^{bil}(m) \leq t \Longrightarrow m^\omega \leq t$

In our new terminology: $R(\langle m, m, m\rangle) \leq t \Longrightarrow m^\omega \leq t$

Theorem 1

$$R(\langle m, n, p\rangle) \leq t \Longrightarrow (mnp)^{\omega/3} \leq t$$

Proof: consider $T = \langle m, n, p\rangle \otimes \langle n, p, m\rangle \otimes \langle p, m, n\rangle$

$$T \cong \langle mnp, mnp, mnp\rangle$$

$$R(T) \leq R(\langle m, n, p\rangle) \times R(\langle n, p, m\rangle) \times R(\langle p, m, n\rangle) = R(\langle m, n, p\rangle)^3 \leq t^3$$

$$\Longrightarrow R(\langle mnp, mnp, mnp\rangle) \leq t^3$$

The first Inequality

Remember:

Proposition 1

Let m and t be any positive integers. Suppose that there exists a bilinear algorithm that computes the product of two $m \times m$ matrices with bilinear complexity t. Then

$$\omega \leq \log_m(t).$$

In short: $\mathcal{C}^{bil}(m) \leq t \Longrightarrow \omega \leq \log_m(t)$ or $\mathcal{C}^{bil}(m) \leq t \Longrightarrow m^\omega \leq t$

In our new terminology: $R(\langle m, m, m \rangle) \leq t \Longrightarrow m^\omega \leq t$

Theorem 1

$$R(\langle m, n, p \rangle) \leq t \Longrightarrow (mnp)^{\omega/3} \leq t$$

Proof: consider $T = \langle m, n, p \rangle \otimes \langle n, p, m \rangle \otimes \langle p, m, n \rangle$

$$T \cong \langle mnp, mnp, mnp \rangle$$

$$R(T) \leq R(\langle m, n, p \rangle) \times R(\langle n, p, m \rangle) \times R(\langle p, m, n \rangle) = R(\langle m, n, p \rangle)^3 \leq t^3$$

$$\Longrightarrow R(\langle mnp, mnp, mnp \rangle) \leq t^3 \quad \overset{\text{Prop 1}}{\Longrightarrow} (mnp)^\omega \leq t^3$$

QED

The first Inequality: applications

Theorem 1

$$R(\langle m, n, p\rangle) \leq t \implies (mnp)^{\omega/3} \leq t$$

Strassen 1969: $R(\langle 2, 2, 2\rangle) \leq 7 \implies \omega < 2.81$

The first Inequality: applications

Theorem 1

$$R(\langle m,n,p\rangle) \leq t \implies (mnp)^{\omega/3} \leq t$$

Strassen 1969:

$$R(\langle 2,2,2\rangle) \leq 7 \implies \omega < 2.81$$
$$R(\langle 2,3,3\rangle) \leq 15 \implies \omega < 2.82$$

The first Inequality: applications

Theorem 1

$$R(\langle m,n,p\rangle) \le t \implies (mnp)^{\omega/3} \le t$$

Strassen 1969:

$$R(\langle 2,2,2\rangle) \le 7 \implies \omega < 2.81$$

$$R(\langle 2,3,3\rangle) \le 15 \implies \omega < 2.82$$

Laderman 1976:

$$R(\langle 3,3,3\rangle) \le 23 \implies \omega < 2.86$$

The first Inequality: applications

Theorem 1

$$R(\langle m,n,p\rangle) \le t \implies (mnp)^{\omega/3} \le t$$

Strassen 1969: $R(\langle 2,2,2\rangle) \le 7 \implies \omega < 2.81$

$R(\langle 2,3,3\rangle) \le 15 \implies \omega < 2.82$

Laderman 1976: $R(\langle 3,3,3\rangle) \le 23 \implies \omega < 2.86$

Pan 1978: $R(\langle 70,70,70\rangle) \le 143640 \implies \omega < 2.795...$

using “trilinear aggregation”

The first Inequality: applications

Theorem 1

$$R(\langle m, n, p\rangle) \leq t \Longrightarrow (mnp)^{\omega/3} \leq t$$

Strassen 1969: $R(\langle 2, 2, 2\rangle) \leq 7 \Longrightarrow \omega < 2.81$

$R(\langle 2, 3, 3\rangle) \leq 15 \Longrightarrow \omega < 2.82$

Laderman 1976: $R(\langle 3, 3, 3\rangle) \leq 23 \Longrightarrow \omega < 2.86$

Pan 1978: $R(\langle 70, 70, 70\rangle) \leq 143640 \Longrightarrow \omega < 2.795...$

using “trilinear aggregation”

Pan 1979: $R(\langle ?, ?, ?\rangle) \leq ? \Longrightarrow \omega < 2.781...$

History of the main improvements on the exponent of square matrix multiplication

Upper bound	Year	Authors
$\omega \leq 3$		
$\omega < 2.81$	1969	Strassen
$\omega < 2.79$	1979	Pan
$\omega < 2.78$	1979	Bini, Capovani, Romani and Lotti
$\omega < 2.55$	1981	Schönhage
$\omega < 2.53$	1981	Pan
$\omega < 2.52$	1982	Romani
$\omega < 2.50$	1982	Coppersmith and Winograd
$\omega < 2.48$	1986	Strassen
$\omega < 2.376$	1987	Coppersmith and Winograd
$\omega < 2.373$	2010	Stothers
$\omega < 2.3729$	2012	Vassilevska Williams
$\omega < 2.3728639$	2014	LG

rank and Theorem 1

General Tensors (Section 3.1)

Consider three vector spaces U, V and W over $\mathbb{F}$

General Tensors (Section 3.1)

Consider three vector spaces U, V and W over $\mathbb{F}$

Take bases of U, V and W:

$$U = span\{x_1, \ldots, x_{\mathrm{dim}(U)}\}$$
$$V = span\{y_1, \ldots, y_{\mathrm{dim}(V)}\}$$
$$W = span\{z_1, \ldots, z_{\mathrm{dim}(W)}\}$$

General Tensors (Section 3.1)

Consider three vector spaces U, V and W over $\mathbb{F}$

Take bases of U, V and W:

$$U = span\{x_1, \ldots, x_{\dim(U)}\}$$
$$V = span\{y_1, \ldots, y_{\dim(V)}\}$$
$$W = span\{z_1, \ldots, z_{\dim(W)}\}$$

A tensor over (U, V, W) is an element of $U \otimes V \otimes W$

General Tensors (Section 3.1)

Consider three vector spaces U, V and W over $\mathbb{F}$

Take bases of U, V and W:

$$U = span\{x_1, \ldots, x_{\dim(U)}\}$$
$$V = span\{y_1, \ldots, y_{\dim(V)}\}$$
$$W = span\{z_1, \ldots, z_{\dim(W)}\}$$

A tensor over (U, V, W) is an element of $U \otimes V \otimes W$

i.e., a formal sum $T = \sum_{u=1}^{\dim(U)} \sum_{v=1}^{\dim(V)} \sum_{w=1}^{\dim(W)} d_{uvw} \, x_u \otimes y_v \otimes z_w$

General Tensors (Section 3.1)

Consider three vector spaces U, V and W over $\mathbb{F}$

Take bases of U, V and W:

$$U = span\{x_1, \ldots, x_{\dim(U)}\}$$
$$V = span\{y_1, \ldots, y_{\dim(V)}\}$$
$$W = span\{z_1, \ldots, z_{\dim(W)}\}$$

A tensor over (U, V, W) is an element of $U \otimes V \otimes W$

i.e., a formal sum $T = \sum_{u=1}^{\dim(U)} \sum_{v=1}^{\dim(V)} \sum_{w=1}^{\dim(W)} \underbrace{d_{uvw}}_{\in \mathbb{F}} x_u \otimes y_v \otimes z_w$

General Tensors (Section 3.1)

Consider three vector spaces U, V and W over $\mathbb{F}$

Take bases of U, V and W:

$$U = span\{x_1, \ldots, x_{\dim(U)}\}$$
$$V = span\{y_1, \ldots, y_{\dim(V)}\}$$
$$W = span\{z_1, \ldots, z_{\dim(W)}\}$$

A tensor over (U, V, W) is an element of $U \otimes V \otimes W$

i.e., a formal sum $T = \sum_{u=1}^{\dim(U)} \sum_{v=1}^{\dim(V)} \sum_{w=1}^{\dim(W)} \underbrace{d_{uvw}}_{\in \mathbb{F}} x_u \otimes y_v \otimes z_w$

"a three-dimension array with $\dim(U) \times \dim(V) \times \dim(W)$ entries in $\mathbb{F}$"

General Tensors (Section 3.1)

A tensor over (U, V, W) is an element of $U \otimes V \otimes W$

i.e., a formal sum $T = \sum_{u=1}^{\dim(U)} \sum_{v=1}^{\dim(V)} \sum_{w=1}^{\dim(W)} \underbrace{d_{uvw}}_{\in \mathbb{F}} x_u \otimes y_v \otimes z_w$

General Tensors (Section 3.1)

A tensor over (U, V, W) is an element of $U \otimes V \otimes W$

i.e., a formal sum $T = \sum_{u=1}^{\dim(U)} \sum_{v=1}^{\dim(V)} \sum_{w=1}^{\dim(W)} \underbrace{d_{uvw}}_{\in \mathbb{F}} x_u \otimes y_v \otimes z_w$

Definition 1

The tensor corresponding to the multiplication of an $m \times n$ matrix by an $n \times p$ matrix is

$$\langle m, n, p \rangle = \sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} a_{ik} \otimes b_{kj} \otimes c_{ij}.$$

General Tensors (Section 3.1)

A tensor over (U, V, W) is an element of $U \otimes V \otimes W$

i.e., a formal sum $T = \sum_{u=1}^{\dim(U)} \sum_{v=1}^{\dim(V)} \sum_{w=1}^{\dim(W)} \underbrace{d_{uvw}}_{\in \mathbb{F}} x_u \otimes y_v \otimes z_w$

Definition 1

The tensor corresponding to the multiplication of an $m \times n$ matrix by an $n \times p$ matrix is

$$\langle m, n, p \rangle = \sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} a_{ik} \otimes b_{kj} \otimes c_{ij}.$$

General Tensors (Section 3.1)

A tensor over (U, V, W) is an element of $U \otimes V \otimes W$

i.e., a formal sum $T = \sum_{u=1}^{\dim(U)} \sum_{v=1}^{\dim(V)} \sum_{w=1}^{\dim(W)} \underbrace{d_{uvw}}_{\in \mathbb{F}} x_u \otimes y_v \otimes z_w$

$\dim(U) = mn$, $\dim(V) = np$ and $\dim(W) = mp$

Definition 1

The tensor corresponding to the multiplication of an $m \times n$ matrix by an $n \times p$ matrix is

$$\langle m, n, p \rangle = \sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} a_{ik} \otimes b_{kj} \otimes c_{ij}.$$

General Tensors (Section 3.1)

A tensor over (U, V, W) is an element of $U \otimes V \otimes W$

i.e., a formal sum $T = \sum_{u=1}^{\dim(U)} \sum_{v=1}^{\dim(V)} \sum_{w=1}^{\dim(W)} \underbrace{d_{uvw}}_{\in \mathbb{F}} x_u \otimes y_v \otimes z_w$

$\dim(U) = mn$, $\dim(V) = np$ and $\dim(W) = mp$

$$U = span\left\{ \{a_{ik}\}_{1 \le i \le m, 1 \le k \le n} \right\}$$

$$V = span\left\{ \{b_{k'j}\}_{1 \le k' \le n, 1 \le j \le p} \right\}$$

$$W = span\left\{ \{c_{i'j'}\}_{1 \le i' \le m, 1 \le j' \le p} \right\}$$

Definition 1

The tensor corresponding to the multiplication of an $m \times n$ matrix by an $n \times p$ matrix is

$$\langle m, n, p \rangle = \sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} a_{ik} \otimes b_{kj} \otimes c_{ij}.$$

General Tensors (Section 3.1)

A tensor over (U, V, W) is an element of $U \otimes V \otimes W$

i.e., a formal sum $T = \sum_{u=1}^{\dim(U)} \sum_{v=1}^{\dim(V)} \sum_{w=1}^{\dim(W)} \underbrace{d_{uvw}}_{\in \mathbb{F}} x_u \otimes y_v \otimes z_w$

$\dim(U) = mn$, $\dim(V) = np$ and $\dim(W) = mp$

$$U = span\left\{\{a_{ik}\}_{1 \le i \le m, 1 \le k \le n}\right\}$$

$$V = span\left\{\{b_{k'j}\}_{1 \le k' \le n, 1 \le j \le p}\right\}$$

$$W = span\left\{\{c_{i'j'}\}_{1 \le i' \le m, 1 \le j' \le p}\right\}$$

$$d_{ikk'ji'j'} = \begin{cases} 1 & \text{if } i = i', j = j', k = k' \\ 0 & \text{otherwise} \end{cases}$$

Definition 1

The tensor corresponding to the multiplication of an $m \times n$ matrix by an $n \times p$ matrix is

$$\langle m, n, p \rangle = \sum_{i=1}^{m} \sum_{j=1}^{p} \sum_{k=1}^{n} a_{ik} \otimes b_{kj} \otimes c_{ij}.$$

The rank of a tensor (Section 3.2)

Definition 2

Let T be a tensor over (U, V, W). The rank of T, denoted $R(T)$, is the minimal integer t for which T can be written as

$$T = \sum_{s=1}^{t} \left[\left(\sum_{u=1}^{\dim(U)} \alpha_{su} x_u \right) \otimes \left(\sum_{v=1}^{\dim(V)} \beta_{sv} y_v \right) \otimes \left(\sum_{w=1}^{\dim(W)} \gamma_{sw} z_w \right) \right],$$

for some constants $\alpha_{su}, \beta_{sv}, \gamma_{sw}$ in $\mathbb{F}$.

The rank of a tensor (Section 3.2)

Definition 2

Let T be a tensor over (U, V, W). The rank of T, denoted $R(T)$, is the minimal integer t for which T can be written as

$$T = \sum_{s=1}^{t} \left[\underbrace{\boxed{\left(\sum_{u=1}^{\dim(U)} \alpha_{su} x_u \right)}}_{\text{lin. comb. of the } x_u\text{'s}} \otimes \underbrace{\boxed{\left(\sum_{v=1}^{\dim(V)} \beta_{sv} y_v \right)}}_{\text{lin. comb. of the } y_v\text{'s}} \otimes \underbrace{\boxed{\left(\sum_{w=1}^{\dim(W)} \gamma_{sw} z_w \right)}}_{\text{lin. comb. of the } z_w\text{'s}} \right],$$

for some constants $\alpha_{su}, \beta_{sv}, \gamma_{sw}$ in $\mathbb{F}$.

The rank of a tensor (Section 3.2)

Definition 2

Let T be a tensor over (U, V, W). The rank of T, denoted $R(T)$, is the minimal integer t for which T can be written as

$$T = \sum_{s=1}^{t} \Bigg[\underbrace{\underbrace{\left(\sum_{u=1}^{\dim(U)} \alpha_{su} x_u \right)}_{\text{lin. comb. of the } x_u\text{'s}} \otimes \underbrace{\left(\sum_{v=1}^{\dim(V)} \beta_{sv} y_v \right)}_{\text{lin. comb. of the } y_v\text{'s}} \otimes \underbrace{\left(\sum_{w=1}^{\dim(W)} \gamma_{sw} z_w \right)}_{\text{lin. comb. of the } z_w\text{'s}}}_{\text{tensor of rank 1}} \Bigg],$$

for some constants $\alpha_{su}, \beta_{sv}, \gamma_{sw}$ in $\mathbb{F}$.

Overview of the Lectures

✓ Fundamental techniques for fast matrix multiplication (1969~1987)

- ➢ Basics of bilinear complexity theory: exponent of matrix multiplication, Strassen's algorithm, bilinear algorithms
- ➢ First technique: tensor rank and recursion
- ➢ Second technique: border rank
- ➢ Third technique: the asymptotic sum inequality
- ➢ Fourth technique: the laser method

Lecture 1

✓ Recent progress on matrix multiplication (1987~)

- ➢ Laser method on powers of tensors
- ➢ Other approaches
- ➢ Lower bounds
- ➢ Rectangular matrix multiplication

Lecture 2

✓ Applications of matrix multiplications, open problems

Lecture 3

The border rank of a tensor (Section 4.1)

Let λ be an indeterminate

$\mathbb{F}[\lambda]$ denotes the ring of polynomials in λ with coefficients in $\mathbb{F}$

Definition 3

Let T be a tensor over (U, V, W). The border rank of T, denoted $\underline{R}(T)$, is the minimal integer t for which there exist an integer $c \geq 0$ and a tensor T'' such that T can be written as

$$\lambda^c T = \sum_{s=1}^{t} \left[\left(\sum_{u=1}^{\dim(U)} \alpha_{su} x_u \right) \otimes \left(\sum_{v=1}^{\dim(V)} \beta_{sv} y_v \right) \otimes \left(\sum_{w=1}^{\dim(W)} \gamma_{sw} z_w \right) \right] + \lambda^{c+1} T'',$$

for some constants $\alpha_{su}, \beta_{sv}, \gamma_{sw}$ in $\mathbb{F}[\lambda]$.

The border rank of a tensor (Section 4.1)

Let λ be an indeterminate

$\mathbb{F}[\lambda]$ denotes the ring of polynomials in λ with coefficients in $\mathbb{F}$

Definition 3

Let T be a tensor over (U, V, W). The border rank of T, denoted $\underline{R}(T)$, is the minimal integer t for which there exist an integer $c \geq 0$ and a tensor T'' such that T can be written as

$$\lambda^c T = \sum_{s=1}^{t} \left[\boxed{\left(\sum_{u=1}^{\dim(U)} \alpha_{su} x_u \right)} \otimes \boxed{\left(\sum_{v=1}^{\dim(V)} \beta_{sv} y_v \right)} \otimes \boxed{\left(\sum_{w=1}^{\dim(W)} \gamma_{sw} z_w \right)} \right] + \lambda^{c+1} T'',$$

for some constants $\alpha_{su}, \beta_{sv}, \gamma_{sw}$ in $\mathbb{F}[\lambda]$.

lin. comb. of the x_u's

lin. comb. of the y_v's

lin. comb. of the z_w's

The border rank of a tensor (Section 4.1)

Let λ be an indeterminate

$\mathbb{F}[\lambda]$ denotes the ring of polynomials in λ with coefficients in $\mathbb{F}$

Definition 3

Let T be a tensor over (U, V, W). The border rank of T, denoted $\underline{R}(T)$, is the minimal integer t for which there exist an integer $c \geq 0$ and a tensor T'' such that T can be written as

$$\lambda^c T = \sum_{s=1}^{t} \left[\left(\sum_{u=1}^{\dim(U)} \alpha_{su} x_u \right) \otimes \left(\sum_{v=1}^{\dim(V)} \beta_{sv} y_v \right) \otimes \left(\sum_{w=1}^{\dim(W)} \gamma_{sw} z_w \right) \right] + \lambda^{c+1} T'',$$

for some constants $\alpha_{su}, \beta_{sv}, \gamma_{sw}$ in $\mathbb{F}[\lambda]$.

The border rank of a tensor (Section 4.1)

Let λ be an indeterminate

$\mathbb{F}[\lambda]$ denotes the ring of polynomials in λ with coefficients in $\mathbb{F}$

Definition 3

Let T be a tensor over (U, V, W). The border rank of T, denoted $\underline{R}(T)$, is the minimal integer t for which there exist an integer $c \geq 0$ and a tensor T'' such that T can be written as

$$\lambda^c T = \sum_{s=1}^{t} \left[\left(\sum_{u=1}^{\dim(U)} \alpha_{su} x_u \right) \otimes \left(\sum_{v=1}^{\dim(V)} \beta_{sv} y_v \right) \otimes \left(\sum_{w=1}^{\dim(W)} \gamma_{sw} z_w \right) \right] + \lambda^{c+1} T'',$$

for some constants $\alpha_{su}, \beta_{sv}, \gamma_{sw}$ in $\mathbb{F}[\lambda]$.

Obviously, $\underline{R}(T) \leq R(T)$ for any tensor T.

Example

Construction by Bini, Capovani, Romani and Lotti (1979):

$$
\begin{aligned}
T_{\mathsf{Bini}} &= \sum_{\substack{1\le i,j,k\le 2\\ (i,k)\neq(2,2)}} a_{ik}\otimes b_{kj}\otimes c_{ij} \\
&= \quad a_{11}\otimes b_{11}\otimes c_{11} + a_{12}\otimes b_{21}\otimes c_{11} + a_{11}\otimes b_{12}\otimes c_{12} \\
&\quad + a_{12}\otimes b_{22}\otimes c_{12} + a_{21}\otimes b_{11}\otimes c_{21} + a_{21}\otimes b_{12}\otimes c_{22}
\end{aligned}
$$

Example

Construction by Bini, Capovani, Romani and Lotti (1979):

$$\begin{aligned} T_{\text{Bini}} &= \sum_{\substack{1\le i,j,k\le 2\\(i,k)\neq(2,2)}} a_{ik}\otimes b_{kj}\otimes c_{ij} \\ &= a_{11}\otimes b_{11}\otimes c_{11} + a_{12}\otimes b_{21}\otimes c_{11} + a_{11}\otimes b_{12}\otimes c_{12} \\ &\quad + a_{12}\otimes b_{22}\otimes c_{12} + a_{21}\otimes b_{11}\otimes c_{21} + a_{21}\otimes b_{12}\otimes c_{22} \end{aligned}$$

same as $\langle 2,2,2\rangle$, but without $a_{22}\otimes b_{21}\otimes c_{21}$ and $a_{22}\otimes b_{22}\otimes c_{22}$

Example

Construction by Bini, Capovani, Romani and Lotti (1979):

$$\begin{aligned} T_{\mathsf{Bini}} &= \sum_{\substack{1\le i,j,k\le 2\\(i,k)\neq(2,2)}} a_{ik}\otimes b_{kj}\otimes c_{ij} \\ &= a_{11}\otimes b_{11}\otimes c_{11}+a_{12}\otimes b_{21}\otimes c_{11}+a_{11}\otimes b_{12}\otimes c_{12} \\ &\quad + a_{12}\otimes b_{22}\otimes c_{12}+a_{21}\otimes b_{11}\otimes c_{21}+a_{21}\otimes b_{12}\otimes c_{22} \end{aligned}$$

same as $\langle 2,2,2\rangle$, but $\underbrace{\text{without } a_{22}\otimes b_{21}\otimes c_{21} \text{ and } a_{22}\otimes b_{22}\otimes c_{22}}_{a_{22}=0}$

Example

Construction by Bini, Capovani, Romani and Lotti (1979):

$$\begin{aligned} T_{\mathsf{Bini}} &= \sum_{\substack{1\leq i,j,k\leq 2 \\ (i,k)\neq(2,2)}} a_{ik} \otimes b_{kj} \otimes c_{ij} \\ &= a_{11} \otimes b_{11} \otimes c_{11} + a_{12} \otimes b_{21} \otimes c_{11} + a_{11} \otimes b_{12} \otimes c_{12} \\ &\quad + a_{12} \otimes b_{22} \otimes c_{12} + a_{21} \otimes b_{11} \otimes c_{21} + a_{21} \otimes b_{12} \otimes c_{22} \end{aligned}$$

same as $\langle 2,2,2\rangle$, but $\underbrace{\text{without } a_{22} \otimes b_{21} \otimes c_{21} \text{ and } a_{22} \otimes b_{22} \otimes c_{22}}_{a_{22}=0}$

T_{Bini} represents $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

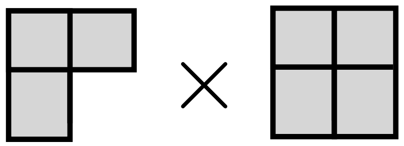
Example

Construction by Bini, Capovani, Romani and Lotti (1979):

$$
\begin{aligned}
T_{\text{Bini}} &= \sum_{\substack{1\le i,j,k\le 2\\ (i,k)\neq(2,2)}} a_{ik}\otimes b_{kj}\otimes c_{ij}\\
&= \quad a_{11}\otimes b_{11}\otimes c_{11} + a_{12}\otimes b_{21}\otimes c_{11} + a_{11}\otimes b_{12}\otimes c_{12}\\
&\quad + a_{12}\otimes b_{22}\otimes c_{12} + a_{21}\otimes b_{11}\otimes c_{21} + a_{21}\otimes b_{12}\otimes c_{22}
\end{aligned}
$$

$$\text{same as } \langle 2,2,2\rangle\text{, but } \underbrace{\text{without } a_{22}\otimes b_{21}\otimes c_{21} \text{ and } a_{22}\otimes b_{22}\otimes c_{22}}_{a_{22}=0}$$

T_{Bini} represents $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$



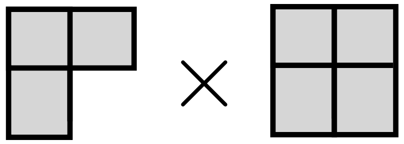
Example

Construction by Bini, Capovani, Romani and Lotti (1979):

$$
\begin{aligned}
T_{\text{Bini}} &= \sum_{\substack{1\le i,j,k\le 2\\(i,k)\neq(2,2)}} a_{ik}\otimes b_{kj}\otimes c_{ij}\\
&= \quad a_{11}\otimes b_{11}\otimes c_{11} + a_{12}\otimes b_{21}\otimes c_{11} + a_{11}\otimes b_{12}\otimes c_{12}\\
&\quad + a_{12}\otimes b_{22}\otimes c_{12} + a_{21}\otimes b_{11}\otimes c_{21} + a_{21}\otimes b_{12}\otimes c_{22}
\end{aligned}
$$

same as $\langle 2,2,2\rangle$, but $\underbrace{\text{without } a_{22}\otimes b_{21}\otimes c_{21} \text{ and } a_{22}\otimes b_{22}\otimes c_{22}}_{a_{22}=0}$

T_{Bini} represents $\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & 0\end{pmatrix}\times\begin{pmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}$



$R(T_{\text{Bini}}) = 6$

Example

Construction by Bini, Capovani, Romani and Lotti (1979):

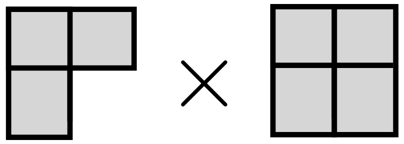
$$T_{\text{Bini}} = \sum_{\substack{1\leq i,j,k\leq 2\\(i,k)\neq(2,2)}} a_{ik}\otimes b_{kj}\otimes c_{ij}$$

$$= a_{11}\otimes b_{11}\otimes c_{11} + a_{12}\otimes b_{21}\otimes c_{11} + a_{11}\otimes b_{12}\otimes c_{12}$$

$$+ a_{12}\otimes b_{22}\otimes c_{12} + a_{21}\otimes b_{11}\otimes c_{21} + a_{21}\otimes b_{12}\otimes c_{22}$$

same as $\langle 2,2,2\rangle$, but $\underbrace{\text{without } a_{22}\otimes b_{21}\otimes c_{21} \text{ and } a_{22}\otimes b_{22}\otimes c_{22}}_{a_{22}=0}$

T_{Bini} represents $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$



$R(T_{\text{Bini}}) = 6$

Bini et al. showed that $\underline{R}(T_{\text{Bini}}) = 5$

Example

Construction by Bini, Capovani, Romani and Lotti (1979): $\underline{R}(T_{\mathsf{Bini}}) \leq 5$

$$\boxed{\begin{aligned} T_{\mathsf{Bini}} = \quad & a_{11} \otimes b_{11} \otimes c_{11} + a_{12} \otimes b_{21} \otimes c_{11} + a_{11} \otimes b_{12} \otimes c_{12} \\ & + a_{12} \otimes b_{22} \otimes c_{12} + a_{21} \otimes b_{11} \otimes c_{21} + a_{21} \otimes b_{12} \otimes c_{22} \end{aligned}}$$

Example

Construction by Bini, Capovani, Romani and Lotti (1979): $\underline{R}(T_{\mathsf{Bini}}) \leq 5$

$$T_{\mathsf{Bini}} = \begin{aligned}[t] & a_{11} \otimes b_{11} \otimes c_{11} + a_{12} \otimes b_{21} \otimes c_{11} + a_{11} \otimes b_{12} \otimes c_{12} \\ & + a_{12} \otimes b_{22} \otimes c_{12} + a_{21} \otimes b_{11} \otimes c_{21} + a_{21} \otimes b_{12} \otimes c_{22} \end{aligned}$$

$$\lambda T_{\mathsf{Bini}} = T' + \lambda^2 T''$$

Example

Construction by Bini, Capovani, Romani and Lotti (1979): $\underline{R}(T_{\text{Bini}}) \leq 5$

$$\boxed{\begin{aligned} T_{\text{Bini}} = \quad & a_{11} \otimes b_{11} \otimes c_{11} + a_{12} \otimes b_{21} \otimes c_{11} + a_{11} \otimes b_{12} \otimes c_{12} \\ & + a_{12} \otimes b_{22} \otimes c_{12} + a_{21} \otimes b_{11} \otimes c_{21} + a_{21} \otimes b_{12} \otimes c_{22} \end{aligned}}$$

$$\boxed{\lambda T_{\text{Bini}} = T' + \lambda^2 T''}$$

where $$\begin{aligned} T' =& (a_{12} + \lambda a_{11}) \otimes (b_{12} + \lambda b_{22}) \otimes c_{12} \\ & + (a_{21} + \lambda a_{11}) \otimes b_{11} \otimes (c_{11} + \lambda c_{21}) \\ & - a_{12} \otimes b_{12} \otimes (c_{11} + c_{12} + \lambda c_{22}) \\ & - a_{21} \otimes (b_{11} + b_{12} + \lambda b_{21}) \otimes c_{11} \\ & + (a_{12} + a_{21}) \otimes (b_{12} + \lambda b_{21}) \otimes (c_{11} + \lambda c_{22}) \end{aligned}$$

and $T'' = a_{11} \otimes b_{22} \otimes c_{12} + a_{11} \otimes b_{11} \otimes c_{21} + (a_{12} + a_{21}) \otimes b_{21} \otimes c_{22}.$

Example

Construction by Bini, Capovani, Romani and Lotti (1979): $\underline{R}(T_{\text{Bini}}) \leq 5$

$$\boxed{\begin{aligned} T_{\text{Bini}} = \quad & a_{11} \otimes b_{11} \otimes c_{11} + a_{12} \otimes b_{21} \otimes c_{11} + a_{11} \otimes b_{12} \otimes c_{12} \\ & + a_{12} \otimes b_{22} \otimes c_{12} + a_{21} \otimes b_{11} \otimes c_{21} + a_{21} \otimes b_{12} \otimes c_{22} \end{aligned}}$$

$$\boxed{\lambda T_{\text{Bini}} = T' + \lambda^2 T''}$$

c = 1

where $$\left.\begin{aligned} T' = & (a_{12} + \lambda a_{11}) \otimes (b_{12} + \lambda b_{22}) \otimes c_{12} \\ & + (a_{21} + \lambda a_{11}) \otimes b_{11} \otimes (c_{11} + \lambda c_{21}) \\ & - a_{12} \otimes b_{12} \otimes (c_{11} + c_{12} + \lambda c_{22}) \\ & - a_{21} \otimes (b_{11} + b_{12} + \lambda b_{21}) \otimes c_{11} \\ & + (a_{12} + a_{21}) \otimes (b_{12} + \lambda b_{21}) \otimes (c_{11} + \lambda c_{22}) \end{aligned}\right]$$

t = 5 rank-one terms

and $$T'' = a_{11} \otimes b_{22} \otimes c_{12} + a_{11} \otimes b_{11} \otimes c_{21} + (a_{12} + a_{21}) \otimes b_{21} \otimes c_{22}.$$

Example

Construction by Bini, Capovani, Romani and Lotti (1979): $\underline{R}(T_{\text{Bini}}) \leq 5$

$$\begin{aligned} T_{\text{Bini}} = \quad & a_{11} \otimes b_{11} \otimes c_{11} + a_{12} \otimes b_{21} \otimes c_{11} + a_{11} \otimes b_{12} \otimes c_{12} \\ & + a_{12} \otimes b_{22} \otimes c_{12} + a_{21} \otimes b_{11} \otimes c_{21} + a_{21} \otimes b_{12} \otimes c_{22} \end{aligned}$$

$$\lambda T_{\text{Bini}} = T' + \lambda^2 T''$$

$c = 1$ $t = 5$

Definition 3

Let T be a tensor over (U, V, W). The border rank of T, denoted $\underline{R}(T)$, is the minimal integer t for which there exist an integer $c \geq 0$ and a tensor T'' such that T can be written as

$$\lambda^c T = \sum_{s=1}^{t} \left[\left(\sum_{u=1}^{\dim(U)} \alpha_{su} x_u \right) \otimes \left(\sum_{v=1}^{\dim(V)} \beta_{sv} y_v \right) \otimes \left(\sum_{w=1}^{\dim(W)} \gamma_{sw} z_w \right) \right] + \lambda^{c+1} T'',$$

for some constants $\alpha_{su}, \beta_{sv}, \gamma_{sw}$ in $\mathbb{F}[\lambda]$.

Example

Construction by Bini, Capovani, Romani and Lotti (1979): $\underline{R}(T_{\text{Bini}}) \leq 5$

$$\boxed{\begin{aligned} T_{\text{Bini}} = \quad & a_{11} \otimes b_{11} \otimes c_{11} + a_{12} \otimes b_{21} \otimes c_{11} + a_{11} \otimes b_{12} \otimes c_{12} \\ & + a_{12} \otimes b_{22} \otimes c_{12} + a_{21} \otimes b_{11} \otimes c_{21} + a_{21} \otimes b_{12} \otimes c_{22} \end{aligned}}$$

$$\boxed{\lambda T_{\text{Bini}} = T' + \lambda^2 T''}$$

$c = 1$ $t = 5$

Definition 3

Let T be a tensor over (U, V, W). The border rank of T, denoted $\underline{R}(T)$, is the minimal integer t for which there exist an integer $c \geq 0$ and a tensor T'' such that T can be written as

$$\lambda^c T = \boxed{\sum_{s=1}^{t} \left[\left(\sum_{u=1}^{\dim(U)} \alpha_{su} x_u \right) \otimes \left(\sum_{v=1}^{\dim(V)} \beta_{sv} y_v \right) \otimes \left(\sum_{w=1}^{\dim(W)} \gamma_{sw} z_w \right) \right]} + \lambda^{c+1} T'',$$

T' that can be computed with t multiplications over $\mathbb{F}[\lambda]$

for some constants $\alpha_{su}, \beta_{sv}, \gamma_{sw}$ in $\mathbb{F}[\lambda]$.

Interpretation of the border rank

Definition 3

Let T be a tensor over (U, V, W). The border rank of T, denoted $\underline{R}(T)$, is the minimal integer t for which there exist an integer $c \geq 0$ and a tensor T'' such that T can be written as

$$\lambda^c T = \underbrace{\boxed{\sum_{s=1}^{t} \left[\left(\sum_{u=1}^{\dim(U)} \alpha_{su} x_u \right) \otimes \left(\sum_{v=1}^{\dim(V)} \beta_{sv} y_v \right) \otimes \left(\sum_{w=1}^{\dim(W)} \gamma_{sw} z_w \right) \right]}}_{\text{T' that can be computed with t multiplications over $\mathbb{F}[\lambda]$}} + \lambda^{c+1} T'',$$

for some constants $\alpha_{su}, \beta_{sv}, \gamma_{sw}$ in $\mathbb{F}[\lambda]$.

Interpretation of the border rank

$$T' = \lambda^c T - \lambda^{c+1} T''$$

Definition 3

Let T be a tensor over (U, V, W). The border rank of T, denoted $\underline{R}(T)$, is the minimal integer t for which there exist an integer $c \geq 0$ and a tensor T'' such that T can be written as

$$\lambda^c T = \boxed{\sum_{s=1}^{t} \left[\left(\sum_{u=1}^{\dim(U)} \alpha_{su} x_u \right) \otimes \left(\sum_{v=1}^{\dim(V)} \beta_{sv} y_v \right) \otimes \left(\sum_{w=1}^{\dim(W)} \gamma_{sw} z_w \right) \right]} + \lambda^{c+1} T'',$$

T' that can be computed with t multiplications over $\mathbb{F}[\lambda]$

for some constants $\alpha_{su}, \beta_{sv}, \gamma_{sw}$ in $\mathbb{F}[\lambda]$.

Interpretation of the border rank

$$T' = \lambda^c T - \lambda^{c+1} T''$$

we get T be computing T' and keeping the terms with the lowest degree in λ

Definition 3

Let T be a tensor over (U, V, W). The border rank of T, denoted $\underline{R}(T)$, is the minimal integer t for which there exist an integer $c \geq 0$ and a tensor T'' such that T can be written as

$$\lambda^c T = \underbrace{\boxed{\sum_{s=1}^{t} \left[\left(\sum_{u=1}^{\dim(U)} \alpha_{su} x_u \right) \otimes \left(\sum_{v=1}^{\dim(V)} \beta_{sv} y_v \right) \otimes \left(\sum_{w=1}^{\dim(W)} \gamma_{sw} z_w \right) \right]}}_{T' \text{ that can be computed with } t \text{ multiplications over } \mathbb{F}[\lambda]} + \lambda^{c+1} T'',$$

for some constants $\alpha_{su}, \beta_{sv}, \gamma_{sw}$ in $\mathbb{F}[\lambda]$.

Interpretation of the border rank

$$T' = \lambda^c T - \lambda^{c+1} T''$$

we get T be computing T' and keeping the terms with the lowest degree in λ

one can think of numerically (e.g., for $\mathbb{F} = \mathbb{R}$) taking λ very small and computing $\lambda^{-c} T'$

Definition 3

Let T be a tensor over (U, V, W). The border rank of T, denoted $\underline{R}(T)$, is the minimal integer t for which there exist an integer $c \geq 0$ and a tensor T'' such that T can be written as

$$\lambda^c T = \boxed{\sum_{s=1}^{t} \left[\left(\sum_{u=1}^{\dim(U)} \alpha_{su} x_u \right) \otimes \left(\sum_{v=1}^{\dim(V)} \beta_{sv} y_v \right) \otimes \left(\sum_{w=1}^{\dim(W)} \gamma_{sw} z_w \right) \right]} + \lambda^{c+1} T'',$$

T' that can be computed with t multiplications over $\mathbb{F}[\lambda]$

for some constants $\alpha_{su}, \beta_{sv}, \gamma_{sw}$ in $\mathbb{F}[\lambda]$.

Interpretation of the border rank

$T' = \lambda^c T - \lambda^{c+1} T''$

we get T be computing T' and keeping the terms with the lowest degree in λ

one can think of numerically (e.g., for $\mathbb{F} = \mathbb{R}$) taking λ very small and computing $\lambda^{-c} T' = T - \lambda T''$

Definition 3

Let T be a tensor over (U, V, W). The border rank of T, denoted $\underline{R}(T)$, is the minimal integer t for which there exist an integer $c \geq 0$ and a tensor T'' such that T can be written as

$$\lambda^c T = \boxed{\sum_{s=1}^{t} \left[\left(\sum_{u=1}^{\dim(U)} \alpha_{su} x_u \right) \otimes \left(\sum_{v=1}^{\dim(V)} \beta_{sv} y_v \right) \otimes \left(\sum_{w=1}^{\dim(W)} \gamma_{sw} z_w \right) \right]} + \lambda^{c+1} T'',$$

T' that can be computed with t multiplications over $\mathbb{F}[\lambda]$

for some constants $\alpha_{su}, \beta_{sv}, \gamma_{sw}$ in $\mathbb{F}[\lambda]$.

Interpretation of the border rank

$$T' = \lambda^c T - \lambda^{c+1} T''$$

we get T be computing T' and keeping the terms with the lowest degree in λ

one can think of numerically (e.g., for $\mathbb{F} = \mathbb{R}$) taking λ very small and computing $\lambda^{-c} T' = T - \lambda T'' \approx T$

Definition 3

Let T be a tensor over (U, V, W). The border rank of T, denoted $\underline{R}(T)$, is the minimal integer t for which there exist an integer $c \geq 0$ and a tensor T'' such that T can be written as

$$\lambda^c T = \boxed{\sum_{s=1}^{t} \left[\left(\sum_{u=1}^{\dim(U)} \alpha_{su} x_u \right) \otimes \left(\sum_{v=1}^{\dim(V)} \beta_{sv} y_v \right) \otimes \left(\sum_{w=1}^{\dim(W)} \gamma_{sw} z_w \right) \right]} + \lambda^{c+1} T'',$$

T' that can be computed with t multiplications over $\mathbb{F}[\lambda]$

for some constants $\alpha_{su}, \beta_{sv}, \gamma_{sw}$ in $\mathbb{F}[\lambda]$.

Interpretation of the border rank

$$T' = \lambda^c T - \lambda^{c+1} T''$$

we get T be computing T' and keeping the terms with the lowest degree in λ

one can think of numerically (e.g., for $\mathbb{F} = \mathbb{R}$) taking λ very small and computing $\lambda^{-c} T' = T - \lambda T'' \approx T$

“border rank = complexity of approximate bilinear algorithms”

Definition 3

Let T be a tensor over (U, V, W). The border rank of T, denoted $\underline{R}(T)$, is the minimal integer t for which there exist an integer $c \geq 0$ and a tensor T'' such that T can be written as

$$\lambda^c T = \underbrace{\boxed{\sum_{s=1}^{t} \left[\left(\sum_{u=1}^{\dim(U)} \alpha_{su} x_u \right) \otimes \left(\sum_{v=1}^{\dim(V)} \beta_{sv} y_v \right) \otimes \left(\sum_{w=1}^{\dim(W)} \gamma_{sw} z_w \right) \right]}}_{T' \text{ that can be computed with } t \text{ multiplications over } \mathbb{F}[\lambda]} + \lambda^{c+1} T'',$$

for some constants $\alpha_{su}, \beta_{sv}, \gamma_{sw}$ in $\mathbb{F}[\lambda]$.

Border rank v.s. rank

Obviously, $\underline{R}(T) \leq R(T)$ for any tensor T.

Border rank v.s. rank

Obviously, $\underline{R}(T) \leq R(T)$ for any tensor T.

Proposition 2

There exists a constant a such that $R(T) \leq a \times \underline{R}(T)$ for any tensor T.

Border rank v.s. rank

Obviously, $\underline{R}(T) \leq R(T)$ for any tensor T.

Proposition 2

There exists a constant a such that $R(T) \leq a \times \underline{R}(T)$ for any tensor T.

Remark: the constant a actually depends on the c in the definition of $\underline{R}(T)$ (for instance: $a = 3$ for $c = 1$)

Border rank v.s. rank

Obviously, $\underline{R}(T) \leq R(T)$ for any tensor T.

Proposition 2

There exists a constant a such that $R(T) \leq a \times \underline{R}(T)$ for any tensor T.

Remark: the constant a actually depends on the c in the definition of $\underline{R}(T)$ (for instance: $a = 3$ for $c = 1$)

Consequence: an approximate bilinear algorithm can be converted into an (usual) bilinear algorithm of “similar” complexity

Border rank v.s. rank

Proposition 2

There exists a constant a such that $R(T) \leq a \times \underline{R}(T)$ for any tensor T.

Proof outline (for c = 1):

Border rank v.s. rank

Proposition 2

There exists a constant a such that $R(T) \leq a \times \underline{R}(T)$ for any tensor T.

Proof outline (for c = 1):

assume that

$$\lambda T = \sum_{s=1}^{t} \left[\left(\sum_{u=1}^{\dim(U)} \alpha_{su} x_u \right) \otimes \left(\sum_{v=1}^{\dim(V)} \beta_{sv} y_v \right) \otimes \left(\sum_{w=1}^{\dim(W)} \gamma_{sw} z_w \right) \right] + \lambda^2 T''$$

for some tensor T'' and some constants $\alpha_{su}, \beta_{sv}, \gamma_{sw}$ in $\mathbb{F}[\lambda]$

Border rank v.s. rank

Proposition 2

There exists a constant a such that $R(T) \leq a \times \underline{R}(T)$ for any tensor T.

Proof outline (for c = 1):

assume that

$$\lambda T = \sum_{s=1}^{t} \left[\left(\sum_{u=1}^{\dim(U)} \alpha_{su} x_u \right) \otimes \left(\sum_{v=1}^{\dim(V)} \beta_{sv} y_v \right) \otimes \left(\sum_{w=1}^{\dim(W)} \gamma_{sw} z_w \right) \right] + \lambda^2 T''$$

for some tensor T'' and some constants $\alpha_{su}, \beta_{sv}, \gamma_{sw}$ in $\mathbb{F}[\lambda]$

i.e., $\underline{R}(T) \leq t$

Border rank v.s. rank

Proposition 2

There exists a constant a such that $R(T) \leq a \times \underline{R}(T)$ for any tensor T.

Proof outline (for c = 1):

we get T be computing the coefficient of λ in T'

$$T' = \lambda T - \lambda^2 T''$$

assume that

$$\lambda T = \sum_{s=1}^{t} \left[\left(\sum_{u=1}^{\dim(U)} \alpha_{su} x_u \right) \otimes \left(\sum_{v=1}^{\dim(V)} \beta_{sv} y_v \right) \otimes \left(\sum_{w=1}^{\dim(W)} \gamma_{sw} z_w \right) \right] + \lambda^2 T''$$

for some tensor T'' and some constants $\alpha_{su}, \beta_{sv}, \gamma_{sw}$ in $\mathbb{F}[\lambda]$

i.e., $\underline{R}(T) \leq t$

Border rank v.s. rank

Proposition 2

There exists a constant a such that $R(T) \leq a \times \underline{R}(T)$ for any tensor T.

Proof outline (for c = 1):

we get T be computing the coefficient of λ in T'

$$T' = \lambda T - \lambda^2 T''$$

assume that

$$\lambda T = \sum_{s=1}^{t} \left[\left(\sum_{u=1}^{\dim(U)} \alpha_{su} x_u \right) \otimes \left(\sum_{v=1}^{\dim(V)} \beta_{sv} y_v \right) \otimes \left(\sum_{w=1}^{\dim(W)} \gamma_{sw} z_w \right) \right] + \lambda^2 T''$$

for some tensor T'' and some constants $\alpha_{su}, \beta_{sv}, \gamma_{sw}$ in $\mathbb{F}[\lambda]$

$$\alpha_{su} = \alpha_{su}^{[0]} + \alpha_{su}^{[1]}\lambda + \alpha_{su}^{[2]}\lambda^2 + \cdots$$

i.e., $\underline{R}(T) \leq t$

Border rank v.s. rank

Proposition 2

There exists a constant a such that $R(T) \leq a \times \underline{R}(T)$ for any tensor T.

Proof outline (for c = 1):

we get T be computing the coefficient of λ in T'

$$T' = \lambda T - \lambda^2 T''$$

assume that

$$\lambda T = \sum_{s=1}^{t} \left[\left(\sum_{u=1}^{\dim(U)} \alpha_{su} x_u \right) \otimes \left(\sum_{v=1}^{\dim(V)} \beta_{sv} y_v \right) \otimes \left(\sum_{w=1}^{\dim(W)} \gamma_{sw} z_w \right) \right] + \lambda^2 T''$$

for some tensor T'' and some constants $\alpha_{su}, \beta_{sv}, \gamma_{sw}$ in $\mathbb{F}[\lambda]$

$$\alpha_{su} = \alpha_{su}^{[0]} + \alpha_{su}^{[1]} \lambda + \alpha_{su}^{[2]} \lambda^2 + \cdots$$

similarly for β_{sv} and γ_{sw}

i.e., $\underline{R}(T) \leq t$

Border rank v.s. rank

Proposition 2

There exists a constant a such that $R(T) \leq a \times \underline{R}(T)$ for any tensor T.

Proof outline (for c = 1):

we get T be computing the coefficient of λ in T'

$$T = \sum_{s=1}^{t} \left[\left(\sum_{u=1}^{\dim(U)} \alpha_{su}^{[1]} x_u \right) \otimes \left(\sum_{v=1}^{\dim(V)} \beta_{sv}^{[0]} y_v \right) \otimes \left(\sum_{w=1}^{\dim(W)} \gamma_{sw}^{[0]} z_w \right) \right]$$

$$+ \sum_{s=1}^{t} \left[\left(\sum_{u=1}^{\dim(U)} \alpha_{su}^{[0]} x_u \right) \otimes \left(\sum_{v=1}^{\dim(V)} \beta_{sv}^{[1]} y_v \right) \otimes \left(\sum_{w=1}^{\dim(W)} \gamma_{sw}^{[0]} z_w \right) \right]$$

$$+ \sum_{s=1}^{t} \left[\left(\sum_{u=1}^{\dim(U)} \alpha_{su}^{[0]} x_u \right) \otimes \left(\sum_{v=1}^{\dim(V)} \beta_{sv}^{[0]} y_v \right) \otimes \left(\sum_{w=1}^{\dim(W)} \gamma_{sw}^{[1]} z_w \right) \right]$$

$$T' = \lambda T - \lambda^2 T''$$

assume that

$$\lambda T = \sum_{s=1}^{t} \left[\left(\sum_{u=1}^{\dim(U)} \alpha_{su} x_u \right) \otimes \left(\sum_{v=1}^{\dim(V)} \beta_{sv} y_v \right) \otimes \left(\sum_{w=1}^{\dim(W)} \gamma_{sw} z_w \right) \right] + \lambda^2 T''$$

for some tensor T'' and some constants $\alpha_{su}, \beta_{sv}, \gamma_{sw}$ in $\mathbb{F}[\lambda]$

$$\alpha_{su} = \alpha_{su}^{[0]} + \alpha_{su}^{[1]}\lambda + \alpha_{su}^{[2]}\lambda^2 + \cdots$$

similarly for β_{sv} and γ_{sw}

i.e., $\underline{R}(T) \leq t$

Border rank v.s. rank

Proposition 2

There exists a constant a such that $R(T) \leq a \times \underline{R}(T)$ for any tensor T.

Proof outline (for c = 1):

we get T be computing the coefficient of λ in T'

$$T' = \lambda T - \lambda^2 T''$$

$$\boxed{R(T) \leq 3 \times t}$$

$$T = \sum_{s=1}^{t}\left[\left(\sum_{u=1}^{\dim(U)} \alpha^{[1]}_{su} x_u\right) \otimes \left(\sum_{v=1}^{\dim(V)} \beta^{[0]}_{sv} y_v\right) \otimes \left(\sum_{w=1}^{\dim(W)} \gamma^{[0]}_{sw} z_w\right)\right]$$
$$+ \sum_{s=1}^{t}\left[\left(\sum_{u=1}^{\dim(U)} \alpha^{[0]}_{su} x_u\right) \otimes \left(\sum_{v=1}^{\dim(V)} \beta^{[1]}_{sv} y_v\right) \otimes \left(\sum_{w=1}^{\dim(W)} \gamma^{[0]}_{sw} z_w\right)\right]$$
$$+ \sum_{s=1}^{t}\left[\left(\sum_{u=1}^{\dim(U)} \alpha^{[0]}_{su} x_u\right) \otimes \left(\sum_{v=1}^{\dim(V)} \beta^{[0]}_{sv} y_v\right) \otimes \left(\sum_{w=1}^{\dim(W)} \gamma^{[1]}_{sw} z_w\right)\right]$$

assume that

$$\lambda T = \sum_{s=1}^{t}\left[\left(\sum_{u=1}^{\dim(U)} \alpha_{su} x_u\right) \otimes \left(\sum_{v=1}^{\dim(V)} \beta_{sv} y_v\right) \otimes \left(\sum_{w=1}^{\dim(W)} \gamma_{sw} z_w\right)\right] + \lambda^2 T''$$

for some tensor T'' and some constants $\alpha_{su}, \beta_{sv}, \gamma_{sw}$ in $\mathbb{F}[\lambda]$

$$\alpha_{su} = \alpha^{[0]}_{su} + \alpha^{[1]}_{su}\lambda + \alpha^{[2]}_{su}\lambda^2 + \cdots$$

similarly for β_{sv} and γ_{sw}

i.e., $\underline{R}(T) \leq t$

Example

Construction by Bini, Capovani, Romani and Lotti (1979):

$$\boxed{\begin{aligned} T_{\text{Bini}} = \quad & a_{11} \otimes b_{11} \otimes c_{11} + a_{12} \otimes b_{21} \otimes c_{11} + a_{11} \otimes b_{12} \otimes c_{12} \\ & + a_{12} \otimes b_{22} \otimes c_{12} + a_{21} \otimes b_{11} \otimes c_{21} + a_{21} \otimes b_{12} \otimes c_{22} \end{aligned}}$$

$\underline{R}(T_{\text{Bini}}) \leq 5$

Example

Construction by Bini, Capovani, Romani and Lotti (1979):

$$\boxed{\begin{aligned} T_{\mathsf{Bini}} = {} & a_{11} \otimes b_{11} \otimes c_{11} + a_{12} \otimes b_{21} \otimes c_{11} + a_{11} \otimes b_{12} \otimes c_{12} \\ & + a_{12} \otimes b_{22} \otimes c_{12} + a_{21} \otimes b_{11} \otimes c_{21} + a_{21} \otimes b_{12} \otimes c_{22} \end{aligned}}$$

$\underline{R}(T_{\mathsf{Bini}}) \leq 5$

T_{Bini} represents $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

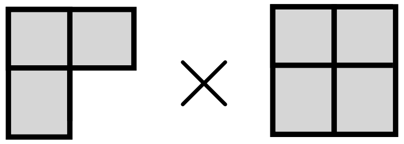
Example

Construction by Bini, Capovani, Romani and Lotti (1979):

$$\boxed{\begin{aligned} T_{\mathsf{Bini}} = \quad & a_{11} \otimes b_{11} \otimes c_{11} + a_{12} \otimes b_{21} \otimes c_{11} + a_{11} \otimes b_{12} \otimes c_{12} \\ & + a_{12} \otimes b_{22} \otimes c_{12} + a_{21} \otimes b_{11} \otimes c_{21} + a_{21} \otimes b_{12} \otimes c_{22} \end{aligned}}$$

$\underline{R}(T_{\mathsf{Bini}}) \leq 5$

T_{Bini} represents $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$



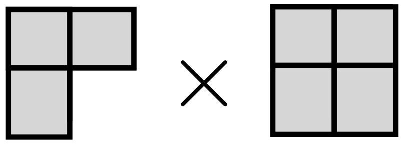
Example

Construction by Bini, Capovani, Romani and Lotti (1979):

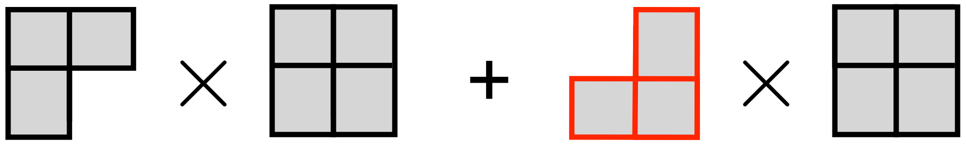
$$\begin{aligned} T_{\text{Bini}} = \quad & a_{11} \otimes b_{11} \otimes c_{11} + a_{12} \otimes b_{21} \otimes c_{11} + a_{11} \otimes b_{12} \otimes c_{12} \\ & + a_{12} \otimes b_{22} \otimes c_{12} + a_{21} \otimes b_{11} \otimes c_{21} + a_{21} \otimes b_{12} \otimes c_{22} \end{aligned}$$

$\underline{R}(T_{\text{Bini}}) \leq 5$

T_{Bini} represents $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$



Two copies of T_{Bini}



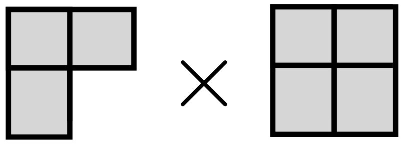
Example

Construction by Bini, Capovani, Romani and Lotti (1979):

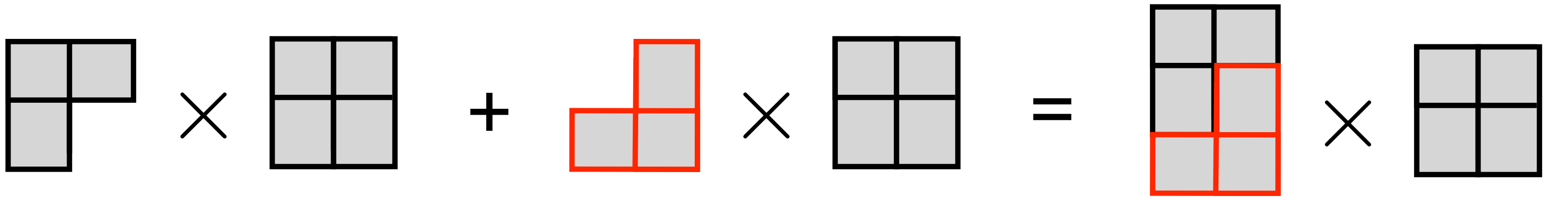
$$\boxed{\begin{aligned} T_{\mathsf{Bini}} = \quad & a_{11} \otimes b_{11} \otimes c_{11} + a_{12} \otimes b_{21} \otimes c_{11} + a_{11} \otimes b_{12} \otimes c_{12} \\ & + a_{12} \otimes b_{22} \otimes c_{12} + a_{21} \otimes b_{11} \otimes c_{21} + a_{21} \otimes b_{12} \otimes c_{22} \end{aligned}}$$

$\underline{R}(T_{\mathsf{Bini}}) \leq 5$

T_{Bini} represents $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$



Two copies of T_{Bini}



$\langle 3, 2, 2 \rangle$

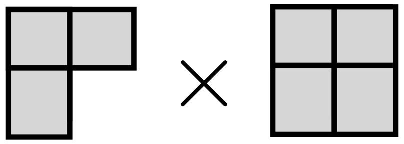
Example

Construction by Bini, Capovani, Romani and Lotti (1979):

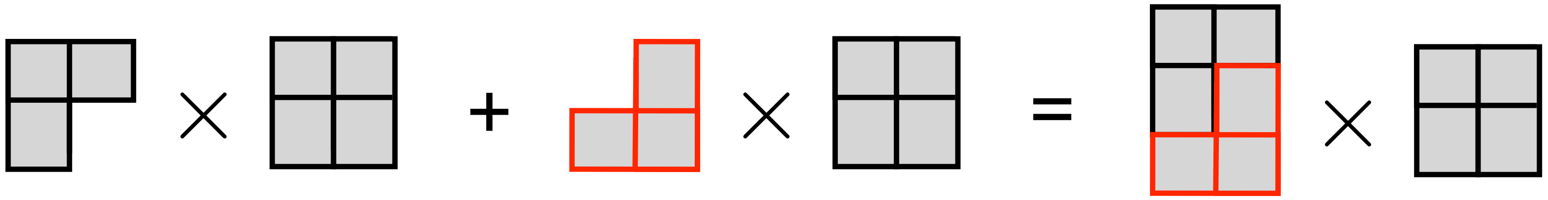
$$\begin{aligned}T_{\text{Bini}} = \quad & a_{11} \otimes b_{11} \otimes c_{11} + a_{12} \otimes b_{21} \otimes c_{11} + a_{11} \otimes b_{12} \otimes c_{12} \\ & + a_{12} \otimes b_{22} \otimes c_{12} + a_{21} \otimes b_{11} \otimes c_{21} + a_{21} \otimes b_{12} \otimes c_{22}\end{aligned}$$

$\underline{R}(T_{\text{Bini}}) \leq 5$

T_{Bini} represents $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$



Two copies of T_{Bini}



$\langle 3, 2, 2 \rangle$

Consequence: $\underline{R}(\langle 3, 2, 2 \rangle) \leq 10$

Example

Consequence: $\underline{R}(\langle 3, 2, 2\rangle) \leq 10$

Proposition 2

There exists a constant a such that $R(T) \leq a \times \underline{R}(T)$ for any tensor T.

Consequence: $\underline{R}(\langle 3,2,2\rangle) \leq 10 \overset{\text{Prop 2}}{\Longrightarrow} R(\langle 3,2,2\rangle) \leq a \times 10$

Proposition 2

There exists a constant a such that $R(T) \leq a \times \underline{R}(T)$ for any tensor T.

Theorem 1

$$R(\langle m,n,p\rangle) \leq t \Longrightarrow (mnp)^{\omega/3} \leq t$$

Consequence: $\underline{R}(\langle 3,2,2\rangle) \leq 10 \overset{\text{Prop 2}}{\Longrightarrow} R(\langle 3,2,2\rangle) \leq a \times 10$

$$\overset{\text{Th 1}}{\Longrightarrow} 12^{\omega/3} \leq a \times 10$$

Proposition 2

There exists a constant a such that $R(T) \leq a \times \underline{R}(T)$ for any tensor T.

Theorem 1

$$R(\langle m, n, p \rangle) \leq t \implies (mnp)^{\omega/3} \leq t$$

Consequence: $\underline{R}(\langle 3,2,2 \rangle) \leq 10 \overset{\text{Prop 2}}{\implies} R(\langle 3,2,2 \rangle) \leq a \times 10$

$$\overset{\text{Th 1}}{\implies} 12^{\omega/3} \leq a \times 10$$

$$\implies \omega \leq 4.106\ldots \text{ (with } a = 3\text{)}$$

Proposition 2

There exists a constant a such that $R(T) \leq a \times \underline{R}(T)$ for any tensor T.

Theorem 1

$$R(\langle m, n, p \rangle) \leq t \Longrightarrow (mnp)^{\omega/3} \leq t$$

Consequence: $\underline{R}(\langle 3, 2, 2 \rangle) \leq 10 \overset{\text{Prop 2}}{\Longrightarrow} R(\langle 3, 2, 2 \rangle) \leq a \times 10$

$\overset{\text{Th 1}}{\Longrightarrow} 12^{\omega/3} \leq a \times 10$

$\Longrightarrow \omega \leq 4.106\ldots$ (with $a = 3$)

$\underline{R}(\langle 3, 2, 2 \rangle) \leq 10$

Proposition 2

There exists a constant a such that $R(T) \leq a \times \underline{R}(T)$ for any tensor T.

Theorem 1

$$R(\langle m,n,p\rangle) \leq t \Longrightarrow (mnp)^{\omega/3} \leq t$$

Consequence: $\underline{R}(\langle 3,2,2\rangle) \leq 10 \overset{\text{Prop 2}}{\Longrightarrow} R(\langle 3,2,2\rangle) \leq a \times 10$

$\overset{\text{Th 1}}{\Longrightarrow} 12^{\omega/3} \leq a \times 10$

$\Longrightarrow \omega \leq 4.106...$ (with $a = 3$)

$\underline{R}(\langle 3,2,2\rangle) \leq 10 \Longrightarrow \underline{R}\left(\langle 3,2,2\rangle^{\otimes N}\right) \leq 10^N$ (submultiplicativity of the border rank)

Proposition 2

There exists a constant a such that $R(T) \leq a \times \underline{R}(T)$ for any tensor T.

Theorem 1

$$R(\langle m, n, p \rangle) \leq t \Longrightarrow (mnp)^{\omega/3} \leq t$$

Consequence: $\underline{R}(\langle 3, 2, 2 \rangle) \leq 10 \xRightarrow{\text{Prop 2}} R(\langle 3, 2, 2 \rangle) \leq a \times 10$

$$\xRightarrow{\text{Th 1}} 12^{\omega/3} \leq a \times 10$$

$$\Longrightarrow \omega \leq 4.106\ldots \text{ (with } a = 3\text{)}$$

$$\underline{R}(\langle 3, 2, 2 \rangle) \leq 10 \Longrightarrow \underline{R}\left(\underbrace{\langle 3, 2, 2 \rangle^{\otimes N}}_{\langle 3^N, 2^N, 2^N \rangle}\right) \leq 10^N$$ (submultiplicativity of the border rank)

Proposition 2

There exists a constant a such that $R(T) \leq a \times \underline{R}(T)$ for any tensor T.

Theorem 1

$$R(\langle m,n,p\rangle) \leq t \Longrightarrow (mnp)^{\omega/3} \leq t$$

Consequence: $\underline{R}(\langle 3,2,2\rangle) \leq 10 \overset{\text{Prop 2}}{\Longrightarrow} R(\langle 3,2,2\rangle) \leq a \times 10$

$$\overset{\text{Th 1}}{\Longrightarrow} 12^{\omega/3} \leq a \times 10$$

$$\Longrightarrow \omega \leq 4.106\ldots \text{ (with } a = 3\text{)}$$

$$\underline{R}(\langle 3,2,2\rangle) \leq 10 \Longrightarrow \underline{R}\left(\underbrace{\langle 3,2,2\rangle^{\otimes N}}_{\langle 3^N,2^N,2^N\rangle}\right) \leq 10^N \quad \text{(submultiplicativity of the border rank)}$$

$$\overset{\text{Prop 2}}{\Longrightarrow} R(\langle 3^N,2^N,2^N\rangle) \leq a \times 10^N$$

Proposition 2

There exists a constant a such that $R(T) \leq a \times \underline{R}(T)$ for any tensor T.

Theorem 1

$$R(\langle m,n,p\rangle) \leq t \Longrightarrow (mnp)^{\omega/3} \leq t$$

Consequence: $\underline{R}(\langle 3,2,2\rangle) \leq 10 \overset{\text{Prop 2}}{\Longrightarrow} R(\langle 3,2,2\rangle) \leq a \times 10$

$$\overset{\text{Th 1}}{\Longrightarrow} 12^{\omega/3} \leq a \times 10$$

$$\Longrightarrow \omega \leq 4.106\ldots \text{ (with } a = 3)$$

$$\underline{R}(\langle 3,2,2\rangle) \leq 10 \Longrightarrow \underline{R}\Big(\underbrace{\langle 3,2,2\rangle^{\otimes N}}_{\langle 3^N, 2^N, 2^N\rangle}\Big) \leq 10^N$$

(submultiplicativity of the border rank)

$$\overset{\text{Prop 2}}{\Longrightarrow} R(\langle 3^N, 2^N, 2^N\rangle) \leq a \times 10^N$$

$$\overset{\text{Th 1}}{\Longrightarrow} 12^{N\omega/3} \leq a \times 10^N$$

Proposition 2

There exists a constant a such that $R(T) \leq a \times \underline{R}(T)$ for any tensor T.

Theorem 1

$$R(\langle m,n,p\rangle) \leq t \implies (mnp)^{\omega/3} \leq t$$

Consequence: $\underline{R}(\langle 3,2,2\rangle) \leq 10 \overset{\text{Prop 2}}{\implies} R(\langle 3,2,2\rangle) \leq a \times 10$

$$\overset{\text{Th 1}}{\implies} 12^{\omega/3} \leq a \times 10$$

$$\implies \omega \leq 4.106\ldots \text{ (with } a = 3\text{)}$$

$$\underline{R}(\langle 3,2,2\rangle) \leq 10 \implies \underline{R}\Big(\underbrace{\langle 3,2,2\rangle^{\otimes N}}_{\langle 3^N,2^N,2^N\rangle}\Big) \leq 10^N$$ (submultiplicativity of the border rank)

$$\overset{\text{Prop 2}}{\implies} R(\langle 3^N,2^N,2^N\rangle) \leq a \times 10^N$$

$$\overset{\text{Th 1}}{\implies} 12^{N\omega/3} \leq a \times 10^N$$

$$\implies 12^{\omega/3} \leq a^{1/N} \times 10$$

Proposition 2

There exists a constant a such that $R(T) \leq a \times \underline{R}(T)$ for any tensor T.

Theorem 1

$$R(\langle m, n, p \rangle) \leq t \Longrightarrow (mnp)^{\omega/3} \leq t$$

Consequence: $\underline{R}(\langle 3, 2, 2 \rangle) \leq 10 \overset{\text{Prop 2}}{\Longrightarrow} R(\langle 3, 2, 2 \rangle) \leq a \times 10$

$$\overset{\text{Th 1}}{\Longrightarrow} 12^{\omega/3} \leq a \times 10$$

$$\Longrightarrow \omega \leq 4.106\ldots \text{ (with } a = 3\text{)}$$

$$\underline{R}(\langle 3, 2, 2 \rangle) \leq 10 \Longrightarrow \underline{R}\Big(\underbrace{\langle 3, 2, 2 \rangle^{\otimes N}}_{\langle 3^N, 2^N, 2^N \rangle}\Big) \leq 10^N$$ (submultiplicativity of the border rank)

$$\overset{\text{Prop 2}}{\Longrightarrow} R(\langle 3^N, 2^N, 2^N \rangle) \leq a \times 10^N$$

$$\overset{\text{Th 1}}{\Longrightarrow} 12^{N\omega/3} \leq a \times 10^N$$

$$\Longrightarrow 12^{\omega/3} \leq a^{1/N} \times 10 \quad \text{(for any } N \geq 1\text{)}$$

Proposition 2

There exists a constant a such that $R(T) \leq a \times \underline{R}(T)$ for any tensor T.

Theorem 1

$$R(\langle m,n,p\rangle) \leq t \Longrightarrow (mnp)^{\omega/3} \leq t$$

Consequence: $\underline{R}(\langle 3,2,2\rangle) \leq 10 \overset{\text{Prop 2}}{\Longrightarrow} R(\langle 3,2,2\rangle) \leq a \times 10$

$$\overset{\text{Th 1}}{\Longrightarrow} 12^{\omega/3} \leq a \times 10$$

$$\Longrightarrow \omega \leq 4.106\ldots \text{ (with } a = 3\text{)}$$

$$\underline{R}(\langle 3,2,2\rangle) \leq 10 \Longrightarrow \underline{R}\Big(\underbrace{\langle 3,2,2\rangle^{\otimes N}}_{\langle 3^N,2^N,2^N\rangle}\Big) \leq 10^N \quad \text{(submultiplicativity of the border rank)}$$

$$\overset{\text{Prop 2}}{\Longrightarrow} R(\langle 3^N,2^N,2^N\rangle) \leq a \times 10^N$$

$$\overset{\text{Th 1}}{\Longrightarrow} 12^{N\omega/3} \leq a \times 10^N$$

$$\Longrightarrow 12^{\omega/3} \leq a^{1/N} \times 10 \quad \text{(for any } N \geq 1\text{)}$$

$$\Longrightarrow 12^{\omega/3} \leq 10 \quad \text{(take } N \rightarrow \infty\text{)}$$

Proposition 2

There exists a constant a such that $R(T) \leq a \times \underline{R}(T)$ for any tensor T.

Theorem 1

$$R(\langle m,n,p\rangle) \leq t \Longrightarrow (mnp)^{\omega/3} \leq t$$

Consequence: $\underline{R}(\langle 3,2,2\rangle) \leq 10 \xRightarrow{\text{Prop 2}} R(\langle 3,2,2\rangle) \leq a \times 10$

$$\xRightarrow{\text{Th 1}} 12^{\omega/3} \leq a \times 10$$

$$\Longrightarrow \omega \leq 4.106\ldots \ (\text{with } a = 3)$$

$$\underline{R}(\langle 3,2,2\rangle) \leq 10 \Longrightarrow \underline{R}\left(\underbrace{\langle 3,2,2\rangle^{\otimes N}}_{\langle 3^N,2^N,2^N\rangle}\right) \leq 10^N \quad \text{(submultiplicativity of the border rank)}$$

$$\xRightarrow{\text{Prop 2}} R(\langle 3^N,2^N,2^N\rangle) \leq a \times 10^N$$

$$\xRightarrow{\text{Th 1}} 12^{N\omega/3} \leq a \times 10^N$$

$$\Longrightarrow 12^{\omega/3} \leq a^{1/N} \times 10 \quad (\text{for any } N \geq 1)$$

$$\Longrightarrow 12^{\omega/3} \leq 10 \quad (\text{take } N \to \infty)$$

$$\Longrightarrow \boxed{\omega \leq 2.779\ldots}$$

[Bini et al. 79]

Proposition 2

There exists a constant a such that $R(T) \le a \times \underline{R}(T)$ for any tensor T.

Theorem 1

$$R(\langle m,n,p\rangle) \le t \implies (mnp)^{\omega/3} \le t$$

The constant a can be “taken as one” when deriving an upper bound on ω using Theorem 1

Consequence: $\underline{R}(\langle 3,2,2\rangle) \le 10 \overset{\text{Prop 2}}{\implies} R(\langle 3,2,2\rangle) \le a \times 10$

$$\overset{\text{Th 1}}{\implies} 12^{\omega/3} \le a \times 10$$

$$\implies \omega \le 4.106\ldots \text{ (with } a = 3\text{)}$$

$$\underline{R}(\langle 3,2,2\rangle) \le 10 \implies \underline{R}\left(\underbrace{\langle 3,2,2\rangle^{\otimes N}}_{\langle 3^N,2^N,2^N\rangle}\right) \le 10^N \quad \text{(submultiplicativity of the border rank)}$$

$$\overset{\text{Prop 2}}{\implies} R(\langle 3^N,2^N,2^N\rangle) \le a \times 10^N$$

$$\overset{\text{Th 1}}{\implies} 12^{N\omega/3} \le a \times 10^N$$

$$\implies 12^{\omega/3} \le a^{1/N} \times 10 \quad \text{(for any } N \ge 1\text{)}$$

$$\implies 12^{\omega/3} \le 10 \quad \text{(take } N \to \infty\text{)}$$

$$\implies \boxed{\omega \le 2.779\ldots}$$ [Bini et al. 79]

Proposition 2

There exists a constant a such that $R(T) \leq a \times \underline{R}(T)$ for any tensor T.

Theorem 1

$$R(\langle m,n,p\rangle) \leq t \Longrightarrow (mnp)^{\omega/3} \leq t$$

The constant a can be "taken as one" when deriving an upper bound on ω using Theorem 1

Theorem 2

$$\underline{R}(\langle m,n,p\rangle) \leq t \Longrightarrow (mnp)^{\omega/3} \leq t$$

$$\underline{R}(\langle 3,2,2\rangle) \leq 10 \Longrightarrow \underline{R}\left(\underbrace{\langle 3,2,2\rangle^{\otimes N}}_{\langle 3^N,2^N,2^N\rangle}\right) \leq 10^N$$ (submultiplicativity of the border rank)

$$\overset{\text{Prop 2}}{\Longrightarrow} R(\langle 3^N,2^N,2^N\rangle) \leq a \times 10^N$$

$$\overset{\text{Th 1}}{\Longrightarrow} 12^{N\omega/3} \leq a \times 10^N$$

$$\Longrightarrow 12^{\omega/3} \leq a^{1/N} \times 10$$

$$\Longrightarrow 12^{\omega/3} \leq 10 \quad (\text{take } N \to \infty)$$

$$\Longrightarrow \boxed{\omega \leq 2.779...}$$ [Bini et al. 79]

Proposition 2

There exists a constant a such that $R(T) \le a \times \underline{R}(T)$ for any tensor T.

Theorem 1

$$R(\langle m,n,p\rangle) \le t \Longrightarrow (mnp)^{\omega/3} \le t$$

The constant a can be "taken as one" when deriving an upper bound on ω using Theorem 1

Theorem 2

$$\underline{R}(\langle m,n,p\rangle) \le t \Longrightarrow (mnp)^{\omega/3} \le t$$

$\underline{R}(\langle 3, 2, 2$ … der rank)

here we used $a^{1/N} \to 1$

$$\overset{\text{Prop 2}}{\Longrightarrow} R(\langle 3^N, 2^N, 2^N\rangle) \le a \times 10^N$$

$$\overset{\text{Th 1}}{\Longrightarrow} 12^{N\omega/3} \le a \times 10^N$$

$$\Longrightarrow 12^{\omega/3} \le a^{1/N} \times 10$$

$$\Longrightarrow 12^{\omega/3} \le 10 \quad \text{(take } N \to \infty\text{)}$$

$$\Longrightarrow \boxed{\omega \le 2.779...}$$

[Bini et al. 79]

Proposition 2

There exists a constant a such that $R(T) \leq a \times \underline{R}(T)$ for any tensor T.

Theorem 1

$$R(\langle m,n,p\rangle) \leq t \Longrightarrow (mnp)^{\omega/3} \leq t$$

The constant a can be "taken as one" when deriving an upper bound on ω using Theorem 1

Theorem 2

$$\underline{R}(\langle m,n,p\rangle) \leq t \Longrightarrow (mnp)^{\omega/3} \leq t$$

$\underline{R}(\langle 3,2,2$ der rank)

here we used $a^{1/N} \to 1$

this is the major source of inefficiency in Theorem 2

$$\overset{\text{Prop 2}}{\Longrightarrow} R(\langle 3^N, \quad {}^N, 2^N\rangle) \leq a \times 10^N$$

$$\overset{\text{Th 1}}{\Longrightarrow} 12^{N\omega/3} \leq a \times 10^N$$

$$\Longrightarrow 12^{\omega/3} \leq a^{1/N} \times 10$$

$$\Longrightarrow 12^{\omega/3} \leq 10 \quad \text{(take } N \to \infty\text{)}$$

$$\Longrightarrow \boxed{\omega \leq 2.779...}$$

[Bini et al. 79]

History of the main improvements on the exponent of square matrix multiplication

Upper bound	Year	Authors
$\omega \leq 3$		
$\omega < 2.81$	1969	Strassen
$\omega < 2.79$	1979	Pan
$\omega < 2.78$	1979	Bini, Capovani, Romani and Lotti
$\omega < 2.55$	1981	Schönhage
$\omega < 2.53$	1981	Pan
$\omega < 2.52$	1982	Romani
$\omega < 2.50$	1982	Coppersmith and Winograd
$\omega < 2.48$	1986	Strassen
$\omega < 2.376$	1987	Coppersmith and Winograd
$\omega < 2.373$	2010	Stothers
$\omega < 2.3729$	2012	Vassilevska Williams
$\omega < 2.3728639$	2014	LG

Border rank and Theorem 2

Overview of the Lectures

✓ Fundamental techniques for fast matrix multiplication (1969~1987)

- Basics of bilinear complexity theory: exponent of matrix multiplication, Strassen's algorithm, bilinear algorithms
- First technique: tensor rank and recursion
- Second technique: border rank
- Third technique: the asymptotic sum inequality
- Fourth technique: the laser method

Lecture 1

✓ Recent progress on matrix multiplication (1987~)

- Laser method on powers of tensors
- Other approaches
- Lower bounds
- Rectangular matrix multiplication

Lecture 2

✓ Applications of matrix multiplications, open problems

Lecture 3

The asymptotic sum inequality (Section 4.2)

Schönhage's construction (1981):

$$T_{\mathsf{Schon}} = \sum_{i,j=1}^{3} a_i \otimes b_j \otimes c_{ij} + \sum_{k=1}^{4} u_k \otimes v_k \otimes w$$

The asymptotic sum inequality (Section 4.2)

Schönhage's construction (1981):

$$T_{\mathsf{Schon}} = \sum_{i,j=1}^{3} a_i \otimes b_j \otimes c_{ij} + \sum_{k=1}^{4} u_k \otimes v_k \otimes w$$

$$\sum_{i,j=1}^{3} a_i \otimes b_j \otimes c_{ij} \cong \langle 3, 1, 3 \rangle$$ 3×1 matrix by 1×3 matrix $$\sum_{i,j=1}^{3} a_{i1} \otimes b_{1j} \otimes c_{ij}$$

$$\sum_{k=1}^{4} u_k \otimes v_k \otimes w \cong \langle 1, 4, 1 \rangle$$ 1×4 matrix by 4×1 matrix $$\sum_{k=1}^{4} u_{1k} \otimes v_{k1} \otimes w_{11}$$

The asymptotic sum inequality (Section 4.2)

Schönhage's construction (1981):

$$T_{\text{Schon}} = \sum_{i,j=1}^{3} a_i \otimes b_j \otimes c_{ij} + \sum_{k=1}^{4} u_k \otimes v_k \otimes w$$

$$\sum_{i,j=1}^{3} a_i \otimes b_j \otimes c_{ij} \cong \langle 3,1,3 \rangle$$

3×1 matrix by 1×3 matrix

$$\sum_{i,j=1}^{3} a_{i{\color{red}1}} \otimes b_{{\color{red}1}j} \otimes c_{ij}$$

$$\sum_{k=1}^{4} u_k \otimes v_k \otimes w \cong \langle 1,4,1 \rangle$$

1×4 matrix by 4×1 matrix

$$\sum_{k=1}^{4} u_{{\color{red}1}k} \otimes v_{k{\color{red}1}} \otimes w_{{\color{red}11}}$$

$$\underline{R}(\langle 3,1,3 \rangle) = 9$$

$$\underline{R}(\langle 1,4,1 \rangle) = 4$$

The asymptotic sum inequality (Section 4.2)

Schönhage’s construction (1981):

$$\boxed{T_{\mathsf{Schon}} = \sum_{i,j=1}^{3} a_i \otimes b_j \otimes c_{ij} + \sum_{k=1}^{4} u_k \otimes v_k \otimes w}$$

$$\sum_{i,j=1}^{3} a_i \otimes b_j \otimes c_{ij} \cong \langle 3,1,3 \rangle$$

3×1 matrix by 1×3 matrix

$$\sum_{i,j=1}^{3} a_{i1} \otimes b_{1j} \otimes c_{ij}$$

$$\sum_{k=1}^{4} u_k \otimes v_k \otimes w \cong \langle 1,4,1 \rangle$$

1×4 matrix by 4×1 matrix

$$\sum_{k=1}^{4} u_{1k} \otimes v_{k1} \otimes w_{11}$$

$$\underline{R}(\langle 3,1,3 \rangle) = 9$$

$$\underline{R}(\langle 1,4,1 \rangle) = 4$$

$$\implies \underline{R}(T_{\mathsf{Schon}}) \leq 13$$

The asymptotic sum inequality (Section 4.2)

Schönhage's construction (1981):

$$T_{\mathsf{Schon}} = \sum_{i,j=1}^{3} a_i \otimes b_j \otimes c_{ij} + \sum_{k=1}^{4} u_k \otimes v_k \otimes w$$

$$\sum_{i,j=1}^{3} a_i \otimes b_j \otimes c_{ij} \cong \langle 3,1,3 \rangle$$ 3×1 matrix by 1×3 matrix $$\sum_{i,j=1}^{3} a_{i1} \otimes b_{1j} \otimes c_{ij}$$

$$\sum_{k=1}^{4} u_k \otimes v_k \otimes w \cong \langle 1,4,1 \rangle$$ 1×4 matrix by 4×1 matrix $$\sum_{k=1}^{4} u_{1k} \otimes v_{k1} \otimes w_{11}$$

$$\underline{R}(\langle 3,1,3 \rangle) = 9$$

$$\underline{R}(\langle 1,4,1 \rangle) = 4$$

$$\Longrightarrow \underline{R}(T_{\mathsf{Schon}}) \leq 13$$

Schönhage showed that $\underline{R}(T_{\mathsf{Schon}}) \leq 10$

The asymptotic sum inequality (Section 4.2)

Schönhage’s construction (1981):

$$\boxed{T_{\mathsf{Schon}} = \sum_{i,j=1}^{3} a_i \otimes b_j \otimes c_{ij} + \sum_{k=1}^{4} u_k \otimes v_k \otimes w}$$

$$\underline{R}(T_{\mathsf{Schon}}) \leq 10$$

The asymptotic sum inequality (Section 4.2)

Schönhage's construction (1981):

$$T_{\mathsf{Schon}} = \sum_{i,j=1}^{3} a_i \otimes b_j \otimes c_{ij} + \sum_{k=1}^{4} u_k \otimes v_k \otimes w$$

$$\underline{R}(T_{\mathsf{Schon}}) \leq 10$$

$$\lambda^2 T_{\mathsf{Schon}} = T' + \lambda^3 T''$$

The asymptotic sum inequality (Section 4.2)

Schönhage's construction (1981):

$$\boxed{T_{\mathsf{Schon}} = \sum_{i,j=1}^{3} a_i \otimes b_j \otimes c_{ij} + \sum_{k=1}^{4} u_k \otimes v_k \otimes w}$$

$\underline{R}(T_{\mathsf{Schon}}) \leq 10$

$$\boxed{\lambda^2 T_{\mathsf{Schon}} = T' + \lambda^3 T''}$$

where
$$\left.\begin{aligned} T' =& (a_1 + \lambda u_1) \otimes (b_1 + \lambda v_1) \otimes (w + \lambda^2 c_{11}) \\ &+ (a_1 + \lambda u_2) \otimes (b_2 + \lambda v_2) \otimes (w + \lambda^2 c_{12}) \\ &+ (a_2 + \lambda u_3) \otimes (b_1 + \lambda v_3) \otimes (w + \lambda^2 c_{21}) \\ &+ (a_2 + \lambda u_4) \otimes (b_2 + \lambda v_4) \otimes (w + \lambda^2 c_{22}) \\ &+ (a_3 - \lambda u_1 - \lambda u_3) \otimes b_1 \otimes (w + \lambda^2 c_{31}) \\ &+ (a_3 - \lambda u_2 - \lambda u_4) \otimes b_2 \otimes (w + \lambda^2 c_{32}) \\ &+ a_1 \otimes (b_3 - \lambda v_1 - \lambda v_2) \otimes (w + \lambda^2 c_{13}) \\ &+ a_2 \otimes (b_3 - \lambda v_3 - \lambda v_4) \otimes (w + \lambda^2 c_{23}) \\ &+ a_3 \otimes b_3 \otimes (w + \lambda^2 c_{33}) \\ &- (a_1 + a_2 + a_3) \otimes (b_1 + b_2 + b_3) \otimes w \end{aligned}\right]$$

10 multiplications

and T'' is some tensor

The asymptotic sum inequality (Section 4.2)

Schönhage's construction (1981):

$$\boxed{T_{\mathsf{Schon}} = \sum_{i,j=1}^{3} a_i \otimes b_j \otimes c_{ij} + \sum_{k=1}^{4} u_k \otimes v_k \otimes w}$$

$$\underline{R}(T_{\mathsf{Schon}}) \leq 10$$

the sum is direct (the two terms do not share variables)

The asymptotic sum inequality (Section 4.2)

Schönhage's construction (1981):

$$\boxed{T_{\mathsf{Schon}} = \sum_{i,j=1}^{3} a_i \otimes b_j \otimes c_{ij} + \sum_{k=1}^{4} u_k \otimes v_k \otimes w} \qquad \underline{R}(T_{\mathsf{Schon}}) \leq 10$$

the sum is direct (the two terms do not share variables)

formally:

$\sum_{i,j=1}^{3} a_i \otimes b_j \otimes c_{ij}$ is a tensor over (U_1, V_1, W_1)

$$U_1 = span\{a_1, a_2, a_3\} \qquad V_1 = span\{b_1, b_2, b_3\} \qquad W_1 = span\{c_{11}, \ldots, c_{33}\}$$

$\sum_{k=1}^{4} u_k \otimes v_k \otimes w$ is a tensor over (U_2, V_2, W_2)

$$U_2 = span\{u_1, \ldots, u_4\} \qquad V_2 = span\{v_1, \ldots, v_4\} \qquad W_2 = span\{w\}$$

T_{Schon} is a tensor over $(U_1 \oplus U_2, V_1 \oplus V_2, W_1 \oplus W_2)$

The asymptotic sum inequality (Section 4.2)

Schönhage's construction (1981):

$$\boxed{T_{\mathsf{Schon}} = \sum_{i,j=1}^{3} a_i \otimes b_j \otimes c_{ij} + \sum_{k=1}^{4} u_k \otimes v_k \otimes w}$$

$$\underline{R}(T_{\mathsf{Schon}}) \leq 10$$

the sum is direct (the two terms do not share variables)

$$\sum_{i,j=1}^{3} a_i \otimes b_j \otimes c_{ij} \cong \langle 3, 1, 3 \rangle \qquad \sum_{k=1}^{4} u_k \otimes v_k \otimes w \cong \langle 1, 4, 1 \rangle$$

The asymptotic sum inequality (Section 4.2)

Schönhage's construction (1981):

$$\boxed{T_{\mathsf{Schon}} = \sum_{i,j=1}^{3} a_i \otimes b_j \otimes c_{ij} + \sum_{k=1}^{4} u_k \otimes v_k \otimes w}$$

$$\underline{R}(T_{\mathsf{Schon}}) \leq 10$$

the sum is direct (the two terms do not share variables)

$$\sum_{i,j=1}^{3} a_i \otimes b_j \otimes c_{ij} \cong \langle 3,1,3 \rangle \qquad \sum_{k=1}^{4} u_k \otimes v_k \otimes w \cong \langle 1,4,1 \rangle$$

$$T_{\mathsf{Schon}} \cong \langle 3,1,3 \rangle \oplus \langle 1,4,1 \rangle$$

The asymptotic sum inequality (Section 4.2)

Schönhage's construction (1981):

$$T_{\mathsf{Schon}} = \sum_{i,j=1}^{3} a_i \otimes b_j \otimes c_{ij} + \sum_{k=1}^{4} u_k \otimes v_k \otimes w$$

$$\underline{R}(T_{\mathsf{Schon}}) \leq 10$$

the sum is direct (the two terms do not share variables)

$$\sum_{i,j=1}^{3} a_i \otimes b_j \otimes c_{ij} \cong \langle 3, 1, 3 \rangle \qquad \sum_{k=1}^{4} u_k \otimes v_k \otimes w \cong \langle 1, 4, 1 \rangle$$

$$T_{\mathsf{Schon}} \cong \langle 3, 1, 3 \rangle \oplus \langle 1, 4, 1 \rangle$$

Theorem (the asymptotic sum inequality, special case) [Schönhage 1981]

$$\underline{R}(\langle m_1, n_1, p_1 \rangle \oplus \langle m_2, n_2, p_2 \rangle) \leq t \Longrightarrow (m_1 n_1 p_1)^{\omega/3} + (m_2 n_2 p_2)^{\omega/3} \leq t$$

The asymptotic sum inequality (Section 4.2)

Schönhage's construction (1981):

$$\boxed{T_{\mathsf{Schon}} = \sum_{i,j=1}^{3} a_i \otimes b_j \otimes c_{ij} + \sum_{k=1}^{4} u_k \otimes v_k \otimes w}$$

$$\underline{R}(T_{\mathsf{Schon}}) \leq 10$$

the sum is direct (the two terms do not share variables)

$$\sum_{i,j=1}^{3} a_i \otimes b_j \otimes c_{ij} \cong \langle 3,1,3\rangle \qquad \sum_{k=1}^{4} u_k \otimes v_k \otimes w \cong \langle 1,4,1\rangle$$

$$T_{\mathsf{Schon}} \cong \langle 3,1,3\rangle \oplus \langle 1,4,1\rangle$$

Theorem (the asymptotic sum inequality, special case) [Schönhage 1981]

$$\underline{R}(\langle m_1,n_1,p_1\rangle \oplus \langle m_2,n_2,p_2\rangle) \leq t \implies (m_1n_1p_1)^{\omega/3} + (m_2n_2p_2)^{\omega/3} \leq t$$

Consequence: $9^{\omega/3} + 4^{\omega/3} \leq 10$

The asymptotic sum inequality (Section 4.2)

Schönhage's construction (1981):

$$\boxed{T_{\mathsf{Schon}} = \sum_{i,j=1}^{3} a_i \otimes b_j \otimes c_{ij} + \sum_{k=1}^{4} u_k \otimes v_k \otimes w}$$

$$\underline{R}(T_{\mathsf{Schon}}) \leq 10$$

the sum is direct (the two terms do not share variables)

$$\sum_{i,j=1}^{3} a_i \otimes b_j \otimes c_{ij} \cong \langle 3,1,3\rangle \qquad \sum_{k=1}^{4} u_k \otimes v_k \otimes w \cong \langle 1,4,1\rangle$$

$$T_{\mathsf{Schon}} \cong \langle 3,1,3\rangle \oplus \langle 1,4,1\rangle$$

Theorem (the asymptotic sum inequality, special case) [Schönhage 1981]

$$\underline{R}(\langle m_1,n_1,p_1\rangle \oplus \langle m_2,n_2,p_2\rangle) \leq t \implies (m_1n_1p_1)^{\omega/3} + (m_2n_2p_2)^{\omega/3} \leq t$$

Consequence: $9^{\omega/3} + 4^{\omega/3} \leq 10 \implies \omega \leq 2.59...$

The asymptotic sum inequality (Section 4.2)

Schönhage's construction (1981):

$$\boxed{T_{\text{Schon}} = \sum_{i,j=1}^{3} a_i \otimes b_j \otimes c_{ij} \underset{\uparrow}{+} \sum_{k=1}^{4} u_k \otimes v_k \otimes w}$$

$$\underline{R}(T_{\text{Schon}}) \leq 10$$

the sum is direct (the two terms do not share variables)

$$\sum_{i,j=1}^{3} a_i \otimes b_j \otimes c_{ij} \cong \langle 3,1,3 \rangle \qquad \sum_{k=1}^{4} u_k \otimes v_k \otimes w \cong \langle 1,4,1 \rangle$$

$$T_{\text{Schon}} \cong \langle 3,1,3 \rangle \oplus \langle 1,4,1 \rangle$$

Theorem (the asymptotic sum inequality, special case) [Schönhage 1981]

$$\underline{R}(\langle m_1, n_1, p_1 \rangle \oplus \langle m_2, n_2, p_2 \rangle) \leq t \Longrightarrow (m_1 n_1 p_1)^{\omega/3} + (m_2 n_2 p_2)^{\omega/3} \leq t$$

Consequence: $9^{\omega/3} + 4^{\omega/3} \leq 10 \quad \Longrightarrow \omega \leq 2.59\ldots$

Using a variant of this construction, Schönhage finally obtained $\boxed{\omega \leq 2.54\ldots}$

History of the main improvements on the exponent of square matrix multiplication

Upper bound	Year	Authors
$\omega \leq 3$		
$\omega < 2.81$	1969	Strassen
$\omega < 2.79$	1979	Pan
$\omega < 2.78$	1979	Bini, Capovani, Romani and Lotti
$\omega < 2.55$	1981	Schönhage
$\omega < 2.53$	1981	Pan
$\omega < 2.52$	1982	Romani
$\omega < 2.50$	1982	Coppersmith and Winograd
$\omega < 2.48$	1986	Strassen
$\omega < 2.376$	1987	Coppersmith and Winograd
$\omega < 2.373$	2010	Stothers
$\omega < 2.3729$	2012	Vassilevska Williams
$\omega < 2.3728639$	2014	LG

Asymptotic sum inequality

The asymptotic sum inequality

Theorem 3 (the asymptotic sum inequality, general form) [Schönhage 1981]

$$\underline{R}\left(\bigoplus_{i=1}^{k}\langle m_i, n_i, p_i\rangle\right) \leq t \Longrightarrow \sum_{i=1}^{k}(m_i n_i p_i)^{\omega/3} \leq t$$

Theorem (the asymptotic sum inequality, special case) [Schönhage 1981]

$$\underline{R}(\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle) \leq t \Longrightarrow (m_1 n_1 p_1)^{\omega/3} + (m_2 n_2 p_2)^{\omega/3} \leq t$$

Consequence: $9^{\omega/3} + 4^{\omega/3} \leq 10 \quad \Longrightarrow \omega \leq 2.59...$

Using a variant of this construction, Schönhage finally obtained $\boxed{\omega \leq 2.54...}$

The asymptotic sum inequality

Theorem (the asympotic sum inequality, special case)

$$\underline{R}(\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle) \leq t \implies (m_1 n_1 p_1)^{\omega/3} + (m_2 n_2 p_2)^{\omega/3} \leq t$$

Proof outline

The asymptotic sum inequality

Theorem (the asymptotic sum inequality, special case)

$$\underline{R}(\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle) \leq t \Longrightarrow (m_1 n_1 p_1)^{\omega/3} + (m_2 n_2 p_2)^{\omega/3} \leq t$$

Proof outline

Take the N-th power, for some large N:

$$t^N \geq \underline{R}\left((\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle)^{\otimes N}\right)$$

The asymptotic sum inequality

Theorem (the asympotic sum inequality, special case)

$$\underline{R}(\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle) \leq t \Longrightarrow (m_1 n_1 p_1)^{\omega/3} + (m_2 n_2 p_2)^{\omega/3} \leq t$$

Proof outline

Take the N-th power, for some large N:

$$\begin{aligned} t^N &\geq \underline{R}\left((\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle)^{\otimes N}\right) \\ &= \underline{R}\left(\sum_{a=0}^{N} \binom{N}{a} \langle m_1, n_1, p_1\rangle^{\otimes a} \otimes \langle m_2, n_2, p_2\rangle^{\otimes (N-a)}\right) \end{aligned}$$

The asymptotic sum inequality

Theorem (the asympotic sum inequality, special case)

$$\underline{R}(\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle) \leq t \Longrightarrow (m_1 n_1 p_1)^{\omega/3} + (m_2 n_2 p_2)^{\omega/3} \leq t$$

<u>Proof outline</u>

Take the N-th power, for some large N:

$$t^N \geq \underline{R}\left((\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle)^{\otimes N}\right)$$

$$= \underline{R}\left(\sum_{a=0}^{N} \binom{N}{a} \langle m_1, n_1, p_1\rangle^{\otimes a} \otimes \langle m_2, n_2, p_2\rangle^{\otimes (N-a)}\right)$$

direct sum of $\binom{N}{a}$ copies of $\langle m_1, n_1, p_1\rangle^{\otimes a} \otimes \langle m_2, n_2, p_2\rangle^{\otimes (N-a)}$

The asymptotic sum inequality

Theorem (the asympotic sum inequality, special case)

$$\underline{R}(\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle) \leq t \Longrightarrow (m_1 n_1 p_1)^{\omega/3} + (m_2 n_2 p_2)^{\omega/3} \leq t$$

Proof outline

Take the N-th power, for some large N:

$$t^N \geq \underline{R}\left((\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle)^{\otimes N}\right)$$

$$= \underline{R}\left(\sum_{a=0}^{N} \binom{N}{a} \langle m_1, n_1, p_1\rangle^{\otimes a} \otimes \langle m_2, n_2, p_2\rangle^{\otimes (N-a)}\right)$$

direct sum of $\binom{N}{a}$ copies of $\langle m_1, n_1, p_1\rangle^{\otimes a} \otimes \langle m_2, n_2, p_2\rangle^{\otimes (N-a)}$

$$= \underline{R}\left(\sum_{a=0}^{N} \binom{N}{a} \left\langle m_1^a m_2^{(N-a)}, n_1^a n_2^{(N-a)}, p_1^a p_2^{(N-a)}\right\rangle\right)$$

The asymptotic sum inequality

Theorem (the asympotic sum inequality, special case)

$$\underline{R}(\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle) \leq t \Longrightarrow (m_1 n_1 p_1)^{\omega/3} + (m_2 n_2 p_2)^{\omega/3} \leq t$$

<u>Proof outline</u>

Take the N-th power, for some large N:

$$t^N \geq \underline{R}\left((\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle)^{\otimes N}\right)$$

$$= \underline{R}\left(\sum_{a=0}^{N} \binom{N}{a} \langle m_1, n_1, p_1\rangle^{\otimes a} \otimes \langle m_2, n_2, p_2\rangle^{\otimes (N-a)}\right)$$

direct sum of $\binom{N}{a}$ copies of $\langle m_1, n_1, p_1\rangle^{\otimes a} \otimes \langle m_2, n_2, p_2\rangle^{\otimes (N-a)}$

$$= \underline{R}\left(\sum_{a=0}^{N} \underbrace{\binom{N}{a} \left\langle m_1^a m_2^{(N-a)}, n_1^a n_2^{(N-a)}, p_1^a p_2^{(N-a)}\right\rangle}_{\mathrm{T}_a}\right)$$

The asymptotic sum inequality

Theorem (the asympotic sum inequality, special case)

$$\underline{R}(\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle) \leq t \Longrightarrow (m_1 n_1 p_1)^{\omega/3} + (m_2 n_2 p_2)^{\omega/3} \leq t$$

<u>Proof outline</u>

Take the N-th power, for some large N:

$$\begin{aligned}
t^N &\geq \underline{R}\left((\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle)^{\otimes N}\right) \\
&= \underline{R}\left(\sum_{a=0}^{N} \binom{N}{a} \langle m_1, n_1, p_1\rangle^{\otimes a} \otimes \langle m_2, n_2, p_2\rangle^{\otimes (N-a)}\right)
\end{aligned}$$

direct sum of $\binom{N}{a}$ copies of $\langle m_1, n_1, p_1\rangle^{\otimes a} \otimes \langle m_2, n_2, p_2\rangle^{\otimes (N-a)}$

$$= \underline{R}\left(\sum_{a=0}^{N} \underbrace{\binom{N}{a} \left\langle m_1^a m_2^{(N-a)}, n_1^a n_2^{(N-a)}, p_1^a p_2^{(N-a)}\right\rangle}_{\mathrm{T}_a}\right)$$

By definition of ω we have $\binom{N}{a} \geq \underline{R}\left(\left\langle \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega} \right\rangle\right)$.

The asymptotic sum inequality

Theorem (the asympotic sum inequality, special case)

$$\underline{R}(\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle) \leq t \Longrightarrow (m_1 n_1 p_1)^{\omega/3} + (m_2 n_2 p_2)^{\omega/3} \leq t$$

<u>Proof outline</u>

Take the N-th power, for some large N:

$$t^N \geq \underline{R}\left((\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle)^{\otimes N}\right)$$

$$= \underline{R}\left(\sum_{a=0}^{N} \binom{N}{a} \langle m_1, n_1, p_1\rangle^{\otimes a} \otimes \langle m_2, n_2, p_2\rangle^{\otimes (N-a)}\right)$$

direc

use $k^\omega \geq \underline{R}(\langle k, k, k\rangle)$

$$= \underline{R}\left(\sum_{a=0}^{N} \underbrace{\binom{N}{a} \left\langle m_1^a m_2^{(N-a)}, n_1^a n_2^{(N-a)}, p_1^a p_2^{(N-a)}\right\rangle}_{\mathrm{T}_a}\right)$$

By definition of ω we have $\binom{N}{a} \geq \underline{R}\left(\left\langle \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega}\right\rangle\right)$.

The asymptotic sum inequality

Theorem (the asympotic sum inequality, special case)

$$\underline{R}(\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle) \leq t \implies (m_1 n_1 p_1)^{\omega/3} + (m_2 n_2 p_2)^{\omega/3} \leq t$$

Proof outline

Take the N-th power, for some large N:

$$t^N \geq \underline{R}\left((\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle)^{\otimes N}\right)$$

$$= \underline{R}\left(\sum_{a=0}^{N} \boxed{\binom{N}{a}} \langle m_1, n_1, p_1\rangle^{\otimes a} \otimes \langle m_2, n_2, p_2\rangle^{\otimes(N-a)}\right)$$

direc

use $k^{\omega+\varepsilon} \geq \underline{R}(\langle k, k, k\rangle)$ for a small $\varepsilon > 0$

$$= \underline{R}\left(\sum_{a=0}^{N} \underbrace{\binom{N}{a}\left\langle m_1^a m_2^{(N-a)}, \quad n_2^{(N-a)}, p_1^a p_2^{(N-a)}\right\rangle}_{T_a}\right)$$

By definition of ω we have $\binom{N}{a} \geq \underline{R}\left(\left\langle \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega}\right\rangle\right)$.

The asymptotic sum inequality

Theorem (the asympotic sum inequality, special case)

$$\underline{R}(\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle) \le t \Longrightarrow (m_1 n_1 p_1)^{\omega/3} + (m_2 n_2 p_2)^{\omega/3} \le t$$

<u>Proof outline</u>

Take the N-th power, for some large N:

$$t^N \ge \underline{R}\left((\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle)^{\otimes N}\right)$$

$$= \underline{R}\left(\sum_{a=0}^{N} \boxed{\binom{N}{a}} \langle m_1, n_1, p_1\rangle^{\otimes a} \otimes \langle m_2, n_2, p_2\rangle^{\otimes (N-a)}\right)$$

direc

use $k^{\omega+\varepsilon} \ge \underline{R}(\langle k, k, k\rangle)$ for a small $\varepsilon > 0$

$$= \underline{R}\left(\sum_{a=0}^{N} \underbrace{\binom{N}{a} \left\langle m_1^a m_2^{(N-a)}, \; n_2^{(N-a)}, p_1^a p_2^{(N-a)}\right\rangle}_{T_a}\right)$$

By definition of ω we have $\binom{N}{a} \ge \underline{R}\left(\left\langle \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega} \right\rangle\right)$.

$$\underline{R}(T_a) \ge \underline{R}\left(\left\langle \binom{N}{a}^{1/\omega} m_1^a m_2^{(N-a)}, \binom{N}{a}^{1/\omega} n_1^a n_2^{(N-a)}, \binom{N}{a}^{1/\omega} p_1^a p_2^{(N-a)} \right\rangle\right)$$

T

$\binom{N}{a}$ multiplications “give” $\left\langle \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega} \right\rangle$

T

$\underline{R}$ … $/3 \leq t$

P

Take the N-t power, for some large N:

$$t^N \geq \underline{R}\left(\left(\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle\right)^{\otimes N}\right)$$

$$= \underline{R}\left(\sum_{a=0}^{N} \boxed{\binom{N}{a}} \langle m_1, n_1, p_1\rangle^{\otimes a} \otimes \langle m_2, n_2, p_2\rangle^{\otimes (N-a)}\right)$$

direc

use $k^{\omega+\varepsilon} \geq \underline{R}(\langle k, k, k\rangle)$ for a small $\varepsilon > 0$

$$= \underline{R}\left(\sum_{a=0}^{N} \binom{N}{a} \underbrace{\left\langle m_1^a m_2^{(N-a)}, \quad n_2^{(N-a)}, p_1^a p_2^{(N-a)} \right\rangle}_{T_a}\right)$$

By definition of ω we have $\binom{N}{a} \geq \underline{R}\left(\left\langle \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega} \right\rangle\right)$.

$$\underline{R}(T_a) \geq \underline{R}\left(\left\langle \binom{N}{a}^{1/\omega} m_1^a m_2^{(N-a)}, \binom{N}{a}^{1/\omega} n_1^a n_2^{(N-a)}, \binom{N}{a}^{1/\omega} p_1^a p_2^{(N-a)} \right\rangle\right)$$

T

$\binom{N}{a}$ multiplications “give” $\left\langle \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega} \right\rangle$

$\binom{N}{a}$ copies of $\left\langle m_1^a m_2^{(N-a)}, n_1^a n_2^{(N-a)}, p_1^a p_2^{(N-a)} \right\rangle$ “give”

$$\left\langle \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega} \right\rangle \otimes \left\langle m_1^a m_2^{(N-a)}, n_1^a n_2^{(N-a)}, p_1^a p_2^{(N-a)} \right\rangle$$

T

$\underline{R}$ $/3 \leq t$

P

Take the N-th power, for some large N:

$$t^N \geq \underline{R}\left(\left(\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle\right)^{\otimes N}\right)$$

$$= \underline{R}\left(\sum_{a=0}^{N} \boxed{\binom{N}{a}} \langle m_1, n_1, p_1\rangle^{\otimes a} \otimes \langle m_2, n_2, p_2\rangle^{\otimes (N-a)}\right)$$

direc

use $k^{\omega+\varepsilon} \geq \underline{R}(\langle k, k, k\rangle)$ for a small $\varepsilon > 0$

$$= \underline{R}\left(\sum_{a=0}^{N} \underbrace{\binom{N}{a} \left\langle m_1^a m_2^{(N-a)}, n_1^a n_2^{(N-a)}, p_1^a p_2^{(N-a)} \right\rangle}_{T_a}\right)$$

By definition of ω we have $\binom{N}{a} \geq \underline{R}\left(\left\langle \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega} \right\rangle\right)$.

$$\underline{R}(T_a) \geq \underline{R}\left(\left\langle \binom{N}{a}^{1/\omega} m_1^a m_2^{(N-a)}, \binom{N}{a}^{1/\omega} n_1^a n_2^{(N-a)}, \binom{N}{a}^{1/\omega} p_1^a p_2^{(N-a)} \right\rangle\right)$$

T

T

$\underline{R}$ $/3 \le t$

P

$\binom{N}{a}$ multiplications "give" $\left\langle \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega} \right\rangle$

$$\overset{T_a}{\boxed{\binom{N}{a} \text{ copies of } \left\langle m_1^a m_2^{(N-a)}, n_1^a n_2^{(N-a)}, p_1^a p_2^{(N-a)} \right\rangle}}$$ "give"

$$\left\langle \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega} \right\rangle \otimes \left\langle m_1^a m_2^{(N-a)}, n_1^a n_2^{(N-a)}, p_1^a p_2^{(N-a)} \right\rangle$$

Take the N-th power, for some large N:

$$t^N \ge \underline{R}\left((\langle m_1, n_1, p_1 \rangle \oplus \langle m_2, n_2, p_2 \rangle)^{\otimes N}\right)$$

$$= \underline{R}\left(\sum_{a=0}^{N} \binom{N}{a} \langle m_1, n_1, p_1 \rangle^{\otimes a} \otimes \langle m_2, n_2, p_2 \rangle^{\otimes (N-a)}\right)$$

direc

use $k^{\omega+\varepsilon} \ge \underline{R}(\langle k, k, k \rangle)$ for a small $\varepsilon > 0$

$$= \underline{R}\left(\sum_{a=0}^{N} \underbrace{\binom{N}{a} \left\langle m_1^a m_2^{(N-a)}, n_1^a n_2^{(N-a)}, p_1^a p_2^{(N-a)} \right\rangle}_{T_a}\right)$$

By definition of ω we have $\binom{N}{a} \ge \underline{R}\left(\left\langle \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega} \right\rangle\right)$.

$$\underline{R}(\boxed{T_a}) \ge \underline{R}\left(\left\langle \binom{N}{a}^{1/\omega} m_1^a m_2^{(N-a)}, \binom{N}{a}^{1/\omega} n_1^a n_2^{(N-a)}, \binom{N}{a}^{1/\omega} p_1^a p_2^{(N-a)} \right\rangle\right)$$

$\binom{N}{a}$ multiplications "give" $\left\langle \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega} \right\rangle$

$$\overset{T_a}{\boxed{\binom{N}{a} \text{ copies of } \left\langle m_1^a m_2^{(N-a)}, n_1^a n_2^{(N-a)}, p_1^a p_2^{(N-a)} \right\rangle}} \text{ "give"}$$

$$\boxed{\left\langle \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega} \right\rangle \otimes \left\langle m_1^a m_2^{(N-a)}, n_1^a n_2^{(N-a)}, p_1^a p_2^{(N-a)} \right\rangle}$$

T

R … $/3 \le t$

P

Take the N-th power, for some large N:

$$t^N \ge \underline{R}\left(\left(\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle\right)^{\otimes N}\right)$$

$$= \underline{R}\left(\sum_{a=0}^{N} \binom{N}{a} \langle m_1, n_1, p_1\rangle^{\otimes a} \otimes \langle m_2, n_2, p_2\rangle^{\otimes (N-a)}\right)$$

direc

use $k^{\omega+\varepsilon} \ge \underline{R}(\langle k, k, k\rangle)$ for a small $\varepsilon > 0$

$$= \underline{R}\left(\sum_{a=0}^{N} \underbrace{\binom{N}{a} \left\langle m_1^a m_2^{(N-a)}, n_1^a n_2^{(N-a)}, p_1^a p_2^{(N-a)} \right\rangle}_{T_a}\right)$$

By definition of ω we have $\binom{N}{a} \ge \underline{R}\left(\left\langle \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega} \right\rangle\right)$.

$$\underline{R}(\boxed{T_a}) \ge \underline{R}\left(\boxed{\left\langle \binom{N}{a}^{1/\omega} m_1^a m_2^{(N-a)}, \binom{N}{a}^{1/\omega} n_1^a n_2^{(N-a)}, \binom{N}{a}^{1/\omega} p_1^a p_2^{(N-a)} \right\rangle}\right)$$

T

$\binom{N}{a}$ multiplications "give" $\left\langle \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega} \right\rangle$

$$\overset{\text{T}_a}{\boxed{\binom{N}{a} \text{ copies of } \left\langle m_1^a m_2^{(N-a)}, n_1^a n_2^{(N-a)}, p_1^a p_2^{(N-a)} \right\rangle}} \text{ "give"}$$

$$\boxed{\left\langle \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega} \right\rangle \otimes \left\langle m_1^a m_2^{(N-a)}, n_1^a n_2^{(N-a)}, p_1^a p_2^{(N-a)} \right\rangle}$$

T

R $/3 \leq t$

P

Take the N-th power, for some large N:

$$t^N \geq \underline{R}\left(\left(\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle\right)^{\otimes N}\right)$$

$$= \underline{R}\left(\sum_{a=0}^{N} \binom{N}{a} \langle m_1, n_1, p_1\rangle^{\otimes a} \otimes \langle m_2, n_2, p_2\rangle^{\otimes (N-a)}\right)$$

direc

use $k^{\omega+\varepsilon} \geq \underline{R}(\langle k, k, k\rangle)$ for a small $\varepsilon > 0$

$$= \underline{R}\left(\sum_{a=0}^{N} \underbrace{\binom{N}{a} \left\langle m_1^a m_2^{(N-a)}, n_1^a n_2^{(N-a)}, p_1^a p_2^{(N-a)} \right\rangle}_{\text{T}_a}\right)$$

By definition of ω we have $\binom{N}{a} \geq \underline{R}\left(\left\langle \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega}, \binom{N}{a}^{1/\omega} \right\rangle\right)$.

$$t^N \geq \underline{R}(\boxed{T_a}) \geq \underline{R}\left(\boxed{\left\langle \binom{N}{a}^{1/\omega} m_1^a m_2^{(N-a)}, \binom{N}{a}^{1/\omega} n_1^a n_2^{(N-a)}, \binom{N}{a}^{1/\omega} p_1^a p_2^{(N-a)} \right\rangle}\right)$$

The asymptotic sum inequality

Theorem (the asymptotic sum inequality, special case)

$$\underline{R}(\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle) \leq t \Longrightarrow (m_1 n_1 p_1)^{\omega/3} + (m_2 n_2 p_2)^{\omega/3} \leq t$$

$$t^N \geq \underline{R}\left(\sum_{a=0}^{N} \underbrace{\binom{N}{a}\left\langle m_1^a m_2^{(N-a)}, n_1^a n_2^{(N-a)}, p_1^a p_2^{(N-a)}\right\rangle}_{T_a}\right)$$

For any a: $t^N \geq \underline{R}(T_a) \geq \underline{R}\left(\left\langle \binom{N}{a}^{1/\omega} m_1^a m_2^{(N-a)}, \binom{N}{a}^{1/\omega} n_1^a n_2^{(N-a)}, \binom{N}{a}^{1/\omega} p_1^a p_2^{(N-a)}\right\rangle\right)$

The asymptotic sum inequality

Theorem (the asymptotic sum inequality, special case)

$$\underline{R}(\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle) \le t \Longrightarrow (m_1n_1p_1)^{\omega/3} + (m_2n_2p_2)^{\omega/3} \le t$$

Theorem 2

$$\underline{R}(\langle m, n, p\rangle) \le t \Longrightarrow (mnp)^{\omega/3} \le t$$

$$\underbrace{p_1^a p_2^{(N-a)}\Big\rangle}_{T_a}\Bigg)$$

For any a: $t^N \ge \underline{R}(T_a) \ge \underline{R}\left(\left\langle \binom{N}{a}^{1/\omega} m_1^a m_2^{(N-a)}, \binom{N}{a}^{1/\omega} n_1^a n_2^{(N-a)}, \binom{N}{a}^{1/\omega} p_1^a p_2^{(N-a)} \right\rangle\right)$

$$\overset{\text{Th1}}{\Longrightarrow} \; t^N \ge \left(\binom{N}{a}^{3/\omega} (m_1n_1p_1)^a (m_2n_2p_2)^{N-a}\right)^{\omega/3}$$

The asymptotic sum inequality

Theorem (the asymptotic sum inequality, special case)

$$\underline{R}(\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle) \le t \Longrightarrow (m_1 n_1 p_1)^{\omega/3} + (m_2 n_2 p_2)^{\omega/3} \le t$$

Theorem 2

$$\underline{R}(\langle m, n, p\rangle) \le t \Longrightarrow (mnp)^{\omega/3} \le t$$

$$\underbrace{p_1^a p_2^{(N-a)}\Big\rangle\Bigg)}_{T_a}$$

For any a: $t^N \ge \underline{R}(T_a) \ge \underline{R}\left(\left\langle \binom{N}{a}^{1/\omega} m_1^a m_2^{(N-a)}, \binom{N}{a}^{1/\omega} n_1^a n_2^{(N-a)}, \binom{N}{a}^{1/\omega} p_1^a p_2^{(N-a)} \right\rangle\right)$

$$\overset{\text{Th1}}{\Longrightarrow} t^N \ge \left(\binom{N}{a}^{3/\omega} (m_1 n_1 p_1)^a (m_2 n_2 p_2)^{N-a}\right)^{\omega/3} = \binom{N}{a} \left((m_1 n_1 p_1)^a (m_2 n_2 p_2)^{N-a}\right)^{\omega/3}$$

The asymptotic sum inequality

Theorem (the asymptotic sum inequality, special case)

$$\underline{R}(\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle) \leq t \Longrightarrow (m_1 n_1 p_1)^{\omega/3} + (m_2 n_2 p_2)^{\omega/3} \leq t$$

$$t^N \geq \underline{R}\left(\sum_{a=0}^{N} \underbrace{\binom{N}{a} \left\langle m_1^a m_2^{(N-a)}, n_1^a n_2^{(N-a)}, p_1^a p_2^{(N-a)} \right\rangle}_{\mathrm{T}_a}\right)$$

For any a: $t^N \geq \underline{R}(T_a) \geq \underline{R}\left(\left\langle \binom{N}{a}^{1/\omega} m_1^a m_2^{(N-a)}, \binom{N}{a}^{1/\omega} n_1^a n_2^{(N-a)}, \binom{N}{a}^{1/\omega} p_1^a p_2^{(N-a)} \right\rangle\right)$

$$\overset{\text{Th1}}{\Longrightarrow} \; t^N \geq \left(\binom{N}{a}^{3/\omega} (m_1 n_1 p_1)^a (m_2 n_2 p_2)^{N-a}\right)^{\omega/3} = \binom{N}{a} \left((m_1 n_1 p_1)^a (m_2 n_2 p_2)^{N-a}\right)^{\omega/3}$$

Summing over all $a \in \{0, \ldots, N\}$:

$$(N+1) \times t^N \geq \left((m_1 n_1 p_1)^{\omega/3} + (m_2 n_2 p_2)^{\omega/3}\right)^N$$

The asymptotic sum inequality

Theorem (the asymptotic sum inequality, special case)

$$\underline{R}(\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle) \leq t \Longrightarrow (m_1n_1p_1)^{\omega/3} + (m_2n_2p_2)^{\omega/3} \leq t$$

$$t^N \geq \underline{R}\left(\sum_{a=0}^{N} \underbrace{\binom{N}{a}\left\langle m_1^a m_2^{(N-a)}, n_1^a n_2^{(N-a)}, p_1^a p_2^{(N-a)}\right\rangle}_{\mathrm{T}_a}\right)$$

For any a: $t^N \geq \underline{R}(T_a) \geq \underline{R}\left(\left\langle \binom{N}{a}^{1/\omega} m_1^a m_2^{(N-a)}, \binom{N}{a}^{1/\omega} n_1^a n_2^{(N-a)}, \binom{N}{a}^{1/\omega} p_1^a p_2^{(N-a)}\right\rangle\right)$

$$\overset{\text{Th1}}{\Longrightarrow} \quad t^N \geq \left(\binom{N}{a}^{3/\omega} (m_1n_1p_1)^a (m_2n_2p_2)^{N-a}\right)^{\omega/3} = \binom{N}{a}\left((m_1n_1p_1)^a (m_2n_2p_2)^{N-a}\right)^{\omega/3}$$

Summing over all $a \in \{0, \ldots, N\}$:

$$(N+1) \times t^N \geq \left((m_1n_1p_1)^{\omega/3} + (m_2n_2p_2)^{\omega/3}\right)^N$$

Taking power 1/N: $\quad t \geq (m_1n_1p_1)^{\omega/3} + (m_2n_2p_2)^{\omega/3}$ QED

The asymptotic sum inequality

Theorem (the asymptotic sum inequality, special case)

$$\underline{R}(\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle) \leq t \Longrightarrow (m_1 n_1 p_1)^{\omega/3} + (m_2 n_2 p_2)^{\omega/3} \leq t$$

here we used $(N+1)^{1/N} \to 1$

T_a

For any a: $t^N \geq \underline{R}(T_a)$ $\underline{R}\left(\left\langle \binom{N}{a}^{1/\omega} m_1^a m_2^{(N-a)}, \binom{N}{a}^{1/\omega} n_1^a n_2^{(N-a)}, \binom{N}{a}^{1/\omega} p_1^a p_2^{(N-a)} \right\rangle\right)$

$$\overset{\text{Th1}}{\Longrightarrow} \quad t^N \geq \left(\binom{N}{a}^{3/\omega} (m_1 n_1 p_1)^a (m_2 n_2 p_2)^{N-a} \right)^{\omega/3} = \binom{N}{a} \left((m_1 n_1 p_1)^a (m_2 n_2 p_2)^{N-a}\right)^{\omega/3}$$

Summing over all $a \in \{0, \dots, N\}$:

$$(N+1) \times t^N \geq \left((m_1 n_1 p_1)^{\omega/3} + (m_2 n_2 p_2)^{\omega/3} \right)^N$$

Taking power $1/N$: $\quad t \geq (m_1 n_1 p_1)^{\omega/3} + (m_2 n_2 p_2)^{\omega/3}$ QED

The asymptotic sum inequality

Theorem (the asymptotic sum inequality, special case)

$$\underline{R}(\langle m_1, n_1, p_1\rangle \oplus \langle m_2, n_2, p_2\rangle) \leq t \Longrightarrow (m_1n_1p_1)^{\omega/3} + (m_2n_2p_2)^{\omega/3} \leq t$$

here we used $(N+1)^{1/N} \to 1$

this is the major source of inefficiency in the asymptotic sum inequality

For any a: $t^N \geq \underline{R}(T_a)$ $\underline{R}\left(\left\langle \binom{N}{a}^{1/\omega} m_1^a m_2^{(N-a)}, \binom{N}{a}^{1/\omega} n_1^a n_2^{(N-a)}, \binom{N}{a}^{1/\omega} p_1^a p_2^{(N-a)} \right\rangle\right)$

$$\overset{\text{Th1}}{\Longrightarrow} t^N \geq \left(\binom{N}{a}^{3/\omega} (m_1n_1p_1)^a (m_2n_2p_2)^{N-a}\right)^{\omega/3} = \binom{N}{a} \left((m_1n_1p_1)^a (m_2n_2p_2)^{N-a}\right)^{\omega/3}$$

Summing over all $a \in \{0, \ldots, N\}$:

$$(N+1) \times t^N \geq \left((m_1n_1p_1)^{\omega/3} + (m_2n_2p_2)^{\omega/3}\right)^N$$

Taking power 1/N: $t \geq (m_1n_1p_1)^{\omega/3} + (m_2n_2p_2)^{\omega/3}$ QED

History of the main improvements on the exponent of square matrix multiplication

Upper bound	Year	Authors
$\omega \leq 3$		
$\omega < 2.81$	1969	Strassen
$\omega < 2.79$	1979	Pan
$\omega < 2.78$	1979	Bini, Capovani, Romani and Lotti
$\omega < 2.55$	1981	Schönhage
$\omega < 2.53$	1981	Pan
$\omega < 2.52$	1982	Romani
$\omega < 2.50$	1982	Coppersmith and Winograd
$\omega < 2.48$	1986	Strassen
$\omega < 2.376$	1987	Coppersmith and Winograd
$\omega < 2.373$	2010	Stothers
$\omega < 2.3729$	2012	Vassilevska Williams
$\omega < 2.3728639$	2014	LG

Asymptotic sum inequality

Overview of the Lectures

✓ Fundamental techniques for fast matrix multiplication (1969~1987)

- Basics of bilinear complexity theory: exponent of matrix multiplication, Strassen's algorithm, bilinear algorithms
- First technique: tensor rank and recursion
- Second technique: border rank
- Third technique: the asymptotic sum inequality
- Fourth technique: the laser method

Lecture 1

✓ Recent progress on matrix multiplication (1987~)

- Laser method on powers of tensors
- Other approaches
- Lower bounds
- Rectangular matrix multiplication

Lecture 2

✓ Applications of matrix multiplications, open problems

Lecture 3