Auction Design: Max Revenue

Wednesday, August 26, 2020 9:28 AM

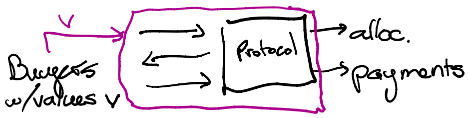
How to sell a used car?
- negotiate
- posted price
- run an auction

} · market research to see what it's worth
· advertise

Given:
- 1 item
- n buyers, $v_i \sim F_i$

Sell item to max revenue

Mechanism:



Bayesian assumption: Buyer's value $v_i \sim F_i$, F_i is common knowledge.

Example. 1 buyer, $v \sim U[0,1]$
optimal posted price?

$rev(p) = p \cdot Pr[sale] = p \times (1-p)$

$p^* = \arg\max\ p(1-p) = \arg\max\ (p - p^2)$

$\left(\frac{d}{dp}(p-p^2) = 1 - 2p \Rightarrow p = 1/2\right)$

$= 1/2$

$rev = p \times (1-p) = 1/4$

Bayes-Nash Equil. (BNE)

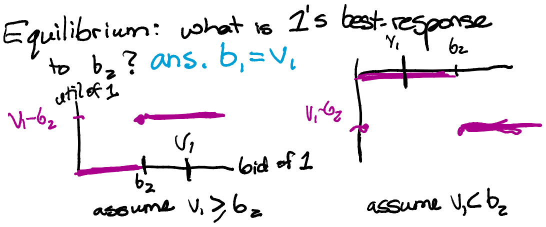
- strategies s_i: {values} $\rightarrow$ {bids}
- common prior $v_i \sim F_i$
- outcomes $x_i^{\hat{}}(s(v)) \equiv x_i(v)$, $p_i^{\hat{}}(s(v)) \equiv p_i(v)$
- interim outcomes $x_i(v_i) \equiv E_{F_{-i}}[x_i(v) \mid v_i]$

  $p_i(v_i) \equiv E_{F_{-i}}[p_i(v) \mid v_i]$
- interim utility $u_i(v_i) = v_i x_i(v_i) - p_i(v_i)$

$$\boxed{\begin{array}{l}\text{defn BNE iff } \forall i, v_i, z \\ v_i x_i(v_i) - p_i(v_i) \geq v_i x_i(z) - p_i(z) \\ (\text{assume } s(\cdot) \text{ is onto})\end{array}}$$

Example A: 2^{nd} price auction
2 bidders $v_i \sim U[0,1]$
Mechanism: solicit bids b_i
- if $b_1 \geq b_2$, 1 win and pays b_2
  else $b_2 > b_1$, 2 win " b_1

Equilibrium: what is 1's best-response to b_2? ans. $b_1 = v_1$



• A truthful dominant strategy equil.

Example B: 1^{st} price auction
2 bidders $v_i \sim U[0,1]$
Mechanism: solicit bids b_i
- if $b_1 > b_2$, 1 wins and pays b_1
  else $b_2 > b_1$, 2 " b_2

Equilibrium: Guess and check, $s(v) = v/2$
- if I bid b, $Pr[I \text{ win}] = Pr_{v \sim U[0,1]}[b > s(v)]$

  $= Pr_v[b > v/2] = Pr_v[v < 2b] = 2b$
- best response: given v^I, pick b^I st

  $b^I = \arg\max (v^I - p^I) Pr[I \text{ win}]$

  $= \arg\max (2b^I(v^I - b^I))$

  $= v^I/2$

Question: $E_{v_1,v_2}[Rev(A)] = 1/3 \equiv E_{v_1,v_2}[Rev(B)] = 1/3$

Better Revenue? Example C.

2^{nd} price auction w/reserve r: if higher bid $\geq r$, win + pay $\max(r, 2^{nd}$ highest bid$)$

Revenue: label bidders st. $v_1 \geq v_2$

case 1:	$r \geq v_1 \geq v_2$	r^2	0
case 2:	$v_1 \geq v_2 \geq r$	$(1-r)^2$	$\frac{2}{3}r + \frac{1}{3}$
case 3:	$v_1 \geq r \geq v_2$	$2(1-r)r$	r
		prob of	rev in

$\Big\}\ (1-r)^2\left(\frac{2}{3}r + \frac{1}{3}\right) + 2(1-r)r^2$

optimized at $r = 1/2$

$rev = 5/12$

case 3: $v_1 \geq r \geq v_2$ $2(1-r)r$ r rev $= 5/12$

case   prob of case   rev in case

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def. A mechanism is *direct* if {bid} = {values}.

Revelation Principle Any outcome (x, p) implemented by some mech. in an equilibrium can be implemented by an incentive-compatible direct mech.

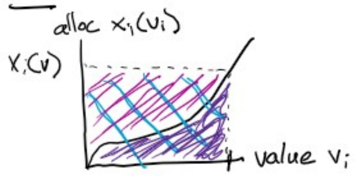
Prf. (sketch) Given a mech, strategies $s(\cdot)$, direct mech inputs v + feeds $s(v)$ to original mech.

Characterization Thm. (x, p) are the BNE of a mech iff

(i) monotonicity $x_i(v_i)$ monotone + non-decreasing

(ii) payment identity $p_i(v_i) = v_i x_i(v_i) - \int_0^{v_i} x_i(z)\,dz$

Prf. (i) + (ii) $\rightarrow$ BNE

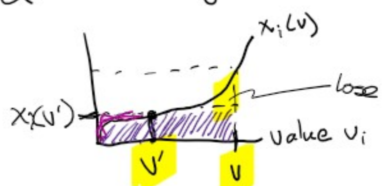


surplus: $v\, x_i(v)$

payment: by (ii)

utility: surplus − payment
$v x_i(v) - p_i(v)$

Q. could agent w/value v benefit by impersonating an agent of value v'?



$v\, x_i(v') - p_i(v')$

$x_i(v_i) \equiv E[x_i(v_1, \dots, v_n) \mid v_i]$

BNE $\longrightarrow$ (i) + (ii): follow incentive constraints

$$\begin{cases} v\,x(v) - p(v) \geq v\,x(v') - p(v') \\ v'x(v') - p(v') \geq v'x(v) - p(v) \end{cases}$$

Consequence (Revenue Equivalence):
Auctions w/same alloc in BNE have the same revenue.

Example 1st price auction: 2 bidders, $v_i \sim [0,1]$

- guess $s(v)$ is monotone in v $\Rightarrow$ same alloc as 2nd price auction
- $p(v) = \cancel{\Pr[v \text{ wins}]}\, s(v)$ their bid

$= E[2^{nd} \text{ price payment} \mid v]$

$= \cancel{\Pr[v \text{ wins in } 2^{nd} \text{ price}]} \times E[2^{nd} \text{ highest value} \mid v \text{ is highest value}]$

$\Rightarrow s(v) = E[2^{nd} \text{ highest val} \mid v \text{ is highest}] = \frac{v}{2}$

Since $\frac{v}{2}$ is monotone in v, it must be a BNE.

$\Rightarrow s(v) = E[2^{nd}$ highest val $| v$ is highest$] = \frac{v}{2}$

Since $\frac{v}{2}$ is monotone in v, it must be a BNE.

This time: optimizing BNE, Myerson's virtue val

Recall Characterization Thm.

(x, p) implementable in BNE of some mech.

$$\Updownarrow$$

monotonicity $x_i(v_i)$ monotone non-decreasing

payment identity $p_i(v_i) = v_i x_i(v_i) - \int_0^{v_i} x_i(z)dz + p(0)$

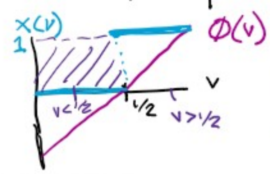
Lemma. [Myerson '81] $E[p_i(v_i)] = E[\phi_i(v_i) x_i(v_i)]$

where $\phi_i(v_i) = v_i - \frac{1-F(v_i)}{f(v_i)}$ is the virtual value.

Approach:
- calculate virtual values ϕ_i
- choose x to max $E[\phi_i(v_i) x_i(v_i)]$
- check that x is monotone
- use payment identity to calc. p

Example A' 1 buyer, $v \sim U[0,1]$

$\phi(v) = v - \frac{1-v}{1} = 2v-1$



$p(v) = vx - \int_0^v x(z)dz$
$= 1/2$ for $v > 1/2$
else 0.

Example E n buyers, $v_i \sim U[0,1]$

$\phi_i(v_i) = 2v_i - 1$

$\underset{\substack{x \\ \sum_i x_i(v_i) \leq 1}}{\operatorname{argmax}} \sum_i \phi_i(v_i) x_i(v_i)$ } allocate to highest v_i if $v_i \geq 1/2$ $\Rightarrow 2^{nd}$ price auction w/reserve $= 1/2$

Prf. (of Myerson's Lemma)

$$E[p_i(v)] = E\left[v x_i(v) - \int_0^v x_i(z)dz\right]$$

$$= \int_0^{\bar{v}} \left(v x(v) - \int_0^v x(z)dz\right) f(v)dv$$

$$= \int_0^{\bar{v}} v x(v) f(v) dv - \int_0^{\bar{v}} \underbrace{\int_0^v x(z)dz}_{h} \underbrace{f(v)dv}_{dg}$$

recall integration by parts: $\int_a^b h\,dg = hg\Big|_a^b - \int_a^b g\,dh$

$$= \int_0^{\bar{v}} v x(v) f(v) dv - \left[\underbrace{\left(\int_0^v x(z)dz\,(F(v)-1)\right)\Big|_0^{\bar{v}}}_{0} - \int_0^{\bar{v}} ((F(v)-1)\, x(v) dv\right]$$

$$= \int_0^{\bar{v}} x(v)\left(v f(v) + (F(v)-1)\right) dv$$

$$= \int_0^{\bar{v}} x(v)\left(v - \frac{1-F(v)}{f(v)}\right) f(v) dv$$

$$= E\left[\left(v - \frac{1-F(v)}{f(v)}\right) x(v)\right]$$