Fair Division

ADFOCS 2020

24 - 28 August 2020

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ILLINOIS
UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

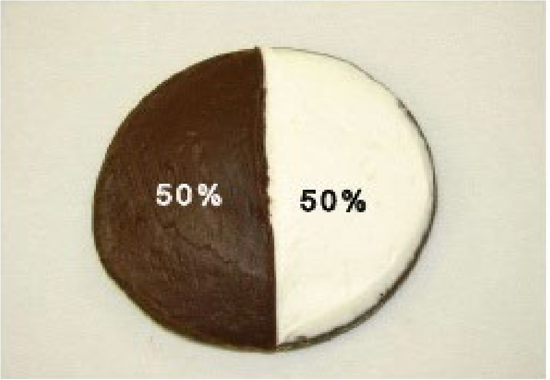
Fair Division

Scares resources

Goal: allocate *fairly and efficiently.*

And do it quickly!

Example: Half moon cookie



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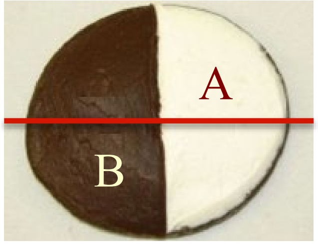

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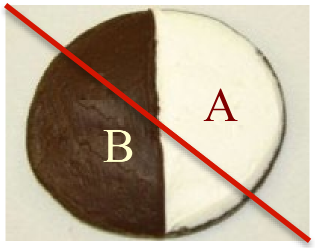

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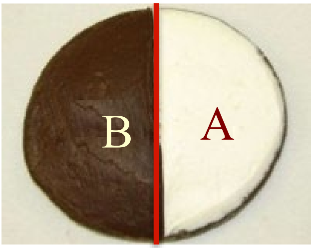
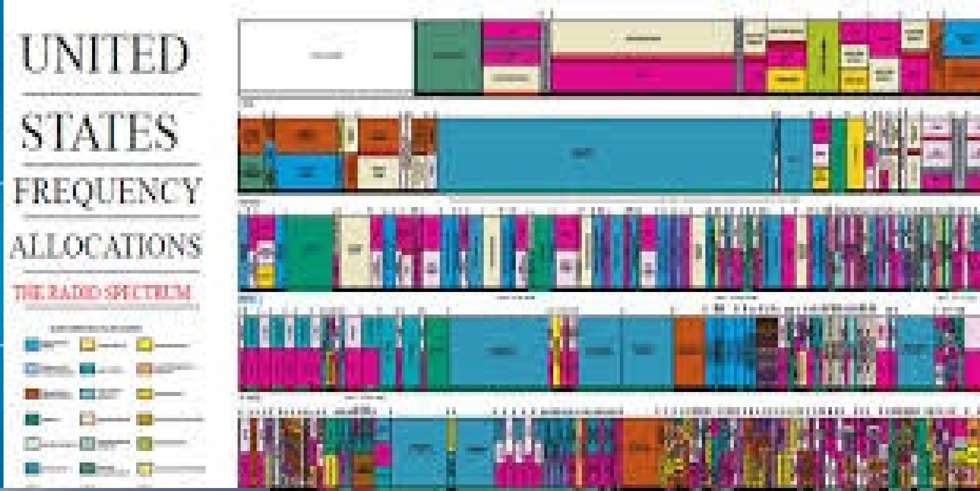


UCLA Kidney Exchan



We *plan* to cover

Part 1: Divisible items (Ruta)

- ☐ Competitive equilibrium and Properties
- ☐ Computation: Fisher, Spending-restricted, Hylland-Zeckhauser

Part 2: Indivisible items (Jugal)

- ☐ Envy-freeness: EF1, EFX
- ☐ Proportionality: MMS, Prop1
- ☐ Nash welfare guarantees

And lots of open questions!

Lecture 1: Competitive Equilibrium

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24^{th} August 2020

Ruta Mehta

ILLINOIS
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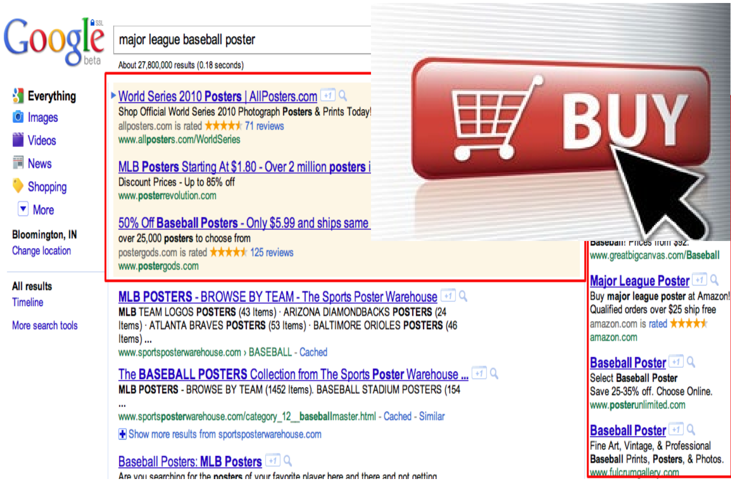
Markets

One of the biggest real-life mechanism that enables (re)distribution of resources.

And they seem to work!

Q: What? Why? And How?



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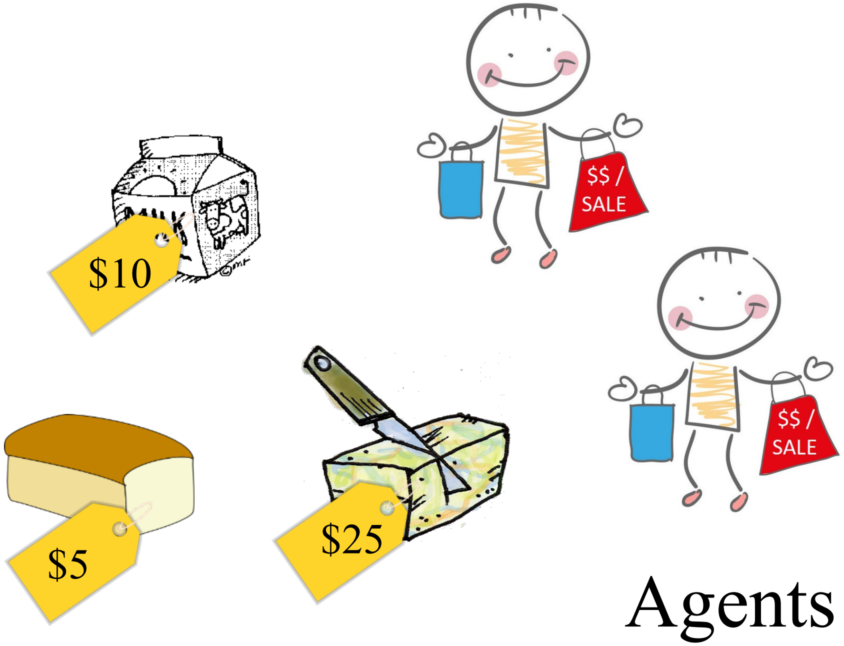
Markets



Goods

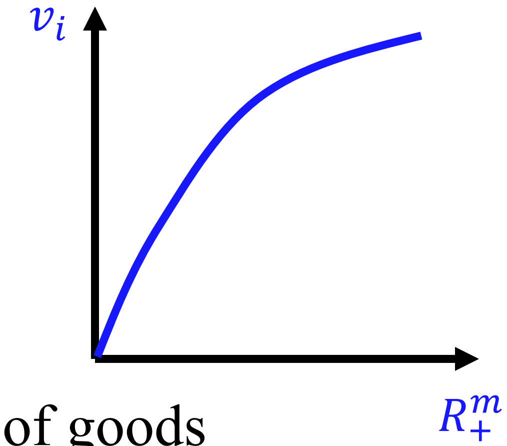
Agents
(buyers)

Buy optimal bundle

Competitive Equilibrium:
Demand = Supply

Fisher's Model (1891)

- A: set of n agents
- G: set of m divisible goods



- Each agent i has
  - ☐ budget of B_i dollars
  - ☐ valuation function $v_i: R_+^m \to R_+$ over bundle of goods (non-decreasing, non-negative)

- Supply of every good is one

Competitive Equilibrium (CE)

Given prices $p = (p_1, \dots, p_m)$ of goods

- Agent i demands an *optimal bundle*, i.e., affordable bundle that maximizes her utility

$$x_i \in argmax_{x:\, p \cdot x \leq B_i}\, v_i(x)$$

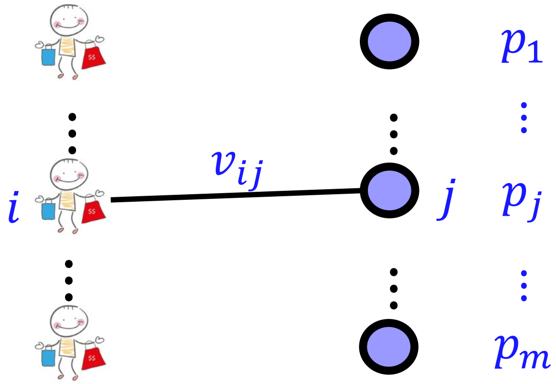
- p is at competitive equilibrium (CE) if *market clears*

Demand = Supply

CE: Linear Valuations

$$v_i(x_i) = \sum_{j \in M} \boxed{v_{ij}} x_{ij}$$

Utility per unit



Optimal bundle: can spend at most B_i dollars.

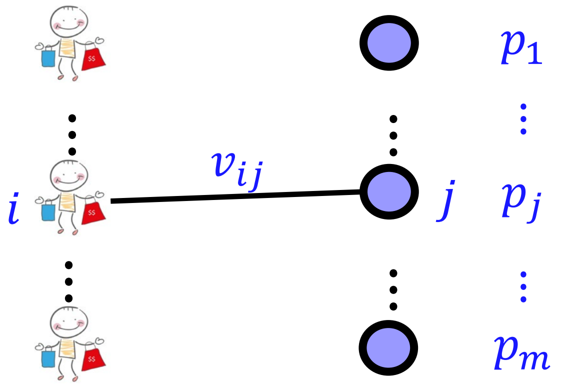
Intuitition

spend wisely: on goods that gives max. utility-per-dollar $\frac{v_{ij}}{p_j}$

CE: Linear Valuations

$$v_i(x_i) = \sum_{j\in M} \boxed{v_{ij}}\, x_{ij}$$

Utility per unit



Optimal bundle: can spend at most B_i dollars.

$$\sum_{j\in M} v_{ij}x_{ij} = \sum_j \boxed{\frac{v_{ij}}{p_j}} \underbrace{(p_j x_{ij})}_{\text{(\$ spent)}} \le \left(\max_{k\in G}\frac{v_{ik}}{p_k}\right)\sum_j p_j x_{ij} \le \boxed{\left(\max_{k\in G}\frac{v_{ik}}{p_k}\right)}\, B_i$$

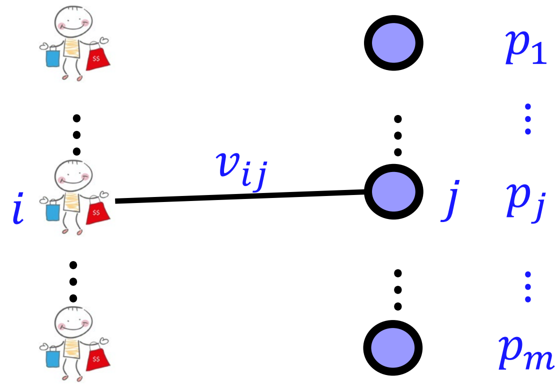
utility per dollar
(bang-per-buck)

MBB
Maximum
bang-per-buck

CE: Linear Valuations

$$v_i(x_i) = \sum_{j\in M} \boxed{v_{ij}}\, x_{ij}$$

Utility per unit



Optimal bundle: can spend at most B_i dollars.

$$\sum_{j\in M} v_{ij}x_{ij} = \sum_j \boxed{\frac{v_{ij}}{p_j}} \underbrace{(p_j x_{ij})}_{(\$\ \text{spent})} \le \left(\max_{k\in G}\frac{v_{ik}}{p_k}\right)\sum_j p_j x_{ij} \underset{=}{\le} \boxed{\left(\max_{k\in G}\frac{v_{ik}}{p_k}\right)} B_i$$

utility per dollar
(bang-per-buck)

MBB
Maximum
bang-per-buck

iff

1. Spends all of $\mathrm{B_i}$.
   $(p.x_i) = B_i$

2. Only on MBB goods
   $x_{ij} > 0 \Rightarrow \frac{v_{ij}}{p_j} = MBB$

CE Characterization

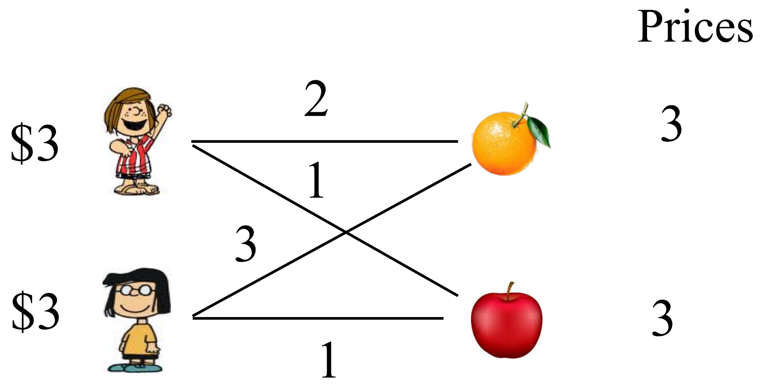
Pirces $p = (p_1, \dots, p_m)$ and allocation $X = (x_1, \dots, x_n)$ are at equilibrium iff

- Optimal bundle (OB): For each agent i
  - ☐ $p \cdot x_i = B_i$
  - ☐ $x_{ij} > 0 \Rightarrow \frac{v_{ij}}{p_j} = \max_{k \in M} \frac{v_{ik}}{p_k}$, for all good j
- Market clears: For each good j,

$$\sum_i x_{ij} = 1.$$

Example

- 2 Buyers (, ), 2 Items (, ) with unit supply
- Each buyer has budget of $1 and a linear utility function



Example

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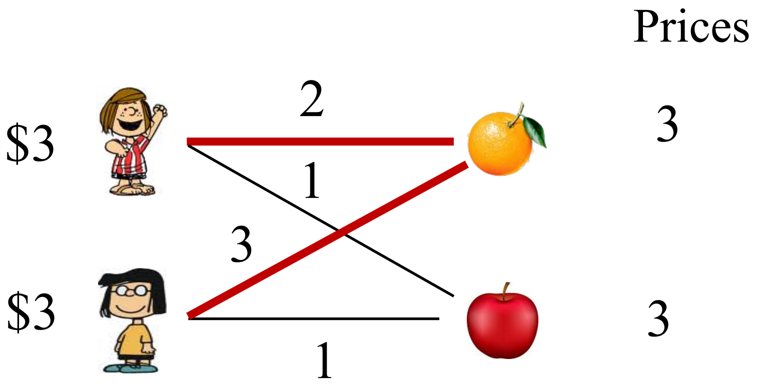


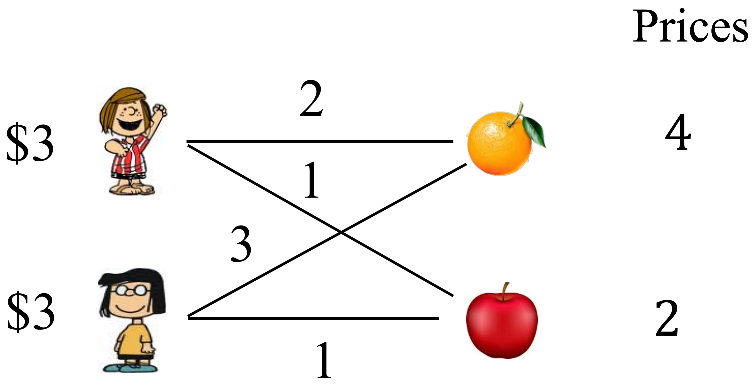
Demand ≠ Supply

MBB

Not an Equilibrium!

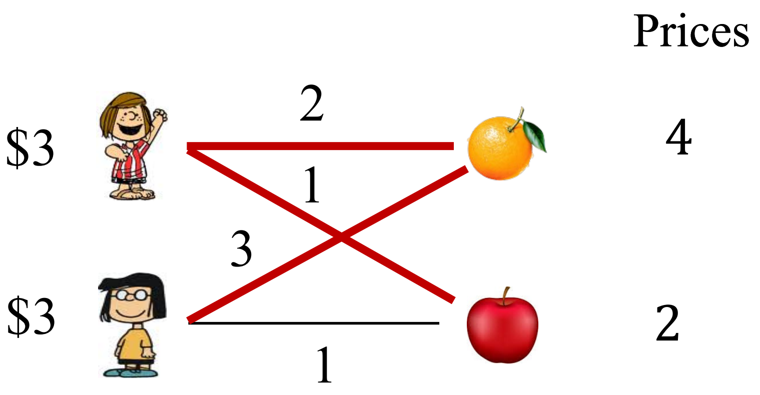
Example

- 2 Buyers (, ), 2 Items (, ) with unit supply
- Each buyer has budget of $1 and a linear utility function



Example

- 2 Buyers (, ), 2 Items (🍊, 🍎) with unit supply
- Each buyer has budget of \$1 and a linear utility function



$(\frac{1}{4}, 1)$ $(\frac{3}{4}, 0)$

Demand = Supply

MBB

Equilibrium!

Existence?

Many ways to prove. We will see one later.

Properties

Efficiency: Pareto optimality

- An allocation $Y = (y_1, y_2, \dots, y_n)$ Pareto dominates another allocation $X = (x_1, x_2, \dots, x_n)$ if
  - $u_i(y_i) \geq u_i(x_i)$, for all buyers i and
  - $u_k(y_k) > u_k(x_k)$ for some buyer k

Efficiency: Pareto optimality

- An allocation $Y = (y_1, y_2, \ldots, y_n)$ Pareto dominates another allocation $X = (x_1, x_2, \ldots, x_n)$ if
  - $u_i(y_i) \geq u_i(x_i)$, for all buyers i and
  - $u_k(y_k) > u_k(x_k)$ for some buyer k

- X is said to be Pareto optimal (PO) if there is no Y that Pareto dominates it

First Welfare Theorem

Theorem: Competitive equilibrium outputs a PO allocation

Proof: (by contradiction)

- Let (p, X) be equilibrium prices and allocations
- Suppose Y Pareto dominates X. That is,
  $v_i(y_i) \geq v_i(x_i), \forall i \in N$, and $v_k(y_k) > v_k(x_k)$ for some k
- Total cost of Y is $\sum_i (p \cdot y_i) \leq \sum_j p_j = \sum_i B_i$
- k demands x_k at prices p and not y_k, because?
- Money *agent i* needs to purchase y_i?

■

CEEI [Foley 1967, Varian 1974]

Competitive Equilibrium with Equal Income

Problem: Fairly allocate a set of goods among agents without involving money

- Give every agent *(fake)* \$1 and compute competitive equilibrium!

Envy-Free (EF)

Allocation X is **envy-free** if every agent prefers her own bundle than anyone else's. That is, for each agent i,

$$v_i(x_i) \geq v_i(x_k), \forall k \in A$$

Theorem: CEEI is envy-free

Proof: Let (p, X) be a CEEI.

- Since the budget of each agent i is \$1, $(p \cdot x_i) = 1$.
- Can agent i afford agent k's bundle (x_k)?

YES

- But she demands x_i instead. Why?

$$v_i(x_i) \geq v_i(x_k)$$

■

Proportionality

Allocation X is **proportional** if every agent gets at least the average of her total value of all goods. That is, for each agent i,

$$v_i(x_i) \geq \frac{v_i(G)}{n}$$

Theorem: CEEI is envy-free

Proof: (EF ⇒ Proportional)

- Let (p, X) be a CEEI.
- X is EF. That is, $v_i(x_i) \geq v_i(x_k), \forall k \in A$. Sum-up over all j

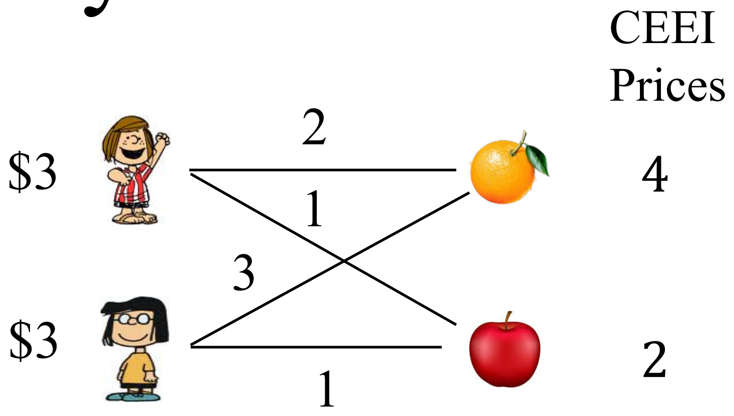
$$n * v_i(x_i) \geq \sum_{k \in A} v_i(x_k) = v_i\left(\sum_{k \in A} x_k\right) = v_i(G)$$

■

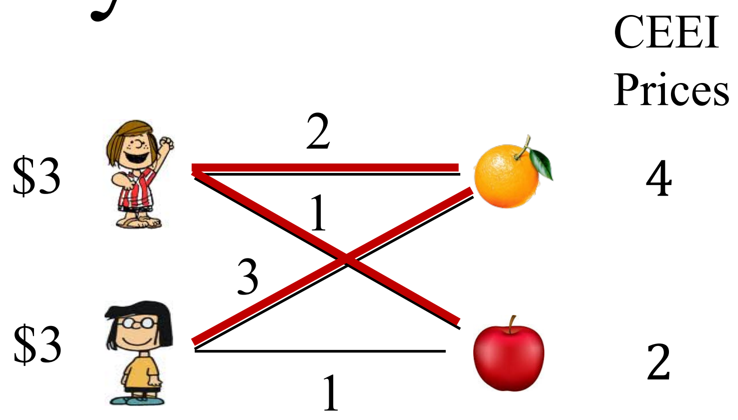
CEEI Properties: Summary

CEEI allocation is

- Pareto optimal (PO)
- Envy-free
- Proportional



CEEI Properties: Summary



CEEI allocation is

- Pareto optimal (PO)
- Envy-free
- Proportional

CEEI Allocation:

$x_1 = \left(\frac{1}{4}, 1\right), x_2 = \left(\frac{3}{4}, 0\right)$

$v_1(x_1) = \frac{3}{2},\ v_2(x_2) = \frac{9}{4}$

$v_1(x_2) = \frac{3}{2},\ v_2(x_1) = \frac{7}{4}$

Next…

- Nash welfare maximizing

Social Welfare

$$\sum_{i \in A} v_i(x_i)$$

Utilitarian

Issues: May assign 0 value to some agents.
Not scale invariant!

Nash Welfare

$$\max: \quad \prod_{i \in A} v_i(x_i)$$

$$\text{s.t.} \quad \sum_{i \in A} x_{ij} \leq 1, \ \forall j \in G$$

$$x_{ij} \geq 0, \qquad \forall i, \forall j$$

Feasible allocations

Max Nash Welfare (MNW)

$$\max: \log\left(\prod_{i\in A} v_i(x_i)\right)$$

$$\text{s.t.} \quad \textstyle\sum_{i\in A} x_{ij} \le 1, \ \forall j \in G$$
$$x_{ij} \ge 0, \qquad \forall i, \forall j$$

Feasible allocations

Max Nash Welfare (MNW)

$$\text{max:} \quad \sum_{i \in A} \log v_i(x_i)$$

$$\text{s.t.} \quad \sum_{i \in A} x_{ij} \leq 1, \ \forall j \in G$$

$$x_{ij} \geq 0, \qquad \forall i, \forall j$$

Feasible allocations

Eisenberg-Gale Convex Program ‘59

$$\text{max:} \quad \sum_{i \in A} \log v_i(x_i)$$

Dual var.

$$\text{s.t.} \quad \textstyle\sum_{i \in A} x_{ij} \leq 1, \;\; \forall j \in G \longrightarrow p_j$$

$$x_{ij} \geq 0, \qquad \forall i, \forall j$$

Theorem. Solutions of EG convex program are exactly the CEEI (p, X).

Proof.

Consequences: CEEI

- **Exists**
- Forms a convex set
- Can be *computed* in polynomial time
- MNW allocations = CEEI allocations

Theorem. Solutions of EG convex program are exactly the CEEI (p, X).

Proof. ⇒(Using KKT)

Recall: CEEI Characterization

Pirces $p = (p_1, \dots, p_m)$ and allocation $X = (x_1, \dots, x_n)$

- Optimal bundle: For each buyer i
  - ☐ $p \cdot x_i = 1$
  - ☐ $x_{ij} > 0 \Rightarrow \frac{v_{ij}}{p_j} = \max_{k \in M} \frac{v_{ik}}{p_k}$, for all good j
- Market clears: For each good j,

$$\sum_i x_{ij} = 1.$$

Theorem. Solutions of EG convex program are exactly the CEEI (p, X).

$$\max: \sum_{i\in A} \log(v_i(x_i)) \quad \xrightarrow{} \sum_j v_{ij}x_{ij}$$

Dual var.

$$\text{s.t.} \quad \sum_{i\in A} x_{ij} \leq 1, \ \forall j \in G \xrightarrow{\quad} p_j \geq 0$$

$$x_{ij} \geq 0, \qquad \forall i, \forall j$$

Proof. ⇒(Using KKT)

$$\forall j,\ p_j > 0 \Rightarrow \sum_i x_{ij} = 1$$

Dual condition to x_{ij}:

$$\frac{v_{ij}}{v_i(x_i)} \leq p_j \Rightarrow \frac{v_{ij}}{p_j} \leq v_i(x_i) \Rightarrow p_j > 0 \Rightarrow \text{market clears}$$

buy only MBB goods

$$\left.\begin{array}{l} \boxed{x_{ij} > 0 \Rightarrow \dfrac{v_{ij}}{p_j} = v_i(x_i)} \\ \downarrow \\ \sum_j v_{ij}x_{ij} = \left(\sum_j p_j x_{ij}\right) v_i(x_i) \\ \qquad \Rightarrow \sum_j p_j x_{ij} = 1 \end{array}\right\} \Rightarrow \text{optimal bundle}$$

Generalizing to CE

Budget of each agent i is B_i (need not be 1)

EG Formulation:

$$\max: \sum_{i \in A} B_i \log v_i(x_i)$$

$$\text{s.t.} \quad \sum_{i \in N} x_{ij} \leq 1, \ \forall j \in G$$

$$x_{ij} \geq 0, \qquad \forall i, \forall j$$

→ Optimal solutions exactly capture CE

CE Properties: Pareto-optimal

- Maximizes *weighted* NSW, $\left(\Pi_i v_i(x_i)^{B_i}\right)^{1/B}$
- *Weighted* envy-free: $\frac{v_i(x_i)}{B_i} \geq \frac{v_i(x_k)}{B_k}, \forall i, k$
- *Weighted* proportional: $v_i(x_i) \geq \frac{B_i}{B} v_i(G), \forall i$

$$B = \sum_i B_i$$

Efficient (Combinatorial) Algorithms

Polynomial time

- Flow based [DPSV'08]
    - ☐ General exchange model (barter system) [DM'15, DGM'16, CM'18]
- Scaling + path following [GM.SV'13]

Strongly polynomial time

- Scaling + flow [Orlin10, Vegh16]
    - ☐ Exchange model (barter system) [GV'19]

We will discuss some in the next lecture

Generalizations

Spending Restricted [CG'18] (for MNW with indivisible goods.)

- CE where total money spent on good j is at most c_j

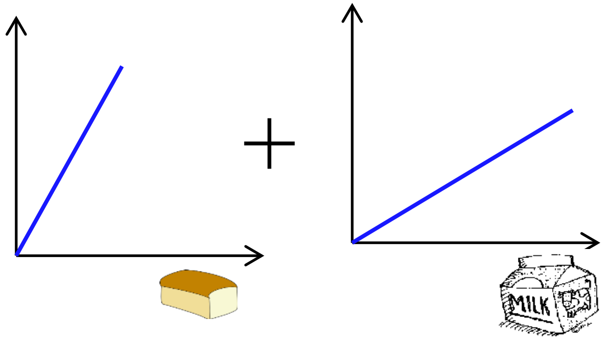
Hylland-Zeckhauser (for PO and strategy-proof matching)

- n agents and n goods
- Every agent has: (a) linear utilities, (b) unit budget,

  (c) wants at most one unit of total allocation
- HZ'79: Equilibrium exists, is PO, and is truthful at large.
  - ☐ For indivisible goods, think of allocation as a probabilities/time-share.

Generalization: Valuation Functions

$v_i : \mathbb{R}^n \rightarrow \mathbb{R}$



Linear

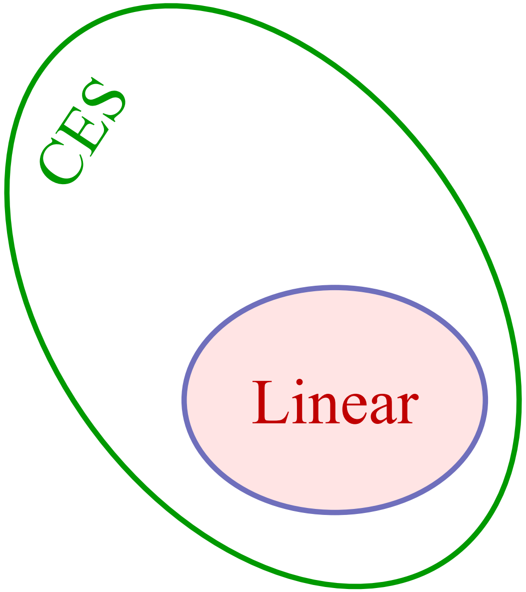
Generalization: Valuation Functions

EG program works!

$v_i : \mathbb{R}^n \rightarrow \mathbb{R}$

$v_i(x_i) = \left(\sum_j v_{ij} x_{ij}^{\rho}\right)^{1/\rho}$

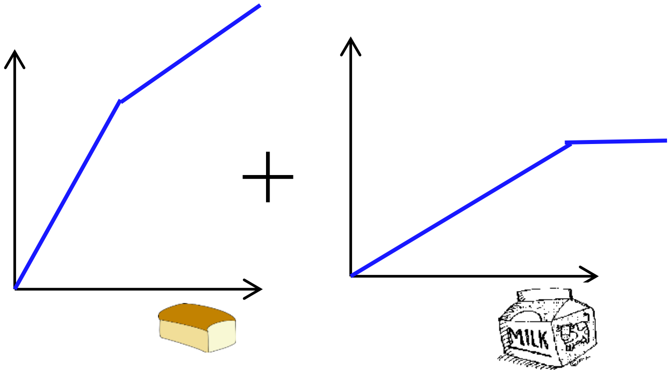
where $\rho \in (-\infty, 1]$

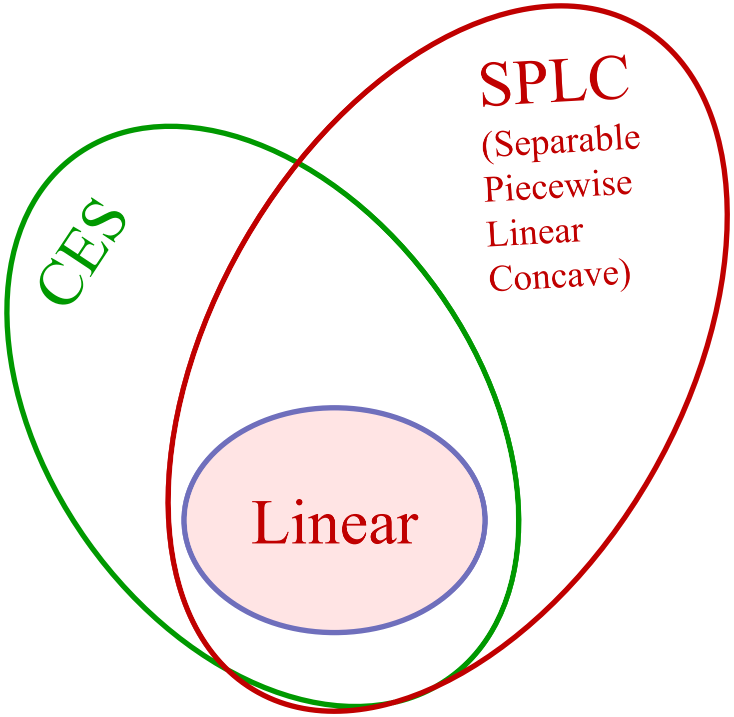


Generalization: Valuation Functions

PPAD-complete [P'94, CT'09, VY'09].
Path-following algorithm
(empirically fast) [GM.SV'12]

$v_i: \mathbb{R}^n \to \mathbb{R}$

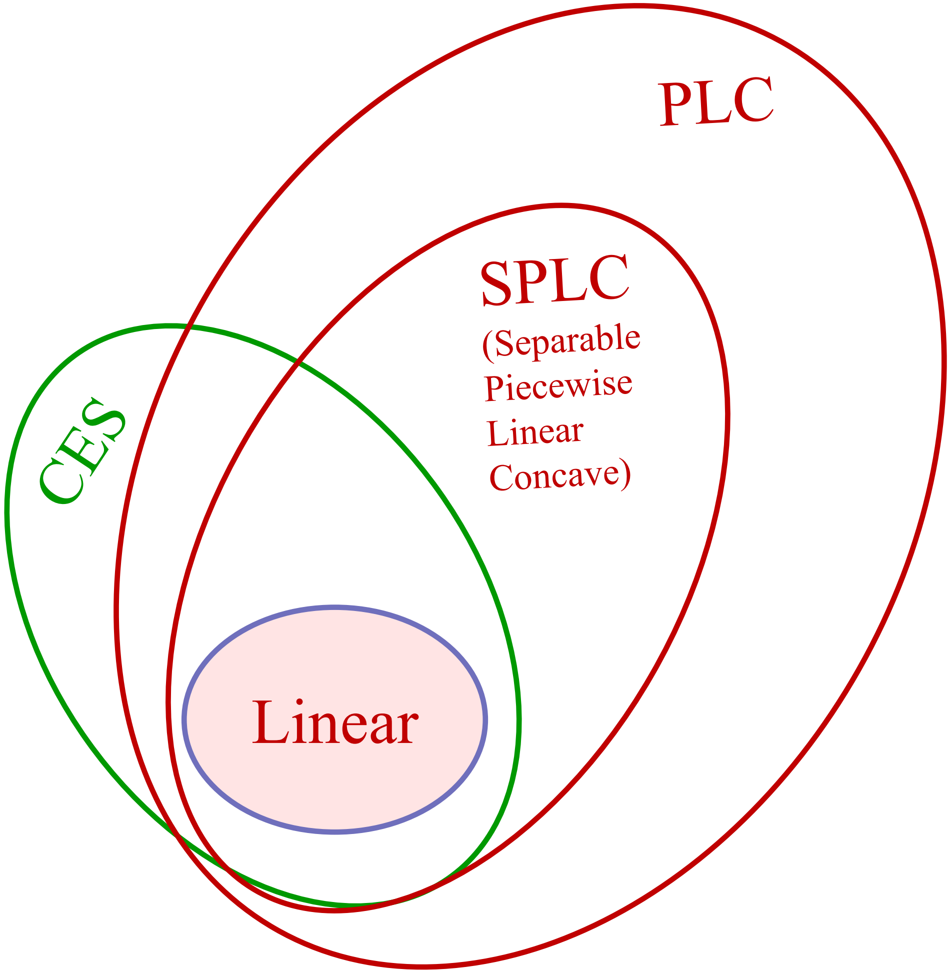




Generalization: Valuation Functions

$v_i: \mathbb{R}^n \to \mathbb{R}$

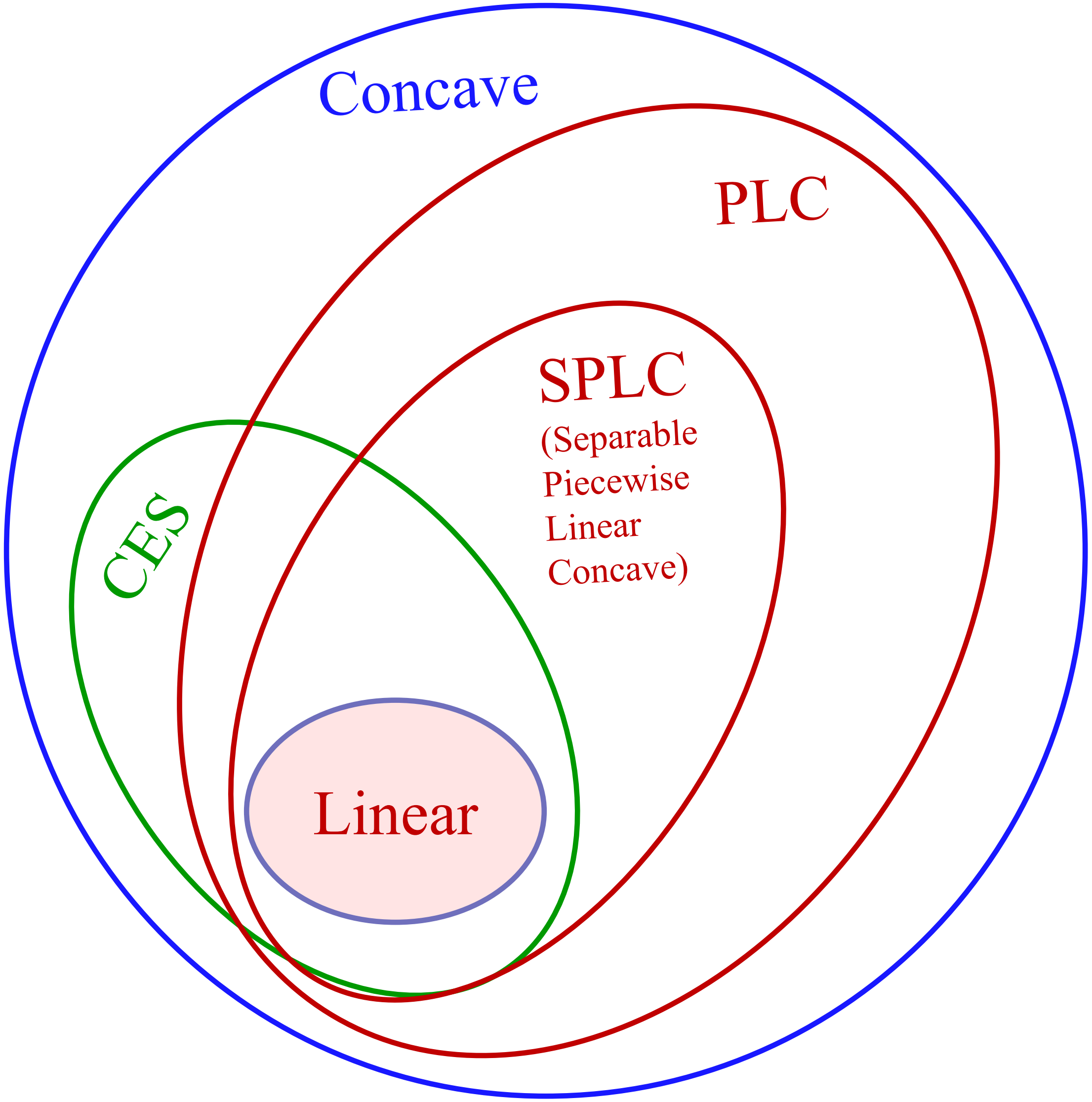
Irrational Eq.
FIXP-complete
[GM.VY'17]



Generalization: Valuation Functions

$v_i: \mathbb{R}^n \to \mathbb{R}$

Irrational Eq.
FIXP-complete
[EY'10]



Tons of other works (we will not cover)

- More generalizations like utility-restriction [CDGJMVY’17, BGHM’17,…]
- Simplex-like path-following algorithms [E’76, GM.SV’12,GM.V’14]
- Auction based algorithms [GKV’04, GK’06, KMV’07, GHV’19]
- Dynamics [WZ’07, Z’11, BDX’11, CCT’18, CCD’19, BNM.’19 …]
- Hardness results [CT’09,VY’09, GM.VY’17,…]
- Strategization and Price-of-Anarchy [ABGM.S’10,CDZ’11, CDZZ’12, BCDF-RFZ’14, M.TVV’14, BGM.’18,…]
- …

Tons of other works (we will not cover)

Cake Cutting

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