ADFOCS: Market Design
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Problem Set #2

1. In this question, we explore the structure of stable matchings.
   (a) Construct an example with n men and n women and $2^{n/2}$ stable matchings.
   (b) Consider your construction for $n = 4$ agents. For each agent x, let $S(x) = \{y : \mu(x) = y, \mu \in M\}$ be the multiset of his/her partners in the stable matchings. Draw a table with 8 rows and 4 columns. In the i'th row, $1 \leq i \leq 4$, list the elements of $S(m_i)$ in *increasing order* of $\succ_{m_i}$ (so man m_i's favorite partner from $S(m_i)$ is in the first column). In the i'th row, $5 \leq i \leq 8$, list the elements of $S(w_{i-4})$ in *decreasing order* of $\succ_{w_{i-4}}$ (so woman w_{i-4}'s favorite partner from $S(w_{i-4})$ is in the last column). What do you notice about the columns of this table?
   (c) Imagine you are designing a two-sided matching market and need to choose a stable matching deterministically. Pick three different matchings from M and compare their potential benefits and drawbacks.

   This exercise hints at the existence of a *median stable matching* in which each individual gets their "middle" stable partner. The existence of such a matching also follows from the fact that the set of stable matchings form a lattice.

2. Suppose that we are matching a set W of workers to a set F of firms. Each worker wants at most one job, but firms can potentially hire multiple workers and have preferences over subsets of workers. A matching is a function $\mu : W \to F$.[1] A blocking pair (w, f) consists of a worker w and firm f such that: the worker prefers the firm to her assigned match *and* the firm would hire w if given its choice of workers from $w \cup \mu(f)$ (i.e., it's favorite subset of $w \cup \mu(f)$ contains w). Worker-proposing DA works as in lecture. For firm-proposing DA, each firm applies to its favorite subset of workers from among the set of workers that have not yet rejected it.
   (a) Give an example in which worker-proposing results in an unstable matching.
   (b) Firms preferences are *responsive* if firms have an integer quota q_f and
      - For any $T \subseteq W$ with $|T| < q_f$, and any $w \in W \setminus T$, f prefers $T \cup \{w\}$ to T iff it prefers $\{w\}$ to no worker.
      - For any $T \subseteq W$ with $|T| < q_f$, and any $w, w' \in W \setminus T$, f prefers $T \cup \{w\}$ to $T \cup \{w'\}$ iff it prefers $\{w\}$ to $\{w'\}$.
      - The empty set is preferred to any subset T with $|T| > q_f$.

      Reduce this setting to the one-to-one matching setting from lecture and conclude both worker-proposing and firm-proposing are stable.
   (c) Consider the following instance: there are two workers and two firms. The preferences are as follows. Workers have $f_1 \succ_{w_1} f_2$ and $f_2 \succ_{w_2} f_1$. Firm f_1 has $\{w_1, w_2\} \succ_{f_1} \{w_2\} \succ_{f_1} \{w_1\}$. Firm f_2 has $\{w_1\} \succ_{f_2} \{w_2\}$.

[1] As in lecture, we overload notation and write $\mu(f)$ to denote $\{w | \mu(w) = f\}$.

- Prove the firms' preferences are responsive.
- Compute the firm-proposing stable matching and find a feasible individually rational matching that both firms prefer.
- Conclude firm-proposing is not strategyproof by exhibiting a profitable deviation.
- Note when preferences are responsive, mechanisms need only solicit the quota and preferences over individual workers to simulate college-proposing. Can colleges manipulate when restricted to this strategy space?

3. Consider the pointing mechanism from the last problem set, adapted for two-sided matching markets. Each man points to his favorite woman, each woman points to her favorite man. While there is a cycle in the resulting graph, pick an arbitrary one and match each man in that cycle to his favorite woman in that cycle. Remove the cycles and repeat.

   Give an example with 3 men and 3 women where the pointing mechanism returns an unstable matching. Use the same example to show that the pointing mechanism is not strategyproof for the women.