Feedback welcome! If you find any typos or anything is unclear or misleading, please email me and know!

For additional detail, see the companion paper, “Box-Simplex Games: Algorithms, Applications, and Algorithmic Graph Theory” on my website.

**Box-Simplex Games**
*Algorithms, Applications, and Algorithmic Graph Theory*

Part I

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Lecture Plan

Today and Tomorrow
• Box-simplex games
• Their structure
• Applications
• Algorithms

Why?
• (Applications) Continuous and combinatorial.
• (Tools) New optimization methods
• (Reinforce) Modifications of common methods

Friday
• Interior point methods
• Introduction of state-of-the-art method
The Problem

Input

• \( n \)-dimensional box: \( B^n_\infty \) \( \overset{\text{def}}{=} \{ x \in \mathbb{R}^n \mid \|x\|_\infty \leq 1 \} \)

• \( m \)-dimensional simplex: \( \Delta^m \) \( \overset{\text{def}}{=} \{ y \in \mathbb{R}^m_{\geq 0} \mid \|y\|_1 = 1 \} \)

Output:

• An approximate solution to

\[
\min_{x \in B^n_\infty} \max_{y \in \Delta^m} f(x, y) \overset{\text{def}}{=} y^T Ax + c^T x - b^T y
\]

Bounded vectors in \( \mathbb{R}^n \)

Probability distributions on \( m \) elements

Box-Simplex Game

\( \ell_1-\ell_\infty \) Game
Key Motivating Questions

**Question #1**

*How can we design efficient methods for solving box-simplex games?*

**Question #2**

*How can we leverage box-simplex solvers to solve continuous and combinatorial optimization problems?*

- Box: $B^n_\infty \triangleq \{ x \in \mathbb{R}^n \mid \|x\|_\infty \leq 1 \}$
- Simplex: $\Delta^m \triangleq \{ y \in \mathbb{R}^m_{\geq 0} \mid \|y\|_1 = 1 \}$
- $\min_{x \in B^n_\infty} \max_{y \in \Delta^m} f(x, y) \triangleq y^T A x + c^T x - b^T y$
Talk Plan (Today and Tomorrow)

Part 1
Structure of box-simplex games
- Primal and dual problems
- Approximate solutions
- Discuss state-of-the-art runtimes

Part 2
Applications
- Box-constrained $\ell_\infty$-regression
- Linear programming
- Maximum cardinality bipartite matching
- Undirected maximum flow

Part 3
Algorithms
- $\ell_\infty$-Gradient Descent (constrained steepest descent)
- $\ell_1$-Mirror Descent (multiplicative weights)
- Mirror prox and primal dual regularizers

Friday
Interior Point Methods
- Box: $B^n_\infty \equiv \{x \in \mathbb{R}^n \mid \|x\|_\infty \leq 1\}$
- Simplex: $\Delta^m \equiv \{y \in \mathbb{R}_{\geq 0}^m \mid \|y\|_1 = 1\}$
- $\min_{x \in B^n_\infty} \max_{y \in \Delta^m} f(x, y) \equiv y^T Ax + c^T x - b^T y$
Primal Problem

Lemma: \( \max_{x \in \Delta^m} d^T x = \max_{i \in [m]} d_i \) for all \( d \in \mathbb{R}^m \) and therefore

\[
 f_{\max}(x) \overset{\text{def}}{=} \max_{y \in \Delta^m} f(x, y) = c^T x + \max_{i \in [m]} [A x - b]_i
\]

Proof:

• Let \( i_* \in \arg\max_{i \in [m]} d_i \). Note that \( \overrightarrow{1}_{i_*} \in \Delta^m \).

  \[
  \Rightarrow \max_{x \in \Delta^m} d^T x \geq d^T \overrightarrow{1}_{i_*} = d_{i_*} = \max_{i \in [m]} d_i
  \]

• \( d_i \leq d_{i_*} \) and \( x_i \geq 0 \) for \( x \in \Delta^m \) and \( i \in [m] \)

  \[
  \Rightarrow \max_{x \in \Delta^m} d^T x = \max_{x \in \Delta^m} \sum_{i \in [m]} d_i x_i \leq \max_{x \in \Delta^m} \sum_{i \in [m]} d_{i_*} x_i = d_{i_*} = \max_{i \in [m]} d_i
  \]
Dual Problem

**Lemma:** \( \min_{x \in B_{\infty}^n} d^T x = -\|d\|_1 \) for all \( d \in \mathbb{R}^n \) and therefore

\[
f_{\min}(y) \overset{\text{def}}{=} \min_{x \in B_{\infty}^n} f(x, y) = -b^T y - \|A^T y - b\|_1
\]

**Proof:**

- Let \( \text{sign}(d) \in \mathbb{R}^n \) with \( \text{sign}(d)_i = 1 \) if \( d_i > 0 \), \(-1\) if \( d_i < 0 \), and 0 otherwise
  - \( \Rightarrow \min_{x \in B_{\infty}^n} d^T x \leq d^T (-\text{sign}(d)) = -\sum_{i \in [n]} |d_i| = -\|d\|_1 \)
- \( |x_i| \leq 1 \) for all \( x \in B_{\infty}^n \) and \( i \in [n] \)
  - \( \Rightarrow \min_{x \in B_{\infty}^n} d^T x = \min_{x \in B_{\infty}^n} \sum_{i \in [n]} |d_i||x_i| \geq \min_{x \in B_{\infty}^n} |d_i| = -\|d\|_1 \)
Primal Dual Relationship

**Primal Problem**
- \( \min_{x \in B^n} f_{\text{max}}(x) = \max_{y \in \Delta^m} f(x, y) \)
- \( f_{\text{max}}(x) = c^T x + \max_{i \in [m]} [Ax - b]_i \)

**Dual Problem**
- \( \max_{y \in \Delta^m} f_{\text{min}}(y) = \min_{x \in B^n} f(x, y) \)
- \( f_{\text{min}}(y) = -b^T y - \|A^T y - b\|_1 \)

**Comparison**
- Trivially: \( f_{\text{max}}(x) \geq f_{\text{min}}(y) \) (weak duality)
- Interestingly: \( \min_{x \in B^n} f_{\text{max}}(x) = \max_{y \in \Delta^m} f_{\text{min}}(y) \) (strong duality)

*We will prove algorithmically later*
Approximate Solutions

Primal Problem

- \( \min_{x \in B_n^\infty} f_{\text{max}}(x) = \max_{y \in \Delta^m} f(x, y) \)
- \( f_{\text{max}}(x) = c^T x + \max_{i \in [m]} [A x - b]_i \)

Dual Problem

- \( \max_{y \in \Delta^m} f_{\text{min}}(y) = \min_{x \in B_n^\infty} f(x, y) \)
- \( f_{\text{min}}(y) = -b^T y - \|A^T y - b\|_1 \)

Approximate Solutions

- Let \( x^* \in \arg\min_{x \in B_n^\infty} f_{\text{max}}(x) \) and \( y^* \in \arg\max_{y \in \Delta^m} f_{\text{min}}(y) \)
- \( \epsilon \)-approximate primal solution: \( x_\epsilon \in B_n^\infty \) with \( f_{\text{max}}(x_\epsilon) \leq f_{\text{max}}(x^*) + \epsilon \)
- \( \epsilon \)-approximate dual solution: \( y_\epsilon \in \Delta^m \) with \( f_{\text{min}}(y_\epsilon) \geq f_{\text{min}}(y^*) - \epsilon \)
- \( \epsilon \)-approximate (primal-dual) saddle point (or equilibrium): \((x_\epsilon, y_\epsilon) \in B_n^\infty \times \Delta^m\)
  \[ f_{\text{max}}(x_\epsilon) - f_{\text{min}}(y_\epsilon) \leq \epsilon \]
**Equilibrium**

**Primal Problem**
- \( \min_{x \in B^n_{\infty}} f_{\max}(x) = \max_{y \in \Delta^m} f(x, y) \)
- \( f_{\max}(x) = c^T x + \max_{i \in [m]} [Ax - b]_i \)

**Dual Problem**
- \( \max_{y \in \Delta^m} f_{\min}(y) = \min_{x \in B^n_{\infty}} f(x, y) \)
- \( f_{\min}(y) = -b^T y - \|A^T y - b\|_1 \)

\( \epsilon \)-approximate (primal-dual) saddle point (or equilibrium)
- **Definition**: \( (x_\epsilon, y_\epsilon) \in B^n_{\infty} \times \Delta^m \) and \( f_{\max}(x_\epsilon) - f_{\min}(y_\epsilon) \leq \epsilon \)
- **Duality gap**: \( \text{gap}(x_\epsilon, y_\epsilon) = f_{\max}(x_\epsilon) - f_{\min}(y_\epsilon) \)
- Total \( f(x_\epsilon, y_\epsilon) \) change by best responses: \( = f_{\max}(x_\epsilon) - f(x_\epsilon, y_\epsilon) + [f(x_\epsilon, y_\epsilon) - f_{\min}(y_\epsilon)] \)
- Sum of \( x_\epsilon \) and \( y_\epsilon \) suboptimality: \( = f_{\max}(x_\epsilon) - f(x_\epsilon, y_\epsilon) + [f(x_\epsilon, y_\epsilon) - f_{\min}(y_\epsilon)] \)

Don’t need \( x_* \) and \( y_* \) to compute!
State-of-the-art

**Theorem**: there is a method which can compute an $\epsilon$-approximate saddle point in time $\tilde{O}(\text{nnz}(A)\|A\|_{\text{op},\infty}/\epsilon)$

**Notation**

- $\text{nnz}(A) \overset{\text{def}}{=} n + m + \text{number of nonzero entries in } A$
- $\|A\|_{\text{op},\infty} \overset{\text{def}}{=} \sup_{x \neq 0} \frac{\|Ax\|_{\infty}}{\|x\|_{\infty}} = \max \ell_1 \text{ norm of row of } A$
- $\tilde{O}(\cdot)$ hides logarithmic factors in $\text{nnz}(A)$, $\|A\|_{\text{op},\infty}/\epsilon$

- Box: $B^n_\infty \overset{\text{def}}{=} \{x \in \mathbb{R}^n \mid \|x\|_\infty \leq 1\}$
- Simplex: $\Delta^m \overset{\text{def}}{=} \{y \in \mathbb{R}^m \mid \|y\|_1 = 1\}$
- $\min_{x \in B^n_\infty} \max_{y \in \Delta^m} f(x, y) \overset{\text{def}}{=} y^T Ax + c^T x - b^T y$

Nearly linear time algorithm

size of the input

"$\ell_\infty$ operator norm" bounds up to constant how suboptimal primal/dual solutions which just optimize $b$ and $c$ are
First-order method

**Theorem**: there is a method which solves box-simplex games to accuracy $\varepsilon$ in time $\tilde{O}\left(\text{nnz}(A)\|A\|_{\text{op},\infty}/\varepsilon\right)$.

- **First order method**: only access objective by evaluating the function and computing the gradient, $\nabla f(x, y) = (A^T y + c, Ax - b)$
- **Note**: only need $b$, $c$, and matrix vector multiplies.
  - Can compute in parallel $\tilde{O}(1)$ depth and $O(\text{nnz}(A))$ work.
- **The method for this theorem?**
  - First order method + matrix vector multiplies with $|A|$
  - Parallel with $\tilde{O}(1)$ depth
History and More State-of-the-art

First Order Methods

• $\tilde{O}(\text{nnz}(A)\|A\|_{op,\infty}/\epsilon)$
  • First in [S17]
  • Later variants (influencing this presentation [JST19,CST21,AJJST21]
• Prior state of the art
  • $\tilde{O}(\text{nnz}(A)\|A\|_{op,\infty}^2/\epsilon^2)$ – folklore / [S13, KLOS14] (influencing this presentation)
  • $\tilde{O}(\text{nnz}(A)\sqrt{n}\|A\|_{op,\infty}/\epsilon)$ - AGD and smoothing.
• [ST18] alternative approach and improvements in sparse case

Interior Point Methods

• [CLS19,B20] $\tilde{O}(\max\{m,n\}^\omega)$ where $\omega < 2.373$ is fast matrix multiplication constant

• [BLLSSW21] $\tilde{O}(mn + \min\{m,n\}^{2.5})$

• [LS14,LS15] $\tilde{O}(\text{nnz}(A)\sqrt{\min\{m,n\} + \min\{m,n\}^{2.5}})$

$w < 2.373$ is current fast matrix multiplication (FMM) constant [W13]
Talk Plan  *(Today and Tomorrow)*

**Part 1**  
Structure of box-simplex games  
- Primal and dual problems  
- Approximate solutions  
- Discuss state-of-the-art runtimes

**Part 2**  
Applications  
- Box constrained $\ell_\infty$-regression  
- Linear programming  
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**Part 3**  
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- $\ell_\infty$-Gradient Descent (constrained steepest descent)  
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*Box:* $B_n^\infty \equiv \{ x \in \mathbb{R}^n \mid \|x\|_\infty \leq 1 \}$  
*Simplex:* $\Delta^m \equiv \{ y \in \mathbb{R}_{\geq 0}^m \mid \|y\|_1 = 1 \}$  
$\min_{x \in B_n^\infty} \max_{y \in \Delta^m} f(x,y) \equiv y^T A x + c^T x - b^T y$
Problem #1: Box-constrained $\ell_\infty$-Regression

**Box-constrained $\ell_\infty$ Regression**

- **Input**: matrix $A \in \mathbb{R}^{m \times n}$ and vector $b \in \mathbb{R}^m$
- **Problem**: $\text{OPT}_\infty = \min_{x \in B^n_\infty} \|Ax - b\|_\infty$
- **Goal**: find $\epsilon$-additive approximation, i.e. $x_\epsilon \in B^n_\infty$ with $\|Ax_\epsilon - b\|_\infty \leq \text{OPT}_\infty + \epsilon$

**Claim**: can compute in $\tilde{O}(\text{nnz}(A)\|A\|_{op,\infty}\epsilon^{-1})$

**Proof**:

- $\|Ax - b\|_\infty = \max_{i \in [m]} \max\{[Ax - b]_i, -[Ax - b]_i\} = \max_{y \in \Delta^{2m}} y^T \left( \begin{array}{c} Ax - b \\ -(Ax - b) \end{array} \right)$
- New matrix has same $\|\cdot\|_{op,\infty}$ and just double nnz
Problem #2: Linear Programming

Approximate Linear Programming

• **Input:** $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, and $\epsilon, \delta, D \geq 0$

• **Problem:** $\text{OPT}_{lp} = \min_{x \in \mathbb{R}^n \mid Ax \geq b} c^T x$

• **Promise:** $\exists x^{lp}_* \in \arg\min_{x \in \mathbb{R}^n \mid Ax \geq b} c^T x$ with $\|x^{lp}_*\| \leq D$

• **Goal:** find $x_{\epsilon,\delta}$ with $c^T x_{\epsilon,\delta} \leq \text{OPT}_{lp}$ and $Ax_{\epsilon,\delta} \geq b - \delta \vec{1}$

Notes

• One of many ways to formulate the problem.

• Key difficulty: how handle that constraint $Ax \geq b$?

• Recurring idea: penalty functions in the objective
Linear Programming

**Approach**

- $p(x) \triangleq M \cdot \max \{0, \max_{i \in [m]} [b - Ax]_i\}$
- $\text{OPT}_p = \min_{x \in \mathbb{R}^n \mid \|x\|_\infty \leq R} c^T x + p(x)$

**Claim:** For $M = (\epsilon + 2\|c\|_1 R)\delta^{-1}$ any $\epsilon$-approximate minimizer to $\text{OPT}_p$ problem is $(\epsilon, \delta)$-approximate linear program solution.

**Theorem:** Can compute $(\epsilon, \delta)$-approximate linear program solution in

$$\tilde{O} \left( \text{nnz}(A) \cdot \frac{D\|A\|_{op,\infty}}{\delta} \max \left\{ 1, \frac{D\|c\|_1}{\epsilon} \right\} \right)$$

Input: $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, and $\epsilon, \delta, D \geq 0$

Problem: $\text{OPT}_{lp} = \min_{x \in \mathbb{R}^n \mid Ax \geq b} c^T x$

Promise: $\exists x_{lp}^* \in \text{argmin}_{x \in \mathbb{R}^n \mid Ax \geq b} \|x_{lp}^*\|_\infty \leq D$

Goal: find $x_{\epsilon, \delta}$ with $c^T x \leq \text{OPT}_{lp}$ and $Ax \geq b - \delta \mathbf{1}$

**Proof of Theorem from Claim**

Can write penalized problem as box-simplex

- $\bar{x} = D^{-1}x$ and $\bar{c} = Dc$
- $\bar{A} = \begin{pmatrix} -DMA \\ 0_n^T \end{pmatrix}$ and $b = \begin{pmatrix} -Mb \\ 0_n^T \end{pmatrix}$

Penalized problem is the same as

$$\min_{\bar{x} \in \mathbb{R}_+^n} \bar{c}^T \bar{x} + \max_{i \in [m+1]} [\bar{A} \bar{x} - b]_i$$

Note that $\|\bar{A}\|_{op,\infty} = O(\max\{1, D\|c\|_1 \epsilon^{-1}\})$ and $M/\epsilon = O(\delta^{-1} \max\{1, D\|c\|_1 \epsilon^{-1}\})$
Linear Programming

Approach
• \( p(x) \equiv M \cdot \max \left\{ 0, \max_{i \in [m]} [b - Ax]_i \right\} \)
• \( \text{OPT}_p = \min_{x \in \mathbb{R}^n \|x\|_\infty \leq R} c^T x + p(x) \)

Claim: For \( M = (\epsilon + 2\|c\|_1 R)\delta^{-1} \) any \( \epsilon \)-approximate minimizer to \( \text{OPT}_p \) problem is \((\epsilon, \delta)\)-approximate linear program solution.

Theorem: Can compute \((\epsilon, \delta)\)-approximate linear program solution in
\[
\tilde{O} \left( \text{nnz}(A) \cdot \frac{D\|A\|_{op,\infty}}{\delta} \max \left\{ 1, \frac{D\|c\|_1}{\epsilon} \right\} \right)
\]

Input: \( A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n, \) and \( \epsilon, \delta, D \geq 0 \)
Problem: \( \text{OPT}_{lp} = \min_{x \in \mathbb{R}^n \mid Ax \geq b} c^T x \)
Promise: \( \exists x_{*}^{lp} \in \arg\min_{x \in \mathbb{R}^n \mid Ax \geq b} \|x_{*}^{lp}\|_\infty \leq D \)
Goal: find \( x_{\epsilon, \delta} \) with \( c^T x \leq \text{OPT}_{lp} \) and \( Ax \geq b - \delta \mathbf{1} \)

Proof of Claim
• Let \( x_\epsilon \) be \( \epsilon \)-approximate minimizer
• Since \( x_{*}^{lp} \) is feasible for penalized problem, \( \text{OPT}_p \leq \text{OPT}_{lp} \)
• \( c^T x_\epsilon + p(x_\epsilon) \leq \text{OPT}_p + \epsilon \leq \text{OPT}_{lp} + \epsilon \)
• \( p(x_\epsilon) \leq \epsilon + c^T (x_{*}^{lp} - x_\epsilon) \)
• \( c^T (x_{*}^{lp} - x_\epsilon) \leq \|c\|_1 \|x_{*}^{lp} - x_\epsilon\|_\infty \)
• \( \|x_{*}^{lp} - x_\epsilon\|_\infty \leq \|x_{*}^{lp}\|_\infty + \|x_\epsilon\|_\infty \)
Problem #3: Bipartite Matching

**Maximum Cardinality (Bipartite) Matching (MCM)**

- **Input**: undirected, bipartite graph $G = (V, E)$
- **Matching**: $M \subseteq E$ such that $e_1 \cap e_2 = \emptyset$ for all $e_1, e_2 \in M$ with $e_1 \neq e_2$
- **Problem**: compute matching $M_\ast$ of maximum cardinality $|M_\ast|$
- **Goal**: find $(1 - \epsilon)$-approximate MCM, i.e. matching $M_\epsilon$ with $|M_\epsilon| \geq (1 - \epsilon)|M_\ast|$
**MCM History**  *Fundamental, incredibly well-studied, notoriously difficult (to improve) problem.*

<table>
<thead>
<tr>
<th>Year</th>
<th>Authors</th>
<th>Runtime $\tilde{O}(\cdot)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1969-1973</td>
<td>Dinic, Karzanov, Hopcroft, Karp</td>
<td>$</td>
</tr>
<tr>
<td>1981</td>
<td>Ibarra, Moran</td>
<td>$</td>
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<tr>
<td>2013</td>
<td>Mądry</td>
<td>$</td>
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<tr>
<td>2020</td>
<td>Liu, S</td>
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<td>2020</td>
<td>Liu, Kathuria, S</td>
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<tr>
<td>2020</td>
<td>Brand, Lee, Nanongkai, Peng, Saranurak, S, Song, Wang</td>
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Note: procedure will use very little graph structure.

• **Result:** can use box-simplex solver to compute $(1 - \epsilon)$-approximate MCM in $\tilde{O}(|E|\epsilon^{-1})$ time and $\tilde{O}(\epsilon^{-1})$ depth
• Time matched by Dinic, Karzanov, Hopcroft, Karp and Allen-Zhu, Orecchia 2015
• Unaware of alternative method that gets this parallelism and this time.
• Alternative method either have large $\epsilon$, $|E|$, or $|V|$ dependence
• Also, implementable semi-streaming (Assadi, Jambulapati, Jin, S, Tian 2021)

$w < 2.373$ is current fast matrix multiplication (FMM) constant [W13]
Approach

$N(a) \equiv \{ b \in V \mid \{a,b\} \in E \}$ denotes the neighbors of $a$

**Fractional Matching**: in the MCM problem $f \in \mathbb{R}^E$ is a fractional matching if for all $a \in V$ it is the case that $\sum_{b \in N(a)} f\{a,b\} \leq 1$.

**Theorem** [GPST91]: There is an algorithm which given any fractional matching $f \in \mathbb{R}^E_{\geq 0}$ can compute an integral matching of cardinality at least $\|f\|_1$ in time $\tilde{O}(|E|)$ and depth $\tilde{O}(1)$.

**Corollary**: The minimum $\ell_1$-norm of a fractional matching is $|M_*|$ and it suffices to compute a fractional matching of $\ell_1$-norm $\geq (1 - \epsilon)|M_*|$.

- **Input**: undirected, bipartite graph $G = (V, E)$
- **Matching**: $M \subseteq E; e_1 \cap e_2 = \emptyset$ for all $e_1, e_2 \in M$ with $e_1 \neq e_2$
- **Problem**: compute matching $M_*$ maximizing $|M_*|$
- **Goal**: matching $M_\epsilon$ with $|M_\epsilon| \geq (1 - \epsilon)|M_*|$
Linear Algebraic Representation

Unsigned (edge-vertex) Incidence Matrix: \( |B| \in \mathbb{R}^{E \times V} \) with
\[
|B|_{\{a,b\},c} = \begin{cases} 
1 & c \in \{a, b\} \\
0 & \text{otherwise}
\end{cases}
\]
for all \( \{a, b\} \in E \) and \( c \in V \)

Lemma: \( f \in \mathbb{R}^E_{\geq 0} \) is a fractional matching if and only if \( |B|^T f \leq 1 \).

Proof: \( [|B|^T f]_a = \sum_{\{b, c\} \in E} f_{\{b, c\}} |B|_{\{a, b\}, c} = \sum_{b \in N(a)} f_{\{a, b\}} \)

Upshot: it suffices to solve
\[
\max_{f \in \mathbb{R}^E_{\geq 0} \mid |B|^T f \leq 1} \mathbf{1}^T f \quad \text{or equivalently} \quad \min_{f \in \mathbb{R}^E_{\geq 0} \mid |B|^T f \leq 1} (-\mathbf{1})^T f
\]
Penalty and Rounding

**Overflow** (excess): \( \text{overflow}(f) \overset{\text{def}}{=} \max\{0, |B|^T f - 1\} \) entrywise

**Note**: \( f \in \mathbb{R}^E \) is a fractional matching if and only if \( \text{overflow}(f) = 0 \)

**Lemma**: given \( f \in \mathbb{R}_{\geq 0}^E \) let \( \tilde{f} \in \mathbb{R}^E \) be defined for all \( \{a, b\} \in E \) with \( \tilde{f}_{\{a,b\}} = 0 \) if \( f_{\{a,b\}} = 0 \) and otherwise

\[
\tilde{f}_{\{a,b\}} = f_{\{a,b\}} \left( 1 - \max \left\{ \frac{[\text{overflow}(f)]_a}{|[B]^T f]_a}, \frac{[\text{overflow}(f)]_b}{|[B]^T f]_b} \right\} \right)
\]

Then \( 0 \leq \tilde{f} \leq f \), \( \tilde{f} \) is a fractional matching, and \( \|f - \tilde{f}\|_1 \leq \|\text{overflow}(f)\|_1 \).

**Proof**: \( f_{\{a,b\}} \cdot \frac{[\text{overflow}(f)]_a}{|[B]^T f]_a} \) is the relative contribution of \( f_{\{a,b\}} \) to overflow

**Upshot**: \(-|M_*| = \min_{f \in \mathbb{R}_+^E} -1^T f + \sum_{a \in V} [\text{overflow}(f)]_a\) and given any \( \epsilon \)-additive minimizer can compute matching of size \( \geq |M_*| - \epsilon \) in time \( \tilde{O}(|E|) \).
The Result

Question #1: how to encode \( \text{overflow}(f) \)?

- Tool: \( \max\{0, a\} = \frac{1}{2} [a + |a|] \)
- Suffices to compute \( \varepsilon |M_*| \) additive approximation to
  \[
  \min_{f \in \mathbb{R}_{\geq 0}^E} -\bar{1}^T f + \frac{1}{2} \bar{1}^T |B|^T f + \frac{1}{2} |V| + \left\| \frac{1}{2} |B|^T f - \bar{1} \right\|_1
  \]

Question #2: how to put \( f \) in simplex?

- Suppose \( \nu \geq |M_*| \), then suffices to work with \( x = \left( \frac{1}{\nu} f, 1 - \frac{1}{\nu} \| f \|_1 \right) \in \Delta^{|E|+1} \)
- Let \( b = (-\frac{\nu}{2} \bar{1}^T |E|, 0) \), and let \( A = \frac{\nu}{2} |B|^T \) with 0 column added
- Suffices to compute \( \varepsilon |M_*| \) additive approximation to
  \[
  \min_{x \in \Delta^{|E|+1}} b^T x + \|Ax - \bar{1}\|_1
  \]
- Suffices to compute \( \varepsilon |M_*| \) additive approximation to
  \[
  \max_{x \in \Delta^{|E|+1}} -b^T x - \|Ax - \bar{1}\|_1
  \]
- Note that \( \|A\|_{\text{op,\infty}} = \nu \) so can solve in \( \tilde{O} \left( \frac{|E| \nu}{\varepsilon |M_*|} \right) \).
- Get result by picking \( \nu \) as every power of 2 between 0 and \( 2|V| \! \)
Theorem: Given any algorithm which compute an $\epsilon$-approximate MCM for any input $\epsilon \in (0,1)$ in time $\tilde{O}(|E|\epsilon^{-\delta})$ for some fixed constant $\delta$, there is an algorithm that computes exact MCM in time $\tilde{O}(|E| \cdot |V|^{1+\delta})$.

Proof

• Given any $\epsilon$-approximate MCM, there are at most $\epsilon |M_*| \leq \epsilon |V|$ more edges that could be matched.

• Augmenting paths finds at least one more matched edge in time $O(|E|)$

• Total time: $\tilde{O}(|E|\epsilon^{-\delta} + \epsilon |E| |V|)$ solving for $\delta$ yields result

Implication: $\tilde{O}(|E|\epsilon^{-1})$ time $(1 - \epsilon)$-approximate MCM yields $\tilde{O}(|E|\sqrt{|V|})$ time exact MCM

Barrier to improving: only improvements known to date use interior point methods
Problem #4: Flow Problems

- Graph $G = (V, E)$
- Vertices $s, t \in V$

**Goal**
Send 1 unit of flow, $f \in \mathbb{R}^E$, between $s$ and $t$ in the "best" way possible.

**What should we minimize?**

- **Length**
  \[
  \sum_{e \in E} |f_e| \quad \|f\|_1
  \]

- **Energy**
  \[
  \sum_{e \in E} |f_e|^2 \quad \|f\|_2
  \]

- **Congestion**
  \[
  \max_{e \in E} |f_e| \quad \|f\|_\infty
  \]

- **Maximum Flow**
  \[
  \tilde{O}(\min\{|E|^{3/2}, |E| \cdot |V|^{2/3})
  \]

- **Shortest Path**
  \[
  \tilde{O}(|E|)
  \]

- **Electric Flow**
  \[
  \tilde{O}(|E|)
  \]

- **Laplacian System Solving**
  \[
  \tilde{O}(|E|)
  \]

See Rasmus Kyng’s talks

Focus for today

No improvement until 2013, will discuss Friday.

Introduce more broadly
The Maximum Flow Problem

Graph $G = (V, E)$
- $n$ vertices $V$
- $m$ edges $E$

Capacities
- $u \in \{1, ..., U\}^E$

Terminals
- Source $s \in V$
- Sink $t \in V$

Value of Flow
- total flow leaving $s$ or entering $t$

Capacity Constraints
- Directed: $f_e \in [0, u_e]$
- Undirected: $f_e \in [-u_e, u_e]$

Flow
- $f \in \mathbb{R}^E$ where $f_e = \text{amount of flow on edge } e$

Goal
- compute maximum $s \to t$ flow

$s \to t$ Flow
- flow in = flow out
- for all $v \notin \{s, t\}$

Introduce maximum flow problem more formally
**Undirected Maxflow**

\[
(1 - \epsilon)\text{-Approximate Flow}
\]

feasible \( s \to t \) flow of value \( \geq (1 - \epsilon)OPT \)

<table>
<thead>
<tr>
<th>Authors</th>
<th>Time for ( \epsilon )-Approximate Undirected Flow</th>
<th>Capacitated (( U \neq 1 ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \ell_1 )-ish</td>
<td>( \tilde{\Omega}(m^{\sqrt{n}\epsilon^{-1}}) )</td>
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<td>[Kar98]</td>
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<tr>
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<td>[LRS13]</td>
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<td>[S13,KLOS14]</td>
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<tr>
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**Step 1 (Combinatorial Advance)**
Build coarse \( \ell_\infty \)-approximator (e.g. oblivious routing or congestion approximator) to change representation.

**Step 2 (Optimization Advance)**
Apply iterative method to boost accuracy (e.g. gradient descent, area-convex dual extrapolation, mirror prox, coordinate descent)

**Note (Further Implications)**
Parallel optimal transport [JST19], streaming matching [JST20], optimization methods [CST21]

Note: there are additional improvements with \( \log(\epsilon^{-1}) \).
Such results give exact directed flow algorithms.
### Talk Plan

**$(1 - \epsilon)$-Approximate Flow**

A feasible $s \rightarrow t$ flow of value $\geq (1 - \epsilon)OPT$

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- Talk 1 & 2: Focus on $\tilde{O}(m \epsilon^{-1})$ runtime.
- Talk 3: Discuss state-of-the-art small $\epsilon$ results
Thank you

Questions?

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