Box-Simplex Games: Algorithms, Applications, and Algorithmic Graph Theory Exercises #2

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Problem #1: Bounding Congestion Approximators

Let $G = (V, E, u)$ be a capacitated graph with $u \in \mathbb{R}^E_{\geq 0}$ and let $\mathrm{OPT}(d) = \min_{\mathbf{B}^\top f = d} \|\mathbf{U}^{-1} f\|_\infty$ for $\mathbf{U} = \mathbf{diag}(u)$. Prove that $\mathbf{R} \in \mathbb{R}^{k \times n}$ is an α-congestion approximator for G if and only if $\mathrm{OPT}(d) \leq \alpha \cdot \|\mathbf{R} d\|_\infty$ for all $d \in \mathbb{R}^n$ and $\|\mathbf{R}\mathbf{B}^\top \mathbf{U}\|_\infty \leq 1$.

Note: this shows that the conditions given in the lecture notes regarding congestion approximators were redundant and is related to [3].

Problem #2: Optimal Congestion Approximator

Let $G = (V, E, u)$ be a capacitated, connected graph with $u \in \mathbb{R}^E_{\geq 0}$ and let $\mathrm{OPT}(d) = \min_{\mathbf{B}^\top f = d} \|\mathbf{U}^{-1} f\|_\infty$ for $\mathbf{U} \stackrel{\text{def}}{=} \mathbf{diag}(u)$.

- Part (a): for $d \in \mathbb{R}^V$ and all $S \subseteq V$ let $d_S \stackrel{\text{def}}{=} \sum_{i \in [S]} d_i$ and $u_S \stackrel{\text{def}}{=} \sum_{\{a,b\} \in E | a \in S, b \notin S} u_{\{a,b\}}$. Show that $\mathrm{OPT}(d)^{-1} = \min_{S \subseteq V \,|\, d_S \neq 0} \frac{u_S}{|d_S|}$.
- Part (b): Prove that there exists a 1-congestion approximator $\mathbf{R} \in \mathbb{R}^{k \times n}$ for G.

Note: this problem is related to *[3]*.

Problem #3: Eulerian Graphs

Provide an algorithm which given any capacitated, Eulerian, directed graph $G = (V, E, u)$ and $s, t \in V$ computes a $(1 - \epsilon)$-approximate maximum s-t flow in time $\tilde{O}(|E|\epsilon^{-1})$.

Note: G is Eulerian if and only if for all for all $a \in V$ it is the case that $\sum_{(a,b) \in E} u_{(a,b)} = \sum_{(b,a) \in E} u_{(b,a)}$. This problem is related to [2].

Problem #4: ℓ_∞-regularization difficulty

Prove that if $r : B_\infty^n \to \mathbb{R}$ is 1-strongly convex with respect $\|\cdot\|_\infty$ then for some universal constant $c > 0$.

$$\sup_{x \in B_\infty^n} r(x) - \inf_{x \in B_\infty^n} r(x) \geq cn \,.$$

Note: this result was discussed and proved formally in [4].

Problem #5: ℓ_2-ℓ_1-Games

Provide an algorithm which given any $\mathbf{A} \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^m$ can compute an ϵ-additive approximation to the problem

$$\min_{x \in \mathbb{R}^n | \|x\|_2 \leq 1} \max_{y \in \Delta^m} y^\top \mathbf{A} x + c^\top x - b^\top y$$

in time $\tilde{O}(\mathrm{nnz}(\mathbf{A}) L \epsilon^{-1})$ where L is the maximum ℓ_2-norm of any row of $\mathbf{A}$.

Hint: you may use without proof that for any $f : \chi \to \mathbb{R}$ that is L-smooth (with respect to ℓ_2) and convex and any closed, convex χ accelerated gradient descent can compute a point x_k with $f(x_k) - \inf_{x \in \chi} f(x) = O(\frac{L\|x_k - x_*\|_2^2}{k^2})$ where $x_* \in \mathrm{argmin}_{x \in \chi} f(x)$ in the time needed to compute k gradients of f and perform k Euclidean projections onto χ.

Problem #6: Minimum Cost Transshipment

In the minimum cost transshipment problem we are given undirected $G = (V, E)$, weights $w \in \mathbb{R}_{>0}^E$, and demands $d \in \mathbb{R}^V$. The problem is to compute $\mathrm{OPT}(d) \stackrel{\mathrm{def}}{=} \min_{f \in \mathbb{R}^E | \mathbf{B}^\top f = d} \sum_{e \in E} w_e |f_e|$. Provide an algorithm which given $\mathbf{R}$ with $\mathrm{nnz}(\mathbf{R}) = \tilde{O}(|E|)$ for which for some $\alpha = \tilde{O}(1)$ it is the case that $\|\mathbf{R}d\|_1 \leq \mathrm{OPT}(d) \leq \alpha \cdot \|\mathbf{R}d\|_1$ computes $\tilde{f}$ with $\mathbf{B}^\top \tilde{f} = d$ and $\sum_{e \in E} w_e |f_e| \leq (1 + \epsilon)\mathrm{OPT}(d)$.

Note: This problem is related to [1] which in turn leverages prior work in this area.

References

[1] Sepehr Assadi, Yujia Jin, Aaron Sidford, and Kevin Tian. Semi-streaming bipartite matching in fewer passes and optimal space. *arXiv preprint arXiv:2011.03495*, 2021.

[2] Alina Ene, Gary L. Miller, Jakub Pachocki, and Aaron Sidford. Routing under balance. In Daniel Wichs and Yishay Mansour, editors, *Proceedings of the 48th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2016, Cambridge, MA, USA, June 18-21, 2016*, pages 598–611. ACM, 2016.

[3] Jonah Sherman. Nearly maximum flows in nearly linear time. In *54th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2013, 26-29 October, 2013, Berkeley, CA, USA*, pages 263–269. IEEE Computer Society, 2013.

[4] Aaron Sidford and Kevin Tian. Coordinate methods for accelerating ℓ_∞ regression and faster approximate maximum flow. In Mikkel Thorup, editor, *59th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2018, Paris, France, October 7-9, 2018*, pages 922–933. IEEE Computer Society, 2018.