Convex Programs

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Given a convex set $K \subseteq \mathbb{R}^n$ and a convex $f : K \to \mathbb{R}$, a convex program is the following optimization problem

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- smooth when $f$ is differentiable with a continuous derivative
- nonsmooth otherwise.
Consider

$$\inf\left\{ \frac{1}{x} : x \in (0, \infty) \right\}.$$
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No \( x \in K \) attains the infimum.

If \( K \subseteq \mathbb{R}^n \) is closed and bounded then the minimum is attained by some \( x \in K \).
Some examples of convex programs

Linear Regression. \( \min_{x \in \mathbb{R}^n} \|Ax - b\|_2 \), where \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^n \).
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In that case

\[
 f(x) = \| Ax - b \|_2^2 = x^T A^T A x - 2b^T A x + b^T b,
\]

and \( \nabla^2 f(x) = 2A^T A \succeq 0 \).
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**Linear programming.**

\[
\min \ c^T x \quad \text{s.t.} \quad Ax \leq b
\]
Computational models

**Oracle Model.** Often we allow oracle access to $f, \nabla f, \nabla^2 f$ and bound the number of oracle calls or iterations that the (usually iterative) algorithm performs.

**word RAM model.** Addition, subtraction, multiplication etc take exactly 1 time step, for numbers that can be stored in a word; usually the input of the problem shall consist of numbers that can fit in a word.
Given a point \( x \in \mathbb{R}^n \) and \( K \subseteq \mathbb{R}^n \), does \( x \in K \)?
Membership for convex sets

**Given a point** $x \in \mathbb{R}^n$ and $K \subseteq \mathbb{R}^n$, does $x \in K$?

* Halfspaces: Let $K := \{ y \in \mathbb{R}^n : \langle a, y \rangle \leq b \}$ where $a \in \mathbb{R}^n$, $b \in \mathbb{R}$. We need to write down $a, b, x$ using finite number of bits to perform membership in $K$. 
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★ Ellipsoids. Let \( K := \{ y \in \mathbb{R}^n : y^T A y \leq 1 \} \) for a PD matrix \( A \in \mathbb{Q}^{n \times n} \).
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- **Ellipsoids**: Let $K := \{ y \in \mathbb{R}^n : y^T A y \leq 1 \}$ for a PD matrix $A \in \mathbb{Q}^{n \times n}$.

- **Intersection of halfspaces (polytopes)**: $K := \{ \langle a_i, y \rangle \leq b_i, i = 1, \ldots, m \}$. 
Membership for convex sets

- \( \ell_1 \) ball: \( K := \{ x \in \mathbb{R}^n : \sum_{i=1}^{n} |x_i| \leq r \} \).
Membership for convex sets

$\ell_1$ ball: $K := \{x \in \mathbb{R}^n : \sum_{i=1}^{n} |x_i| \leq r\}$. It is an intersection of $2^n$ hyperplanes, of all $\{y : \langle y, s \rangle\}$, $s \in \{-, 1 + 1\}^n$. No hyperplane is redundant.
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★ PSD matrices. Given $X \in \mathbb{R}^n$, does $y^T X y \geq 0$ for all $y \in \mathbb{R}^n$? Equivalent to checking whether $\lambda_1(X) \geq 0$. Can only approximately check.
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Some operations that preserve convexity:
  ○ Intersection
  ○ Scaling
  ○ Translation
  ○ Affine transformation
  ○ Set sum
Separation Oracles for convex sets

Theorem (Convexity implies Separating Hyperplane)
For all closed and convex \( K \subseteq \mathbb{R}^n \) and \( x \in \mathbb{R}^n \setminus K \) there exists \( a \in \mathbb{R}^n, b \in \mathbb{R} \) such that

\[
\langle a, x \rangle > b \quad \text{and} \quad \langle a, y \rangle \leq b, \forall y \in K.
\]
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**Theorem (Convexity implies Separating Hyperplane)**

For all closed and convex $K \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n \setminus K$ there exists $a \in \mathbb{R}^n$, $b \in \mathbb{R}$ such that

$$\langle a, x \rangle > b \quad \text{and} \quad \langle a, y \rangle \leq b, \forall y \in K.$$

**Theorem (Separating hyperplanes implies convexity)**

Let $K \subseteq \mathbb{R}^n$ be a convex set. If for every $x \in \mathbb{R}^n \setminus K$ there exists a hyperplane separating $x$ from $K$, then $K$ is convex.
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A separation oracle for a convex set $K \subseteq \mathbb{R}^n$ is a primitive which:

1. given $x \in K$, answers YES
2. given $x \notin K$, answers NO and returns $a \in \mathbb{Q}^n, b \in \mathbb{Q}$ such that the hyperplane $\{y : \langle a, y \rangle = b\}$ separates $x$ from $K$. 

Separation vs. optimization. Constructing efficient (polynomial time) separation oracles for a given family of convex sets is equivalent to constructing algorithms to optimize linear functions over convex sets in this family.
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Back to solving convex programs

Given $c \in \mathbb{Q}$, find whether $\min_{x \in K} f(x) = c$. 

Consider $f(x) = 2x + x$, $K = [1, \infty)$. What happens?

Refined goal: Given $\epsilon > 0$ compute $c \in \mathbb{Q}$ such that $\min_{x \in K} f(x) \in [c - \epsilon, c + \epsilon]$. 
Back to solving convex programs

Given $c \in \mathbb{Q}$, find whether $\min_{x \in K} f(x) = c$.

Consider $f(x) = \frac{2}{x} + x$, $K = [1, \infty)$. What happens?
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$$\min_{x \in K} f(x) \in [c - \epsilon, c + \epsilon].$$
Consider a convex program which has a unique optimal solution $x^* \in K$. Then we can ask for either

1. proximity in value, $f(x) \leq f(x^*) + \epsilon$, or
2. proximity in optimum, $\|x - x^*\|_2 \leq \epsilon$. 
Representing Functions

- Linear and affine. \( f(x) = \langle a, x \rangle + b. \)
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- Linear matrix functions. $f(X) = Tr(XA)$, where $A \in \mathbb{Q}^{n \times n}$ and $X \in \mathbb{R}^{n \times n}$ is a symmetric matrix variable.
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Value Oracle: Given $x \in K$, compute $f(x)$. 
Models of accessing $f$ (again)

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Gradient Oracle: Given $x \in K$, compute $\nabla f(x)$, $\nabla^2 f(x)$, $\nabla^3 f(x)$, ...
Models of accessing $f$ (again)

Value Oracle: Given $x \in K$, compute $f(x)$.
Gradient Oracle: Given $x \in K$, compute $\nabla f(x), \nabla^2 f(x), \nabla^3 f(x), \ldots$.
Measure number of oracle calls, ideally $\text{poly}(n, \log(1/\epsilon))$. 
Examples of how (iterative) algorithms look like

Gradient descent. \( x_{t+1} := x_t - \eta \nabla f(x_t) \).
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Gradient descent. \( x_{t+1} := x_t - \eta \nabla f(x_t). \)

Projected Gradient Descent. \( x_{t+1} := \Pi_K(x_t - \eta \nabla f(x_t)). \)

Newton’s method. \( x_{t+1} := x_t - (\nabla^2 f(x_t))^{-1} \nabla f(x_t). \)
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Mind the gap: Run one step of Newton’s method on
\( f(x) = \frac{1}{2} x^T M x + bx \).
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Mind the gap: Run one step of Newton’s method on \( f(x) = \frac{1}{2} x^T M x + b x \). For all quadratic functions one step of Newton’s method lands on the optimum!
Problem. Given unit capacity graph of $G = (V, E)$, vertices $s, t \in G$
route the maximum amount of flow from $s$ to $t$. 

Let $x \in \mathbb{R}^E$, $\|x\|_{\infty} \leq 1$
with constrains for all $u \in V$:

- $\sum_{(u,v) \in E} x_{uv} - \sum_{(v,u) \in E} x_{uv} = 0$ if $u \neq s, t$
- $\sum_{(u,v) \in E} x_{uv} - \sum_{(v,u) \in E} x_{uv} = F^*$ if $u = s$
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min $\|x\|_{\infty}$
s.t. $Bx = s - 1$

min $\eta \log(\sum_i e^{-x_i}/\eta + e^{x_i}/\eta)$
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Unit capacity Max flow and convex optimization

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$$\min \|x\|_\infty \text{ s.t. } Bx = 1_s - 1_t.$$
**Unit capacity Max flow and convex optimization**

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Langragian Duality

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\begin{align*}
\inf_{x \in \mathbb{R}^n} f(x) \\
\text{s.t. } f_j(x) &\leq 0, \text{ for } j \in [m] \\
h_i(x) &= 0, \text{ for } i \in [p]
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Let \( L(x, \lambda, \mu) := f(x) + \sum_{j \in [m]} \lambda_j f_j(x) + \sum_{i \in [p]} \mu_i h_i(x) \).
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\begin{itemize}
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\[ y^* = \inf_{x \in K} \sup_{\lambda \geq 0, \mu} L(x, \lambda, \mu) = \inf_{x \in \mathbb{R}^n} \sup_{\lambda \geq 0, \mu} L(x, \lambda, \mu) \]
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Let

\[ g(\lambda, \mu) = \inf_{x \in \mathbb{R}^n} L(x, \lambda, \mu) \]

Definition (Dual Program)

\[ \sup_{\lambda \geq 0, \mu} g(\lambda, \mu) \]
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Definition (Dual Program)

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Theorem (Weak Duality)

\[ \sup_{\lambda \geq 0, \mu} g(\lambda, \mu) \leq \inf_{x \in K} f(x). \]
Slater’s condition. There exists $\bar{x}$ such that $h_j(\bar{x}) = 0$ and $f_i(\bar{x}) < 0$. 

Theorem (Slater’s gives strong duality) If all $f_j$, $h_i$ are affine and Slater’s condition holds, then $\sup \lambda \geq 0$, $\mu g(\lambda, \mu) = \inf x \in K f(x)$
Strong Duality

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**Theorem (Slater’s gives strong duality)**

If all $f_j, h_i$ are affine and Slater’s condition holds, then

$$\sup_{\lambda \geq 0, \mu} g(\lambda, \mu) = \inf_{x \in K} f(x)$$
Examples

Linear programming. \( \min c^T x \quad \text{s.t.} \quad Ax \geq b \)
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Let \( L(x, \lambda) = c^T x + \lambda^T (b - Ax) = \langle x, c - A^T \lambda \rangle + \langle b, \lambda \rangle \)
Examples

Linear programming. \( \min c^T x \) s.t. \( Ax \geq b \)

Let \( L(x, \lambda) = c^T x + \lambda^T (b - Ax) = \langle x, c - A^T \lambda \rangle + \langle b, \lambda \rangle \)

What is \( g(\lambda) = \inf_{x \in \mathbb{R}^n} L(x, \lambda) \)?

\[
\begin{align*}
\max & \langle b, \lambda \rangle \\
\text{s.t.} & \quad A^T \lambda = c, \lambda \geq 0
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It is known that in the setting of linear programming strong duality holds.
Examples

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Fact. There exist some convex programs for which strong duality fails, but such programs are not commonly encountered in practice.
Thank you!