P1. In this exercise, we will design an algorithm for $\left(2 - \frac{2}{k}\right)$-p.r. instances of Minimum Multiway Cut.

Algorithm Outline

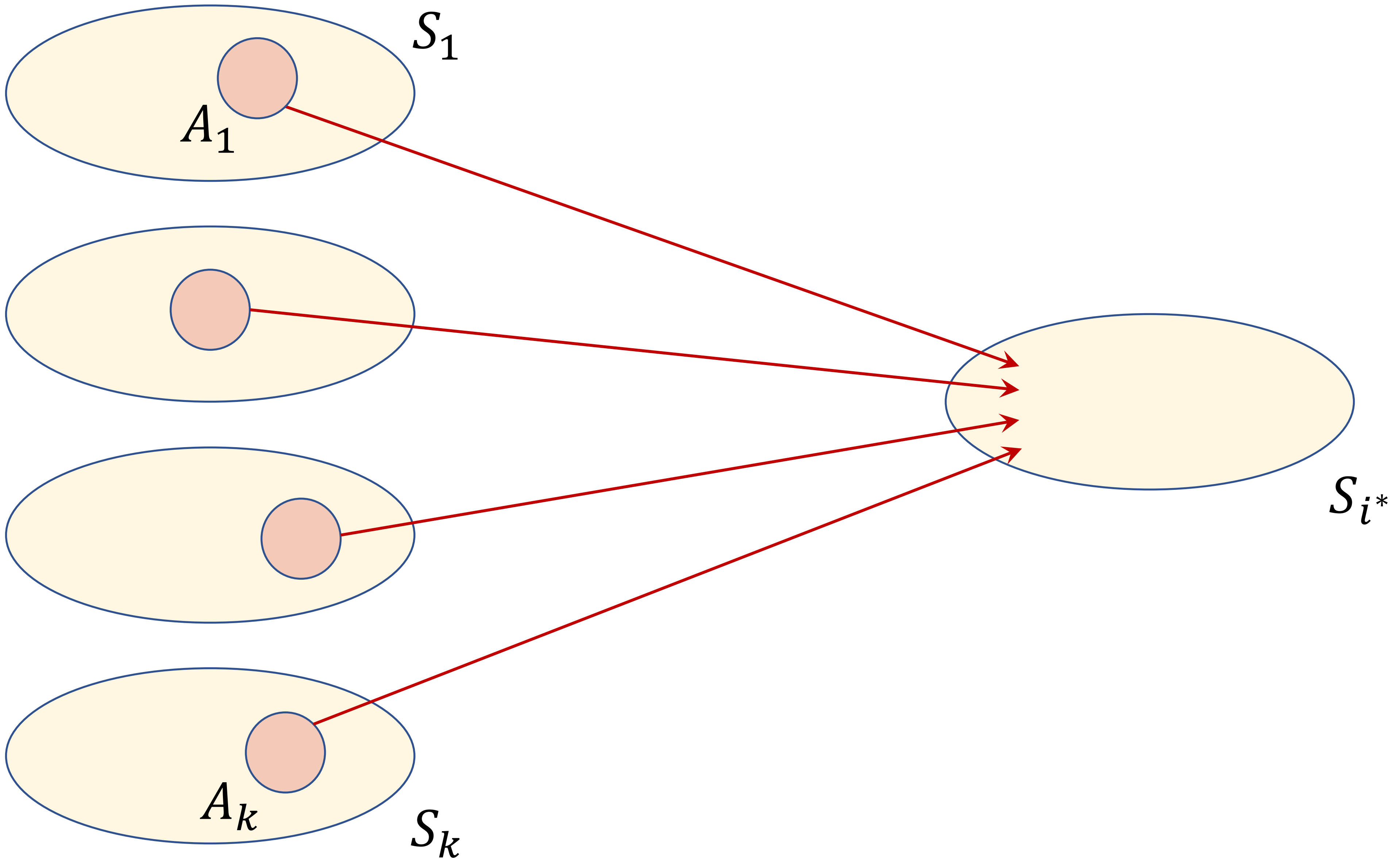
1. Start with an arbitrary partition $S_1, \ldots, S_k$ such that $t_i \in S_i$ for all i.
2. While there is an improvement, improve the partition (see below)

Improvement Procedure Outline

- Choose index i^* that results in best clustering $S'_1, \ldots, S'_k$ below (brute force)
- Find sets $A_i \subseteq S_i \setminus \{t_i\}$ for all $i \neq i^*$
- Consider partition $S'_1, \ldots, S'_k$ obtained by moving points in sets A_i to S_{i^*}:

$$S'_i = S_i \setminus A_i \quad \text{for} \quad i \neq i^* \qquad \text{and} \qquad S'_{i^*} = S_i \cup A_1 \cup \cdots \cup A_k$$

- Return $S'_1, \ldots, S'_k$

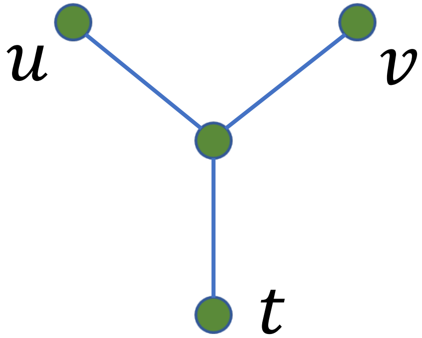

S_1
A_1
S_{i^*}
A_k
S_k

P1 (continued). Implement the *Improvement procedure*: design an efficient algorithm that given i^* finds sets A_i maximizing the improvement

$$cost(S_1, \ldots, S_k) - cost\left(S'_1, \ldots, S'_k\right)$$

Hint: reduce the problem to the minimum s-t cut problem. To this end:

- introduce two new vertices s and t
- connect s to all terminals t_i (with $i \neq i^*$) by edges of infinite cost
- connect t to all points in S_{i^*} by edges of infinite cost
- keep some edges of G as is and replace others with the following gadget:



P2*. Consider the algorithm from **Problem 1**. Given an arbitrary instance of k-Multiway Cut, it may find a "locally optimal" solution which cannot be further improved by the improvement procedure. However, this solution is not necessarily globally optimal.

Prove that the algorithm always finds an optimal solution if the instance is $\left(2 - \frac{2}{k}\right)$-perturbation resilient.

Hint: Let $S_1^*, \dots, S_k^*$ be an optimal solution. Prove that there exist i^* and j^* such that sets $A_i \equiv S_i \cap S_{j^*}^*$ (for $i \neq j^*$) improve $S_1, \dots, S_k$.

P3. Consider an instance I of k-median. A random γ-perturbation I' is defined as follows. It is an instance on the same set of points as I and

$$d'(u, v) = r_{uv} \cdot d(u, v)$$

where $r_{uv} \in_u [1, \gamma]$ are independent random variables.

An instance I is γ-resilient to random perturbation if a random perturbation I' has the same solution as I with probability as least $1 - 1/n$.

Prove that there is no polynomial-time exact algorithm for instances that are 100-resilient to random perturbation.

Hint: given an arbitrary instance I, construct an instance I'that has the same solution and is resilient to random noise. Do so by introducing duplicate points.