Geometry of Equilibria in Bimatrix Games

Bernhard von Stengel

reading material:
Chapter 9 of “Game Theory Basics”

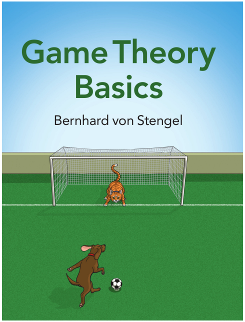
Plan

- recall best-response condition
- upper envelope with 2 and 3 goalposts
- **labels** in best-response diagrams
- equilibria = completely labeled strategy pairs
- the Lemke–Howson algorithm
- labeled polytopes
- complementary pivoting
  - handling degeneracy
  - efficient exact arithmetic

Some Reading Material

B. von Stengel (2021), *Game Theory Basics.* Cambridge University Press.

B. von Stengel (2021), Finding Nash equilibria of two-player games. arXiv:2102.04580.



L. S. Shapley (1974), A note on the Lemke-Howson algorithm. *Mathematical Programming Study 1: Pivoting and Extensions*, 175–189.

Bimatrix Games,

Best-Response Condition

Nash equilibria of bimatrix games

$$A = \begin{bmatrix} 0 & 6 \\ 2 & 5 \\ \boxed{3} & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ \boxed{4} & 3 \end{bmatrix}$$

Nash equilibrium =

pair of strategies x , y with

x best response to y and
y best response to x.

Mixed equilibria

$$A = \begin{bmatrix} 0 & 6 \\ 2 & 5 \\ 3 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 4 & 3 \end{bmatrix}$$

$$x = \begin{bmatrix} 2/3 \\ 1/3 \\ 0 \end{bmatrix} \qquad x^T B = \begin{bmatrix} 5/3 & 5/3 \end{bmatrix}$$

$$Ay = \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix} \qquad y^T = \begin{bmatrix} 1/3 & 2/3 \end{bmatrix}$$

only **pure best responses** can have probability > 0

Best-response condition

Theorem Given: $m \times n$ bimatrix game (A, B) .

Let x be a mixed strategy of player I and
let y be a mixed strategy of player II. Then

$$\boxed{\begin{array}{ll} & x \text{ is a best response to } y \\ \Leftrightarrow & \text{for all pure strategies } i \text{ of player I :} \\ & x_i > 0 \;\Rightarrow\; (Ay)_i = u = \max\{\,(Ay)_k \mid 1 \le k \le m\,\}. \end{array}}$$

Best-response condition

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	x is a best response to y
$\Leftrightarrow$	for all pure strategies i of player I :
	$x_i > 0 \;\; \Rightarrow \;\; (Ay)_i \; = \; u \; = \max\{\, (Ay)_k \mid 1 \le k \le m \,\}.$

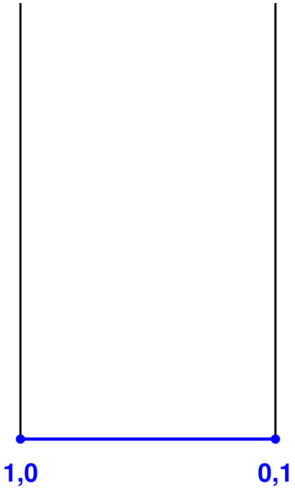
	(x, y) is a mixed equilibrium
$\Leftrightarrow$	for all pure strategies i of player I :
	$x_i = 0 \;\;$ or $\;\; (Ay)_i \; = \; u \; = \max\{\, (Ay)_k \mid 1 \le k \le m \,\},$
	for all pure strategies j of player II :
	$y_j = 0 \;\;$ or $(x^\top B)_j \; = \; v \; = \max\{\, (x^\top B)_\ell \mid 1 \le \ell \le n \,\}.$

Best responses to mixed strategy of player 2

$$\begin{array}{c|cc|} & 4 & 5 \\ \hline 1 & 0 & 6 \\ 2 & 2 & 5 \\ 3 & 3 & 3 \\ \hline \end{array} = A$$

payoffs to
player I

Best responses to mixed strategy of player 2



$$\begin{array}{c|cc|l} & 4 & 5 & \\ \hline 1 & 0 & 6 & \\ 2 & 2 & 5 & = A \\ 3 & 3 & 3 & \end{array}$$

payoffs to
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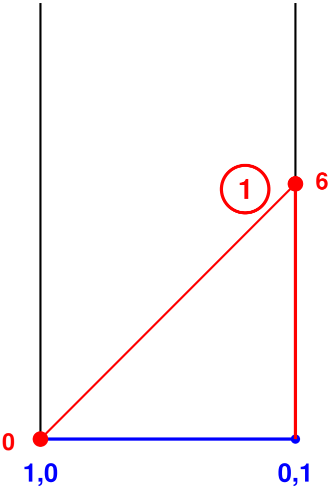
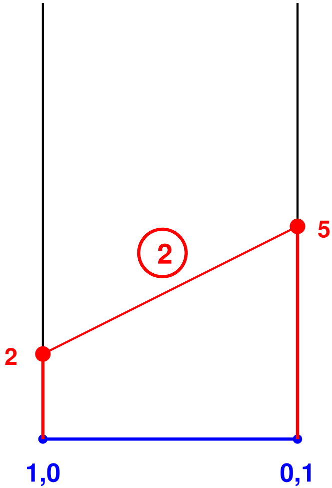


$$\begin{array}{c c} & \begin{array}{cc} 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cc} 0 & 6 \\ 2 & 5 \\ 3 & 3 \end{array}\right] = A \end{array}$$

payoffs to player I

Best responses to mixed strategy of player 2



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payoffs to
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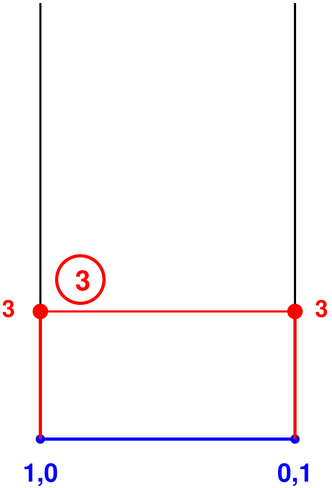

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payoffs to player I

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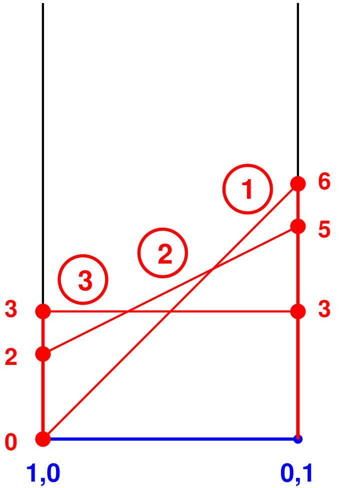

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payoffs to
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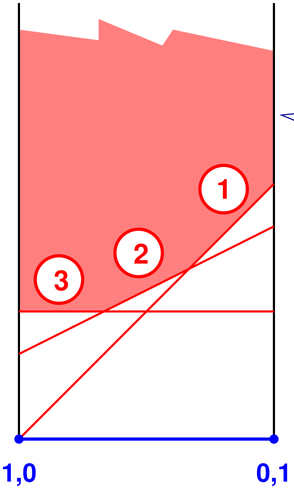
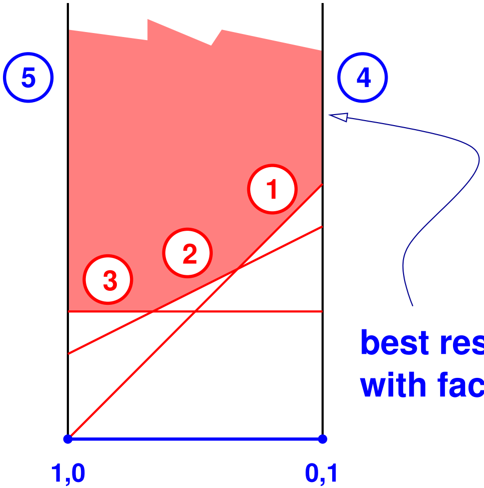


$$\begin{array}{c|cc|l} & 4 & 5 & \\ \hline 1 & 0 & 6 & \\ 2 & 2 & 5 & = A \\ 3 & 3 & 3 & \end{array}$$

payoffs to
player I

best response polyhedron

Best responses to mixed strategy of player 2

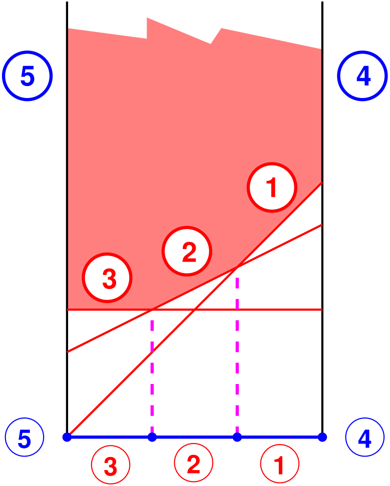


$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} 4 & 5 \end{array} \\ \boxed{\begin{array}{cc} 0 & 6 \\ 2 & 5 \\ 3 & 3 \end{array}} \end{array} = A$$

payoffs to
player I

best response polyhedron with facet labels

Best responses to mixed strategy of player 2



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payoffs to
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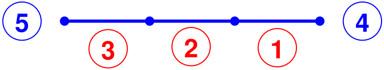
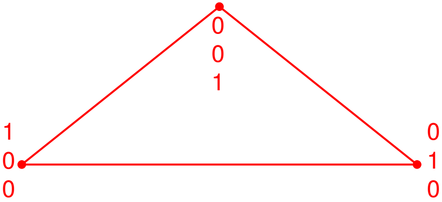
payoffs to
player I



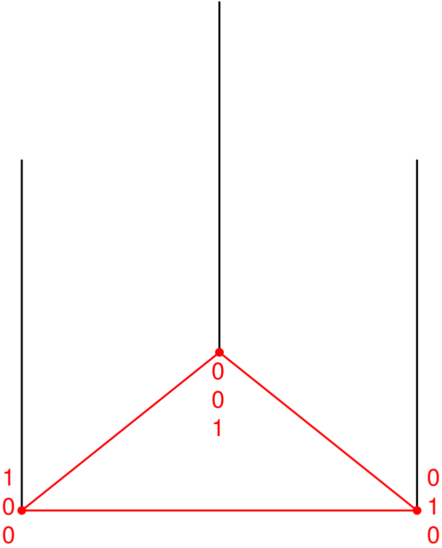
Best responses to mixed strategy of player 1

$$\begin{array}{c|cc} & 4 & 5 \\ \hline 1 & 2 & 1 \\ 2 & 1 & 3 \\ 3 & 4 & 3 \end{array} = B$$

payoffs to
player II



Best responses to mixed strategy of player 1



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payoffs to
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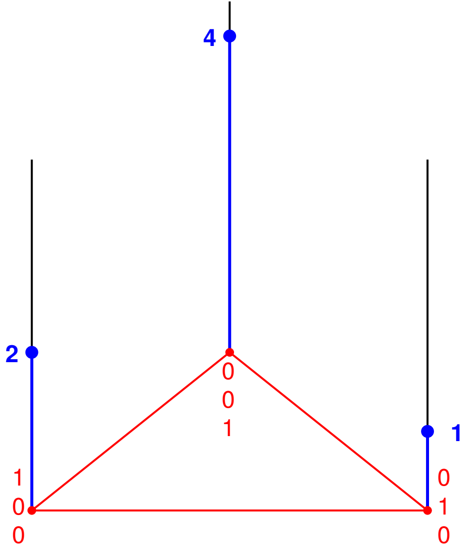
Best responses to mixed strategy of player 1



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payoffs to
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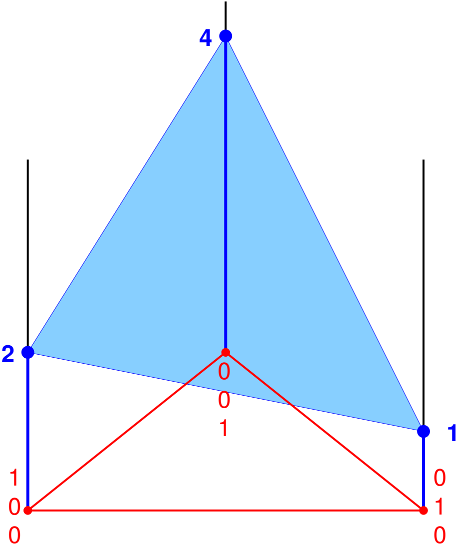
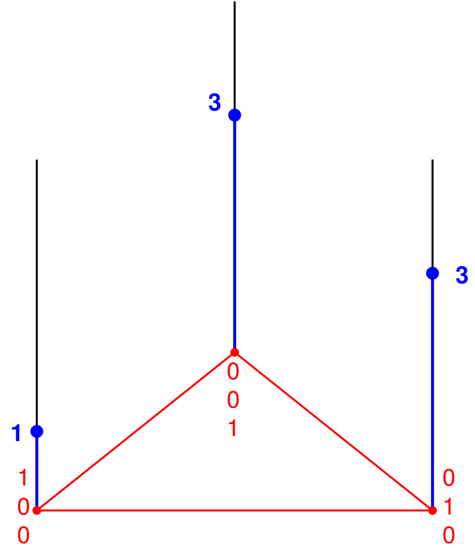


	4	5	
1	2	1	
2	1	3	= B
3	4	3	

payoffs to
player II

Best responses to mixed strategy of player 1



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payoffs to
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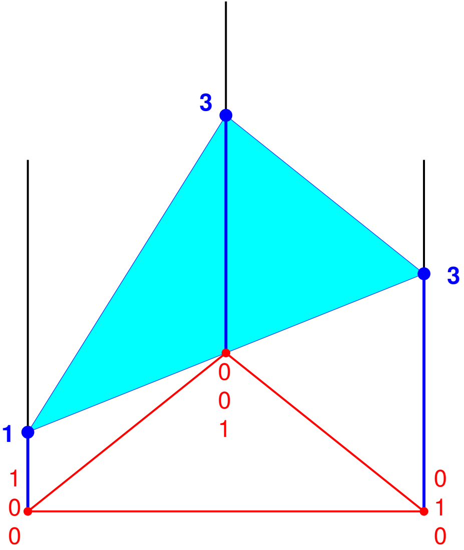

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payoffs to
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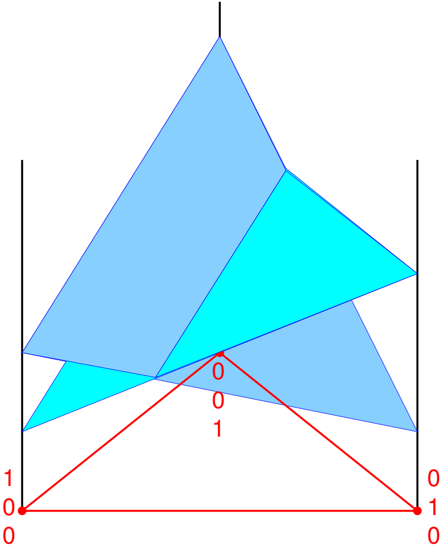

	4	5
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= B

payoffs to
player II

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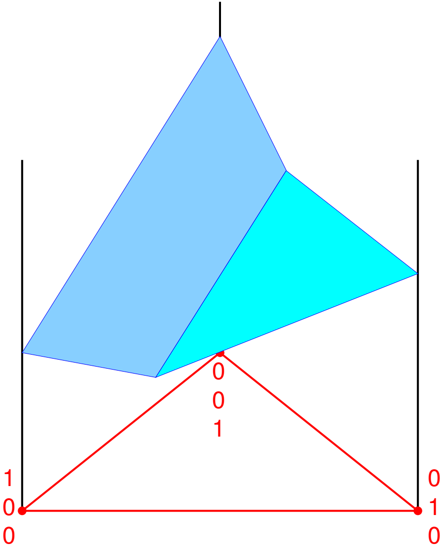

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payoffs to
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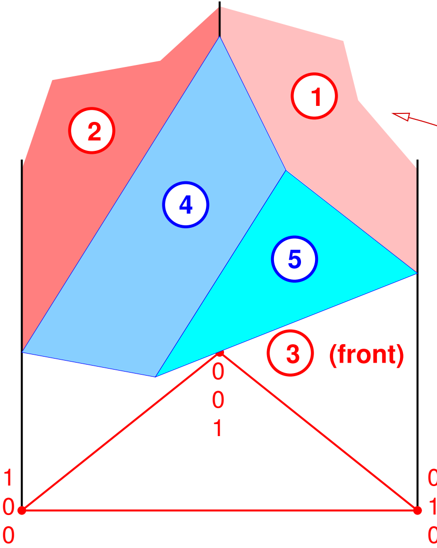

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1	2	1
2	1	3
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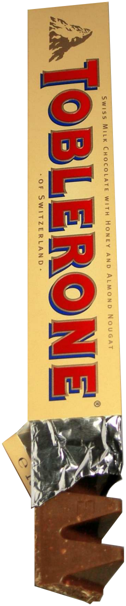
= B

payoffs to player II

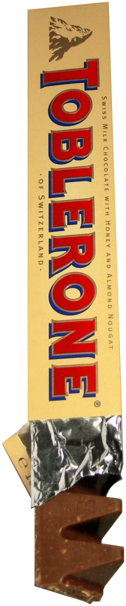
best response polyhedron with facet labels

Alternative view

Chop off Toblerone prism



Chop off Toblerone prism



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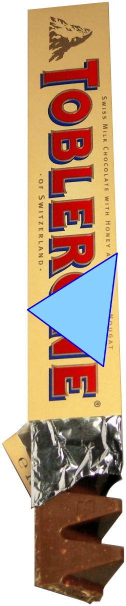

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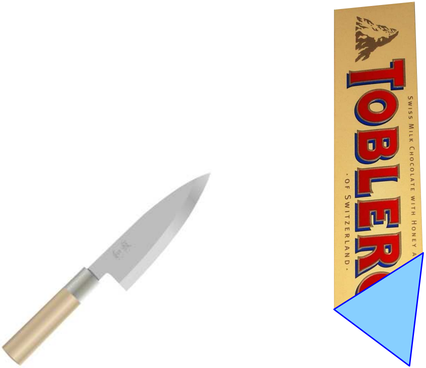
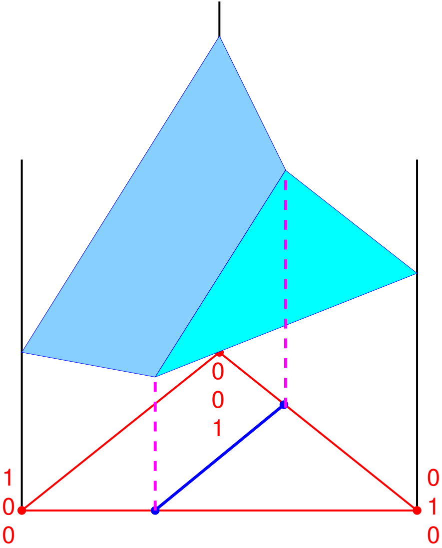


Chop off Toblerone prism

Best responses to mixed strategy of player 1



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payoffs to
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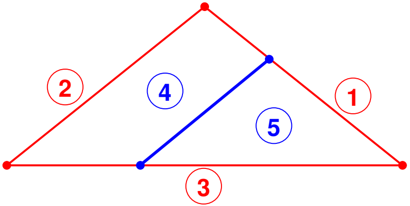
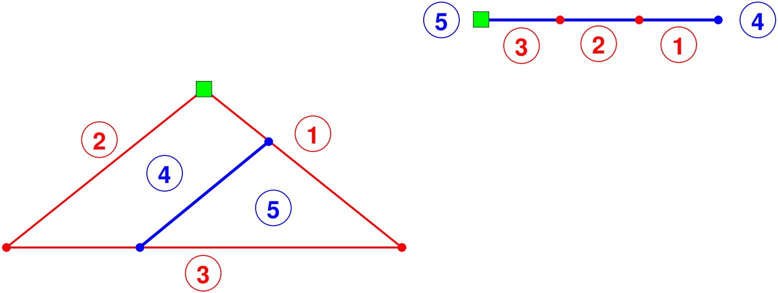
payoffs to
player II



Equilibrium = completely labeled strategy pair



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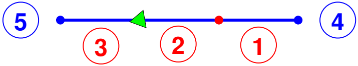

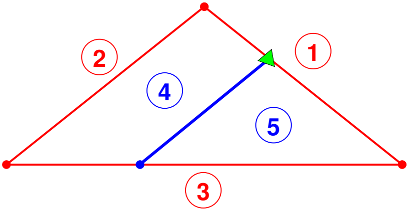

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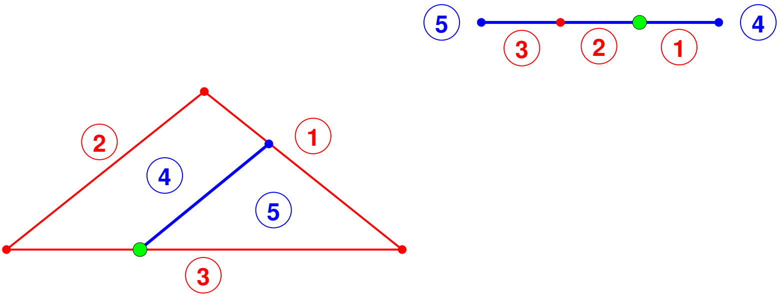

The Lemke-Howson algorithm

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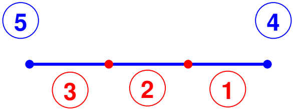

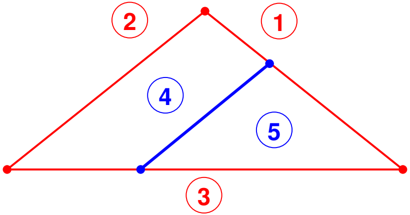

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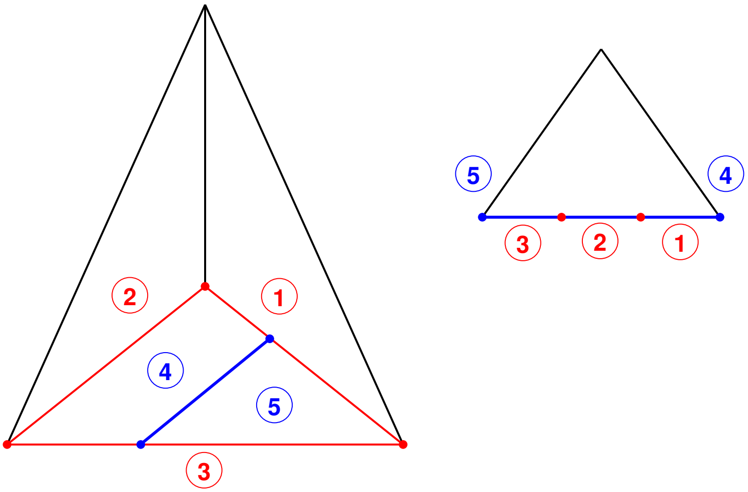

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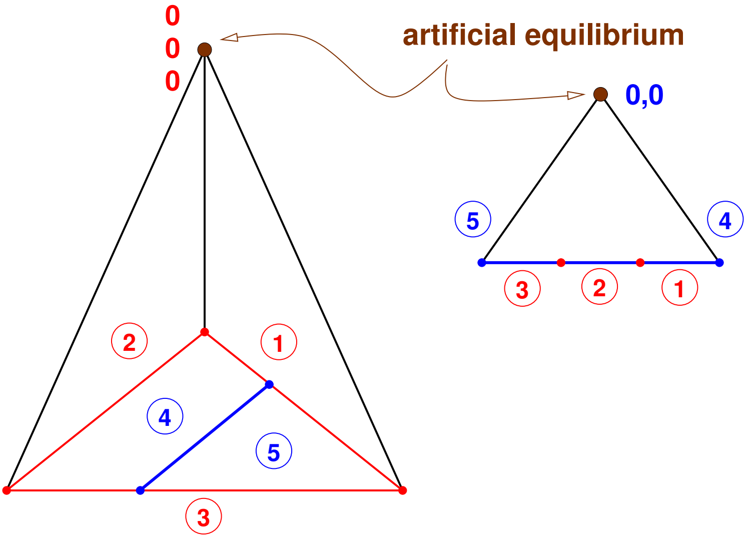

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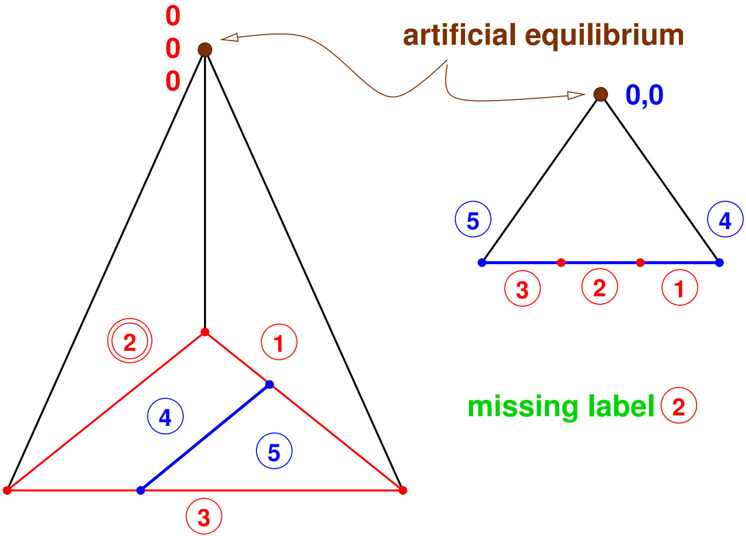

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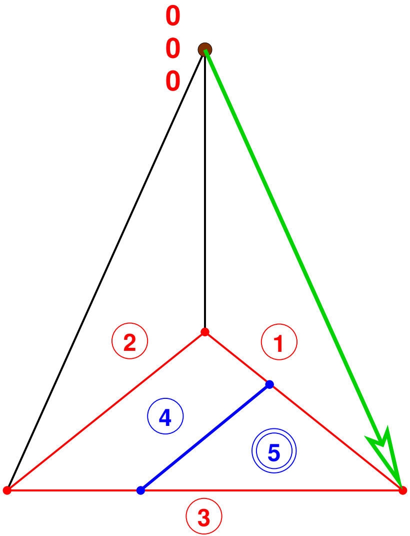
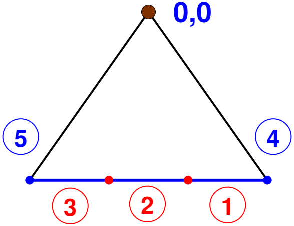
The Lemke–Howson algorithm





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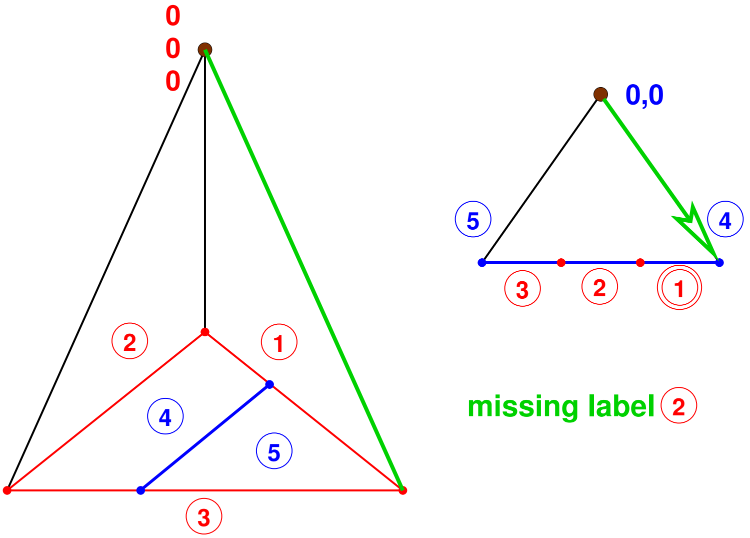

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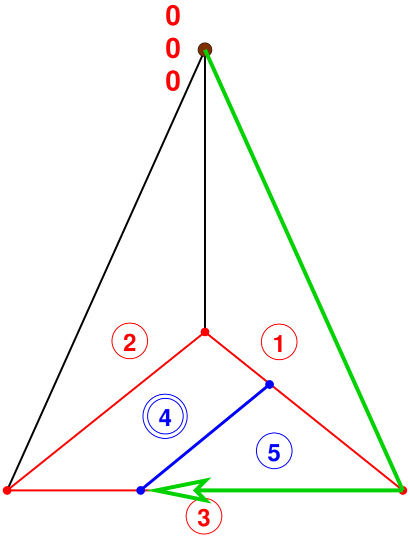

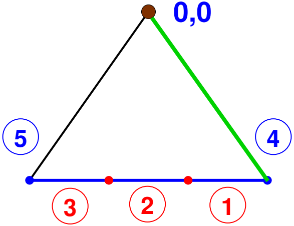

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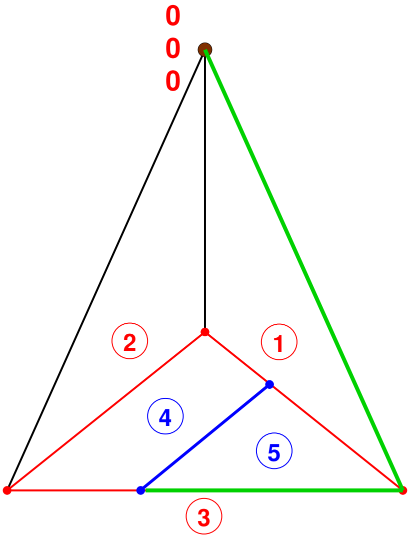

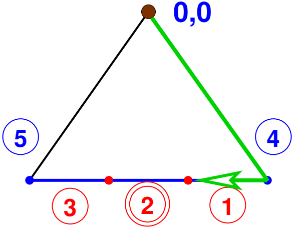

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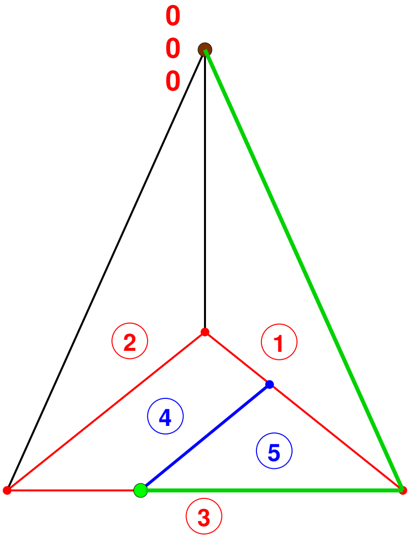
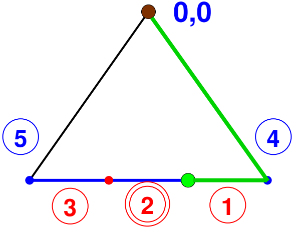




found label 2

Why Lemke-Howson works

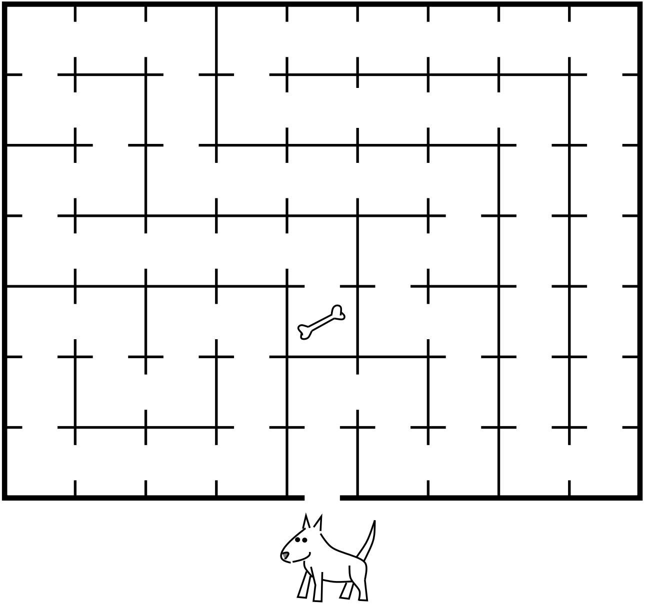
LH finds at least one Nash equilibrium because

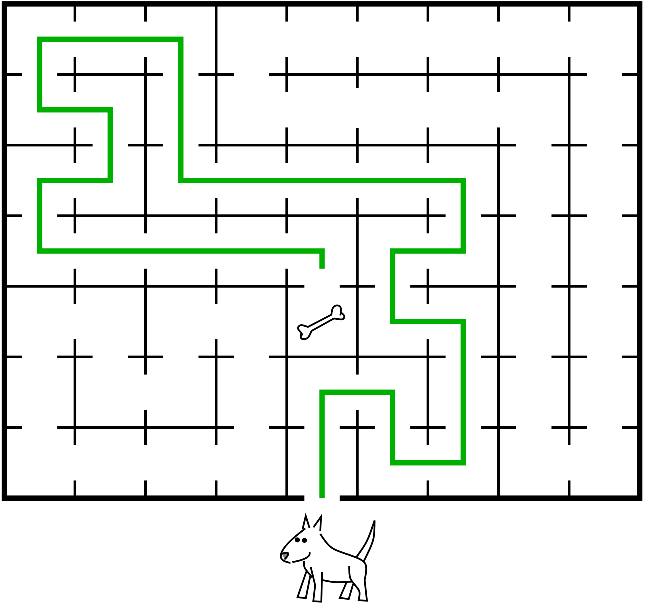
- finitely many "vertices"

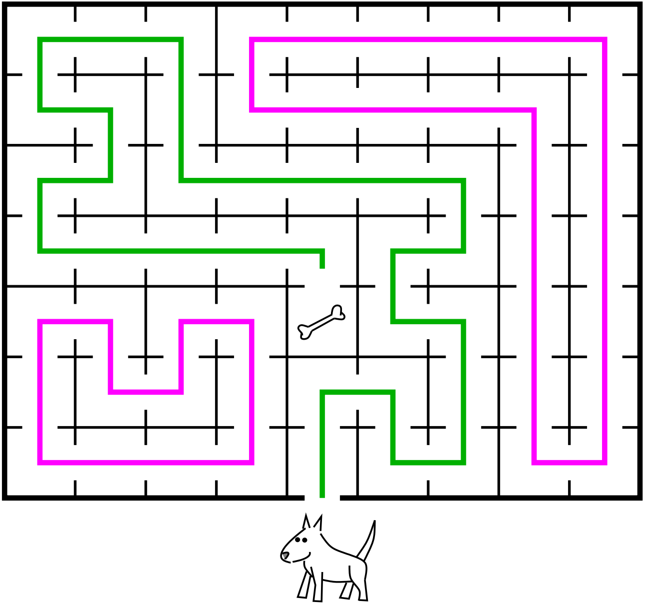
for nondegenerate (generic) games:

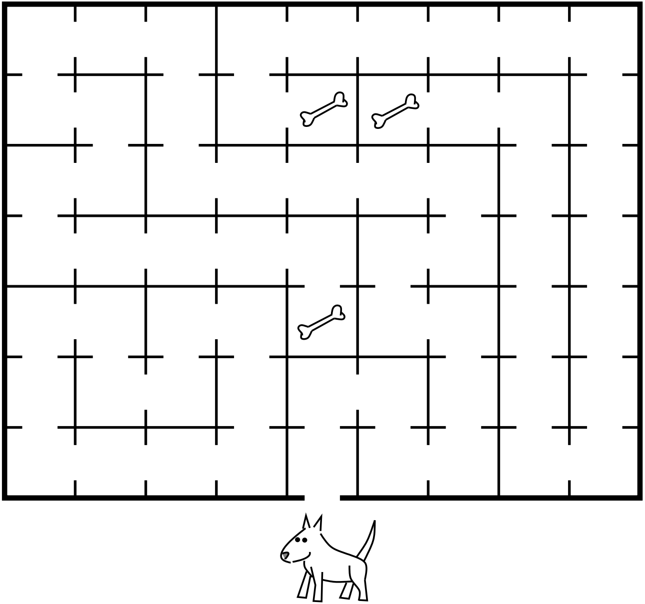
- **unique** starting edge given missing label
- **unique** continuation

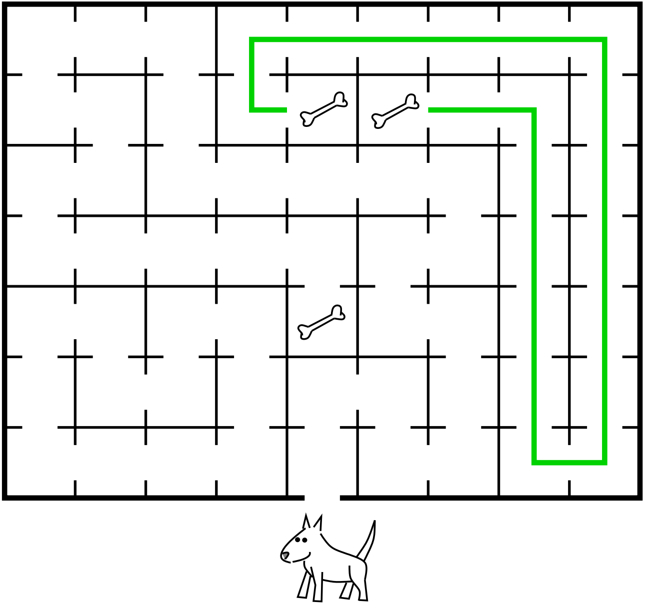
$\Rightarrow$ precludes "coming back" like here:

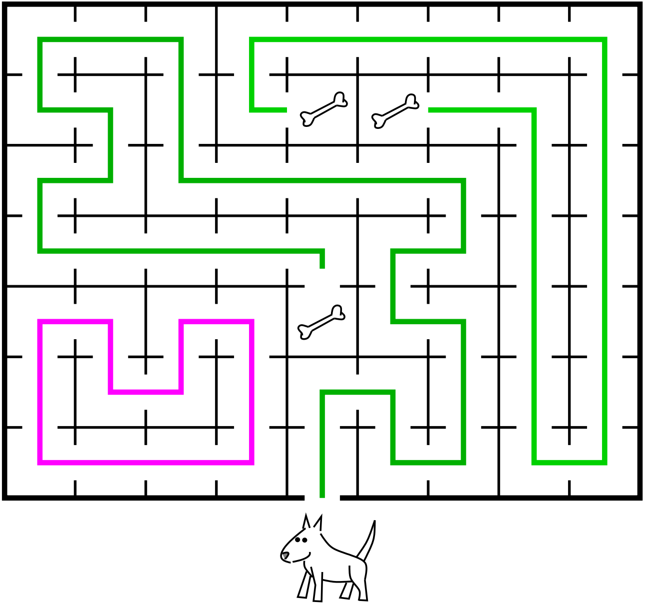












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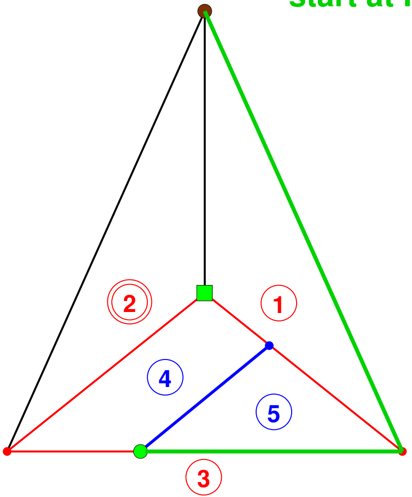
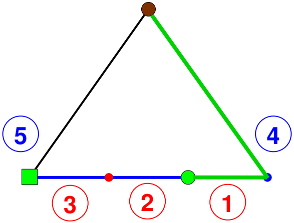


start at Nash equilibrium ■



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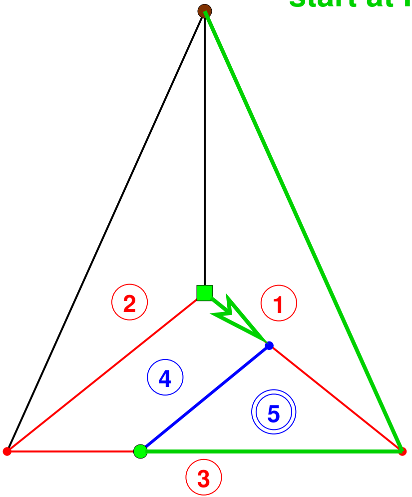
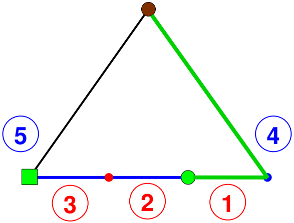


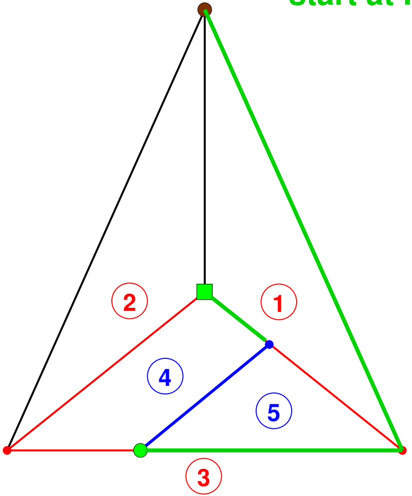
start at Nash equilibrium

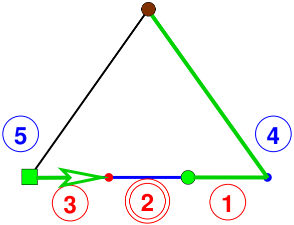


missing label 2

The Lemke–Howson algorithm

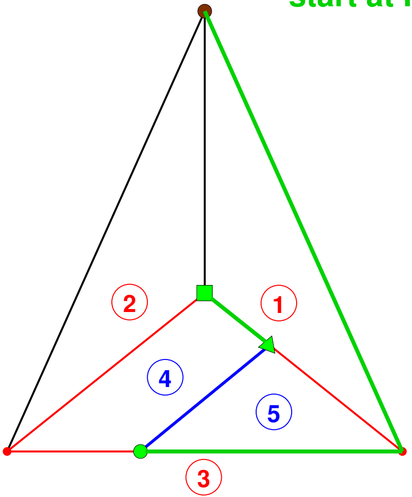
start at Nash equilibrium ■



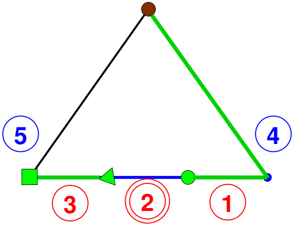


missing label 2

Odd number of Nash equilibria!



start at Nash equilibrium ■



found label 2

Nondegenerate bimatrix games

Given: $m \times n$ bimatrix game (A,B)

$X = \{ x \in \mathbf{R}^m \mid x \geq \mathbf{0},\ x_1 + \ldots + x_m = 1 \}$
$Y = \{ y \in \mathbf{R}^n \mid y \geq \mathbf{0},\ y_1 + \ldots + y_n = 1 \}$

$\mathrm{supp}(x) = \{ i \mid x_i > 0 \}$
$\mathrm{supp}(y) = \{ j \mid y_j > 0 \}$

(A,B) **nondegenerate** $\Leftrightarrow$ $\forall\ x \in X,\ y \in Y$:

$| \{ j \mid j \text{ best response to } x \} | \leq | \mathrm{supp}(x) |,$
$| \{ i \mid i \text{ best response to } y \} | \leq | \mathrm{supp}(y) |.$

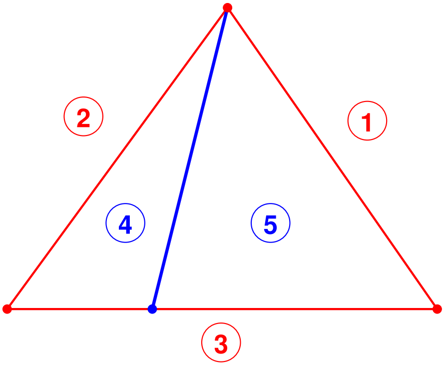
Nondegeneracy via labels

$m \times n$ bimatrix game (A,B) **nondegenerate**

$\Leftrightarrow$ no $x \in X$ has more than m labels,
no $y \in Y$ has more than n labels.

E.g. x with $> m$ labels,
s labels from $\{1, \ldots, m\}$,
$\Rightarrow$ $> m-s$ labels from $\{m+1, \ldots, m+n\}$
$\Leftrightarrow$ $> |\mathrm{supp}(x)|$ best responses to x.
$\Rightarrow$ degenerate.

Example of a degenerate game

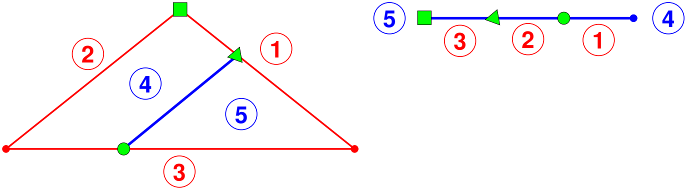


$$\begin{array}{cc} & \begin{array}{cc} 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 4 & 4 \end{bmatrix} \end{array} = B$$

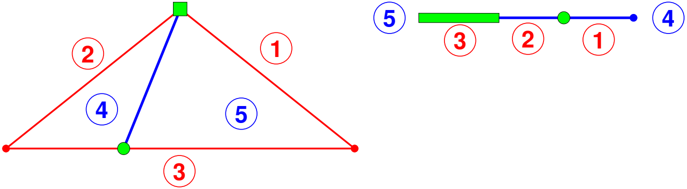
payoffs to
player II

Equilibrium components in a degenerate game

nondegenerate game:



degenerate game, same payoffs for player I:

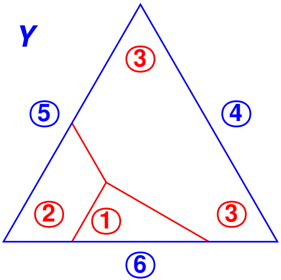


Best-response diagrams for a 3×3 game

Consider the $\mathbf{3} \times \mathbf{3}$ game

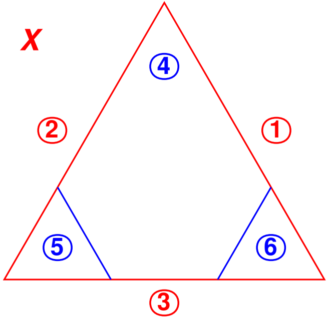
$$A = \begin{array}{c} ① \\ ② \\ ③ \end{array} \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 1 \\ -3 & 4 & 5 \end{bmatrix}, \qquad B = \begin{array}{c} \begin{array}{ccc} ④ & ⑤ & ⑥ \end{array} \\ \begin{bmatrix} 0 & 1 & -2 \\ 2 & 0 & 3 \\ 2 & 1 & 0 \end{bmatrix} \end{array}.$$

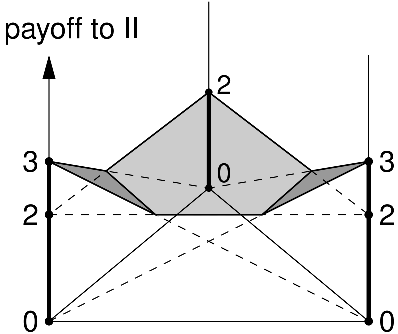
Subdivision of Y into best-response regions:



Upper envelope – with "row shift" of *B*

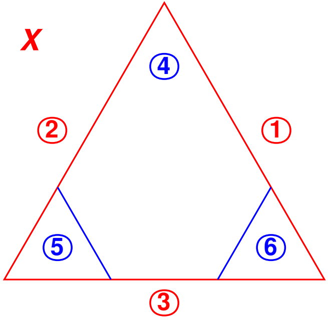
$$
\begin{array}{cc}
 & \begin{array}{ccc} ④ & ⑤ & ⑥ \end{array} \\
B' = & \begin{bmatrix} 2 & 3 & 0 \\ 2 & 0 & 3 \\ 2 & 0 & 0 \end{bmatrix}
\end{array}
$$

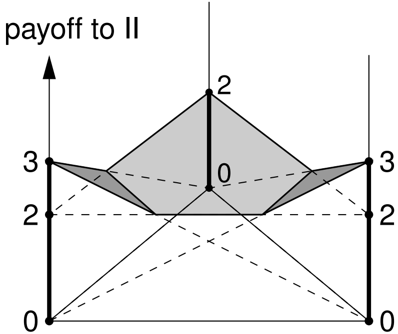




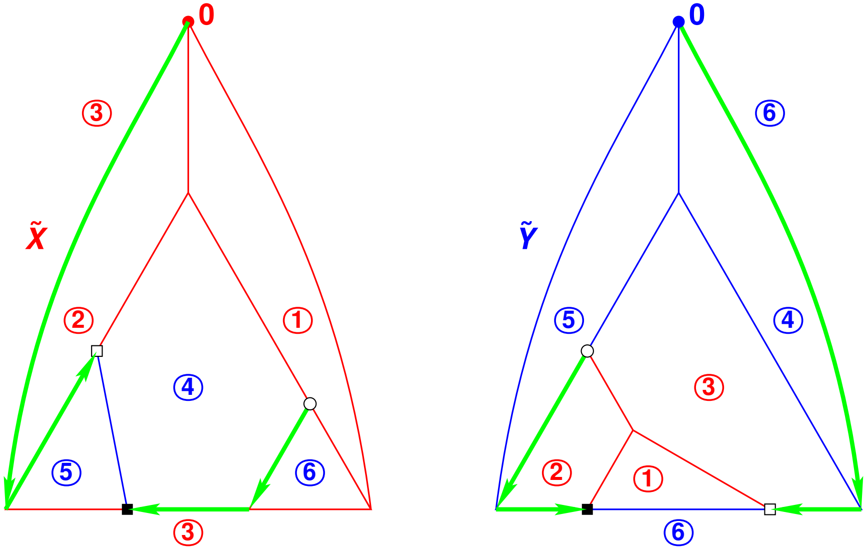
Upper envelope – with "row shift" of B

$$B' = \begin{array}{c} \begin{array}{ccc} ④ & ⑤ & ⑥ \end{array} \\ \begin{bmatrix} 2 & 3 & 0 \\ 2 & 0 & 3 \\ 2 & 0 & 0 \end{bmatrix} \end{array}, \quad B = \begin{bmatrix} 0 & 1 & -2 \\ 2 & 0 & 3 \\ 2 & 0 & 0 \end{bmatrix}, \quad x^\top B' = x^\top B + x_1 [2 \;\; 2 \;\; 2]$$

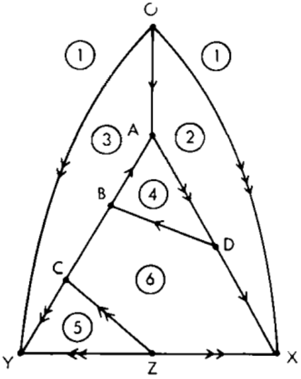


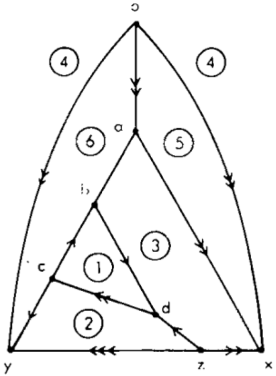


Best-response diagrams X and Y and Lemke-Howson



Diagrams from Shapley (1974)





Key:

ρ^1 OoAaDxX YyCzZ

ρ^2 OoYy XxZz

ρ^3 OoXx YyZz

ρ^4 oOaXx yYcCdZz

ρ^5 oOyY xXzZ ...bAcBdDb...

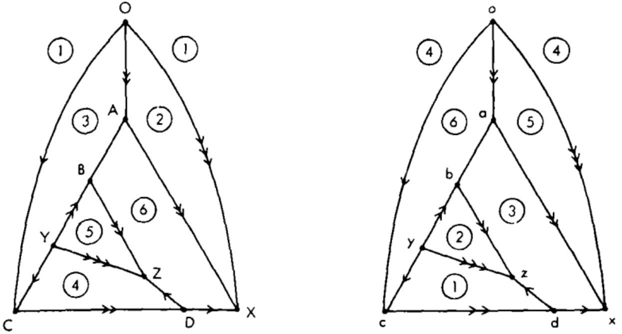
ρ^6 oOxX yYzZ

Payoffs:

$$\begin{bmatrix} 2 & 2 & 0 \\ 0 & 3 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 3 & 0 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

from Robert Wilson, in Shapley (1974)

Lemke-Howson may only find **some** equilibria:



Key:

ρ^1	OoAxX	yYbZz
ρ^2	OoCaDxX	yYcBdZz
ρ^3	OoXx	YyZz
ρ^4	oOaXx	YyBzZ
ρ^5	oOcAdXx	YyCbDzZ
ρ^6	oOxX	yYzZ

Payoffs:

$$\begin{bmatrix} 0 & 3 & 0 \\ 2 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 2 & 3 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Running time of Lemke-Howson

The running time of Lemke-Howson may be **exponential** in the size of the game:

R. Savani and B. von Stengel (2004), Exponentially many steps for finding a Nash equilibrium in a bimatrix game. In: *Proc. 45th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2004)*, 258–267.

R. Savani and B. von Stengel (2006), Hard-to-solve bimatrix games. *Econometrica* 74, 397–429.

R. Savani and B. von Stengel (2016), Unit vector games. *International Journal of Economic Theory* 12, 7–27.

Questions

- how to implement Lemke-Howson as an algorithm
  - **use labeled polytopes**
  - complementary pivoting
- handling degenerate games
- finding one vs. all Nash equilibria
  - possibly exponentially many NE
  - uniqueness is co-NP-complete
- running time of Lemke-Howson
  - worst-case: exponential
  - average case?
  - smoothed analysis?

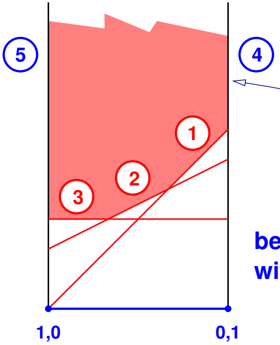
Labeled polytopes and

completely labeled vertex pairs

Best-response polyhedra and polytopes

best-response **polyhedra**:

$$\overline{P} = \{\, (x, v) \in X \times \mathbb{R} \mid B^\top x \le \mathbf{1}v \,\}$$
$$\overline{Q} = \{\, (y, u) \in Y \times \mathbb{R} \mid Ay \le \mathbf{1}u \,\}$$



$$\begin{array}{c|cc|} & 4 & 5 \\ \hline 1 & 0 & 6 \\ 2 & 2 & 5 \\ 3 & 3 & 3 \\ \hline \end{array} = A$$

payoffs to player I

best response polyhedron with facet labels

Best-response polyhedra and polytopes

best-response **polyhedra**:

$$\overline{P} = \{ (x, v) \in X \times \mathbb{R} \mid B^\top x \le \mathbf{1}v \}$$
$$\overline{Q} = \{ (y, u) \in Y \times \mathbb{R} \mid Ay \le \mathbf{1}u \}$$

best-response **polytopes**:

$$P = \{ x \in \mathbb{R}^m \mid x \ge \mathbf{0},\ B^\top x \le \mathbf{1} \}$$
$$Q = \{ y \in \mathbb{R}^n \mid Ay \le \mathbf{1},\ y \ge \mathbf{0} \}$$

obtained from $\overline{P}$, $\overline{Q}$ via $x \mapsto x\frac{1}{v}$, $y \mapsto y\frac{1}{u}$

(requires $u, v > 0$, if needed via adding constants to A, B)

re-normalized to X, Y via $x \mapsto x\frac{1}{\mathbf{1}^\top x}$, $y \mapsto y\frac{1}{\mathbf{1}^\top y}$

Labeled polytopes

$$P = \{\, x \in \mathbb{R}^m \mid \quad x \geq \mathbf{0},\ \ B^\top x \leq \mathbf{1}\}$$

$$Q = \{\, y \in \mathbb{R}^n \mid Ay \leq \mathbf{1}, \qquad y \geq \mathbf{0}\}$$

$(x, y) \in P \times Q$ (re-normalized in $X \times Y$) equilibrium of (A, B)

$\Leftrightarrow$

$x \geq \mathbf{0} \quad \perp \quad Ay \leq \mathbf{1} \quad$ (labels $1, \ldots, m$)

$y \geq \mathbf{0} \quad \perp \quad B^\top x \leq \mathbf{1} \quad$ (labels $m+1, \ldots, m+n$)

Labeled polytopes

$$P = \{\, x \in \mathbb{R}^m \mid \quad x \ge \mathbf{0},\ B^\top x \le \mathbf{1}\}$$
$$Q = \{\, y \in \mathbb{R}^n \mid Ay \le \mathbf{1}, \qquad y \ge \mathbf{0}\}$$

$(x, y) \in P \times Q$ (re-normalized in $X \times Y$) equilibrium of (A, B)

$\Leftrightarrow$

$x \ge \mathbf{0} \quad \perp \quad Ay \le \mathbf{1}$ (labels $1, \ldots, m$)

$y \ge \mathbf{0} \quad \perp \quad B^\top x \le \mathbf{1}$ (labels $m+1, \ldots, m+n$)

artificial equilibrium $(x, y) = (\mathbf{0}, \mathbf{0})$, not in $X \times Y$, not NE.

Only one labeled polytope

$$P = \{\, x \in \mathbb{R}^m \mid \quad x \geq \mathbf{0},\ \ B^\top x \leq \mathbf{1}\}$$

$$Q = \{\, y \in \mathbb{R}^n \mid Ay \leq \mathbf{1}, \qquad y \geq \mathbf{0}\}$$

$$\boxed{R = \{\, z \in \mathbb{R}^k \mid \quad z \geq \mathbf{0}, \quad Cz \leq \mathbf{1}\}}$$

$R = P \times Q$, $\qquad k = m + n$,

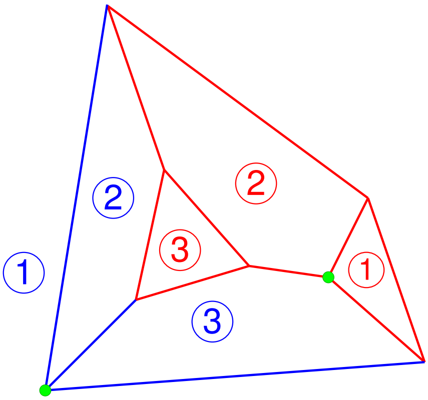
$C = \begin{pmatrix} \mathbf{0} & A \\ B^\top & \mathbf{0} \end{pmatrix} \in \mathbb{R}^{k \times k}$, $\qquad z = (x, y)$

equilibrium $z \quad \Leftrightarrow \quad z \geq \mathbf{0} \ \perp \ Cz \leq \mathbf{1}$ (labels $1, \ldots, k$)

artificial equilibrium $z = \mathbf{0}$, any other $z = (x, y)$ with x re-normalized in X and y in Y is NE of (A, B)

Path of "almost completely labeled" edges

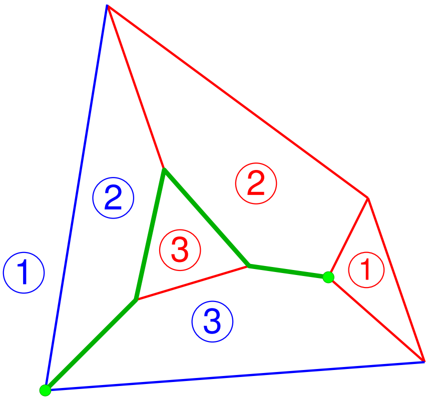
$$R = \{ z \in \mathbb{R}^k \mid z \geq \mathbf{0},\ Cz \leq \mathbf{1} \}$$



Path of "almost completely labeled" edges

$$R = \{\, z \in \mathbb{R}^k \mid z \geq \mathbf{0},\ Cz \leq \mathbf{1} \}$$

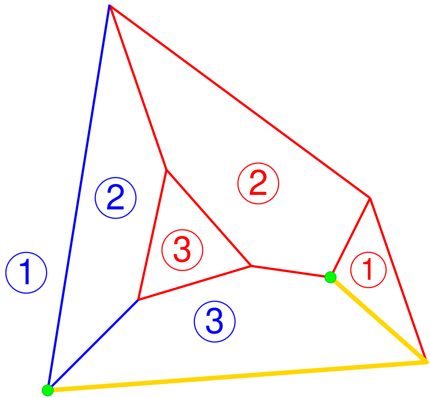
missing label ① :



Path of "almost completely labeled" edges

$$R = \{\, z \in \mathbb{R}^k \mid z \geq \mathbf{0},\ Cz \leq \mathbf{1} \}$$

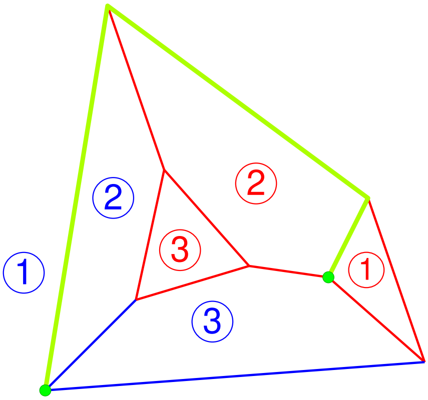
missing label ② :



Path of “almost completely labeled” edges

$$R = \{\, z \in \mathbb{R}^k \mid z \geq \mathbf{0},\ Cz \leq \mathbf{1} \}$$

missing label ③ :



Algebraic implementation
by pivoting

Complementary pivoting

$$
\begin{aligned}
& \boldsymbol{z} \geq \mathbf{0} \quad \perp \quad \boldsymbol{C}\boldsymbol{z} \leq \mathbf{1} \\
\Leftrightarrow \quad & \boldsymbol{z} \geq \mathbf{0} \quad \perp \quad \boldsymbol{s} \geq \mathbf{0}, \qquad \boxed{\boldsymbol{C}\boldsymbol{z} + \boldsymbol{s} = \mathbf{1}}
\end{aligned}
$$

Complementary pivoting

$$z \geq 0 \quad \perp \quad Cz \leq 1$$

$$\Leftrightarrow \quad z \geq 0 \quad \perp \quad s \geq 0, \qquad \boxed{Cz + s = 1}$$

$z \geq 0,\ s \geq 0$ **ℓ-almost complementary** (missing label ℓ)

$$\Leftrightarrow \quad Cz + s = 1, \qquad \boxed{z_i\, s_i = 0} \qquad \text{for } i = 1, \ldots, k,\ \boxed{i \neq \ell}$$

Complementary pivoting

$$z \geq 0 \quad \perp \quad Cz \leq 1$$

$$\Leftrightarrow \quad z \geq 0 \quad \perp \quad s \geq 0, \quad \boxed{Cz + s = 1}$$

$z \geq 0,\ s \geq 0$ **ℓ-almost complementary** (missing label ℓ)

$$\Leftrightarrow \quad Cz + s = 1, \quad \boxed{z_i\, s_i = 0} \quad \text{for } i = 1, \ldots, k, \ \boxed{i \neq \ell}$$

complementary pivoting = algebraic traversal of ℓ-almost complementary edges of $\{\, z \in \mathbb{R}^k \mid z \geq 0,\ Cz \leq 1\}$

starting with $z = 0$, $\quad s = 1 - Cz$.

Complementary pivoting

$$z \geq \mathbf{0} \quad \perp \quad Cz \leq \mathbf{1}$$

$$\Leftrightarrow \quad z \geq \mathbf{0} \quad \perp \quad s \geq \mathbf{0}, \quad \boxed{Cz + s = \mathbf{1}}$$

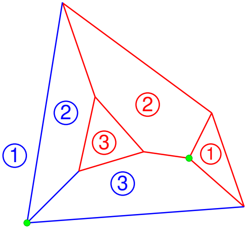
$z \geq \mathbf{0}$, $s \geq \mathbf{0}$ **ℓ-almost complementary** (missing label ℓ)

$$\Leftrightarrow \quad Cz + s = \mathbf{1}, \quad \boxed{z_i s_i = 0} \quad \text{for } i = 1, \ldots, k, \ \boxed{i \neq \ell}$$

complementary pivoting = algebraic traversal of ℓ-almost complementary edges of $\{ z \in \mathbb{R}^k \mid z \geq \mathbf{0}, \ Cz \leq \mathbf{1}\}$

starting with $z = \mathbf{0}$, $s = \mathbf{1} - Cz$.

Example: $C = \begin{bmatrix} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{bmatrix}$



Almost complementary $\boxed{\text{dictionaries}}$

dictionary = any equivalent system to $Cz + s = 1$

basic variables expressed depending on **nonbasic variables**

- nonbasic variables set to **0** :
  - gives **basic solution** = polytope **vertex**,
  - nonbasic variables = binding inequalities = vertex **labels**
- starting dictionary: $s = 1 - Cz$

Almost complementary $\boxed{\text{dictionaries}}$

dictionary = any equivalent system to $\boldsymbol{Cz} + \boldsymbol{s} = \mathbf{1}$

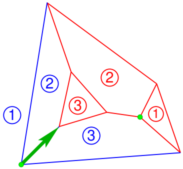
basic variables expressed depending on **nonbasic variables**

- nonbasic variables set to **0** :
  - gives **basic solution** = polytope **vertex**,
  - nonbasic variables = binding inequalities = vertex **labels**
- starting dictionary: $\boldsymbol{s} = \mathbf{1} - \boldsymbol{Cz}$

choose **entering column** = entering nonbasic variable z_ℓ

identify *the* **leaving row** = leaving basic variable, here s_3

$$
\begin{aligned}
s_1 &= 1 \qquad\quad\; - 3z_2 \\
s_2 &= 1 - 2z_1 - 2z_2 - 2z_3 \\
s_3 &= 1 - 3z_1
\end{aligned}
$$

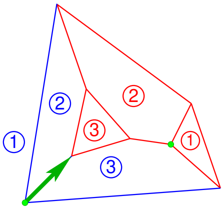


Complementary variables

$$\begin{aligned} s_1 &= 1 \qquad\qquad - 3z_2 \\ s_2 &= 1 - 2z_1 - 2z_2 - 2z_3 \\ s_3 &= 1 - 3z_1 \end{aligned}$$

z_1 enters, $\boxed{s_3}$ leaves:

$$\begin{aligned} s_1 &= 1 \qquad\qquad\quad - 3z_2 \\ s_2 &= \tfrac{1}{3} + \tfrac{2}{3}s_3 - 2z_2 - 2z_3 \\ z_1 &= \tfrac{1}{3} - \tfrac{1}{3}s_3 \end{aligned}$$

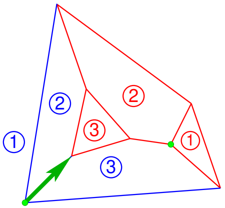


Complementary variables

$$\begin{aligned} s_1 &= 1 \qquad\qquad - 3z_2 \\ s_2 &= 1 - 2z_1 - 2z_2 - 2z_3 \\ s_3 &= 1 - 3z_1 \end{aligned}$$

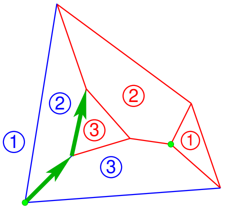
z_1 enters, $\boxed{s_3}$ leaves:

$$\begin{aligned} s_1 &= 1 \qquad\qquad\quad - 3z_2 \\ s_2 &= \tfrac{1}{3} + \tfrac{2}{3}s_3 - 2z_2 - 2z_3 \\ z_1 &= \tfrac{1}{3} - \tfrac{1}{3}s_3 \end{aligned}$$



$\boxed{z_3}$ enters, s_2 leaves:

$$\begin{aligned} s_1 &= 1 \qquad\qquad\quad - 3z_2 \\ z_3 &= \tfrac{1}{6} + \tfrac{1}{3}s_3 - \ \ z_2 - \tfrac{1}{2}s_2 \\ z_1 &= \tfrac{1}{3} - \tfrac{1}{3}s_3 \end{aligned}$$



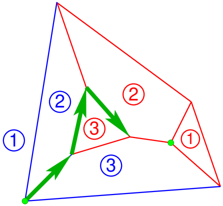
complementary pivoting, continued

... $\boxed{s_2}$ left the basis, $\boxed{z_2}$ enters:

$$
\begin{aligned}
s_1 &= 1 - 3z_2 \\
z_3 &= \tfrac{1}{6} + \tfrac{1}{3}s_3 - z_2 - \tfrac{1}{2}s_2 \\
z_1 &= \tfrac{1}{3} - \tfrac{1}{3}s_3
\end{aligned}
$$

$\boxed{z_3}$ leaves:

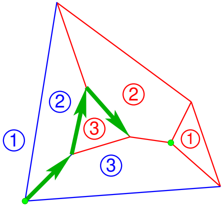
$$
\begin{aligned}
s_1 &= \tfrac{1}{2} - s_3 + 3z_3 + \tfrac{3}{2}s_2 \\
z_2 &= \tfrac{1}{6} + \tfrac{1}{3}s_3 - z_3 - \tfrac{1}{2}s_2 \\
z_1 &= \tfrac{1}{3} - \tfrac{1}{3}s_3
\end{aligned}
$$



complementary pivoting, continued

... $\boxed{s_2}$ left the basis, $\boxed{z_2}$ enters:

$$
\begin{aligned}
s_1 &= 1 \qquad\qquad - 3z_2 \\
z_3 &= \tfrac{1}{6} + \tfrac{1}{3}s_3 - z_2 - \tfrac{1}{2}s_2 \\
z_1 &= \tfrac{1}{3} - \tfrac{1}{3}s_3
\end{aligned}
$$

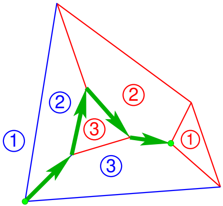


$\boxed{z_3}$ leaves:

$$
\begin{aligned}
s_1 &= \tfrac{1}{2} - s_3 + 3z_3 + \tfrac{3}{2}s_2 \\
z_2 &= \tfrac{1}{6} + \tfrac{1}{3}s_3 - z_3 - \tfrac{1}{2}s_2 \\
z_1 &= \tfrac{1}{3} - \tfrac{1}{3}s_3
\end{aligned}
$$

$\boxed{s_3}$ enters, s_1 leaves, equilibrium found:

$$
\begin{aligned}
s_3 &= \tfrac{1}{2} - s_1 + 3z_3 + \tfrac{3}{2}s_2 \\
z_2 &= \tfrac{1}{3} - \tfrac{1}{3}s_1 \\
z_1 &= \tfrac{1}{6} + \tfrac{1}{3}s_1 - z_3 - \tfrac{1}{2}s_2
\end{aligned}
$$



Labeled polytopes and bimatrix games

Did we solve a game?

Yes!

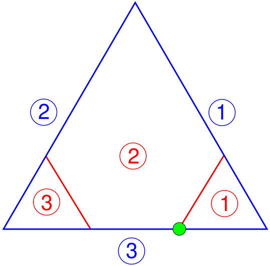
$\boldsymbol{z} = (\frac{1}{6}, \frac{1}{2}, 0)^\top$ is normalized $\bar{\boldsymbol{z}} = (\frac{1}{3}, \frac{2}{3}, 0)^\top$ and a (here unique)

symmetric equilibrium $(\bar{\boldsymbol{z}}, \bar{\boldsymbol{z}})$ of the game $(C, C^\top)$ with

$$C = \begin{bmatrix} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{bmatrix}, \qquad \text{that is,}$$

$$\bar{\boldsymbol{z}} \geq \mathbf{0} \quad \perp \quad C\bar{\boldsymbol{z}} \leq \mathbf{1}u \qquad \text{with payoff } u = 2 = \frac{1}{\mathbf{1}^\top \boldsymbol{z}}$$

	1	2	3
1	0	3	0
2	2	2	2
3	3	0	0



Arbitrary labeled polytopes

simple polytope in $\mathbb{R}^m$ $\Leftrightarrow$ every vertex on only m facets

labeled (simple) polytope in $\mathbb{R}^m$:
every facet has one label in $\{1, \ldots, m\}$

completely labeled vertex = its facets have all labels $1, \ldots, m$

Arbitrary labeled polytopes

simple polytope in $\mathbb{R}^{\boldsymbol{m}}$ $\Leftrightarrow$ every vertex on only $\boldsymbol{m}$ facets

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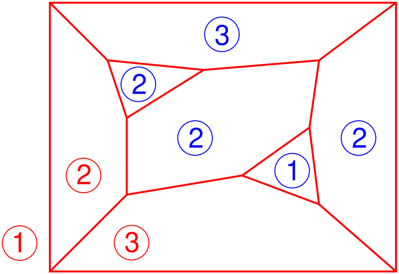
every facet has one label in $\{\mathbf{1}, \ldots, \boldsymbol{m}\}$

completely labeled vertex = its facets have all labels $\mathbf{1}, \ldots, \boldsymbol{m}$

Theorem The number of completely labeled vertices is **even.**



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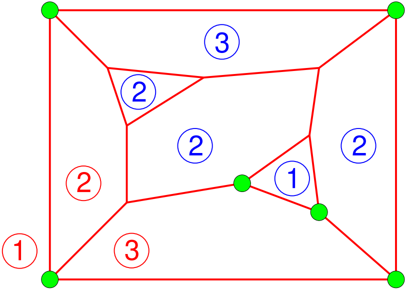

Arbitrary labeled polytopes

simple polytope in $\mathbb{R}^{\boldsymbol{m}}$ $\Leftrightarrow$ every vertex on only $\boldsymbol{m}$ facets

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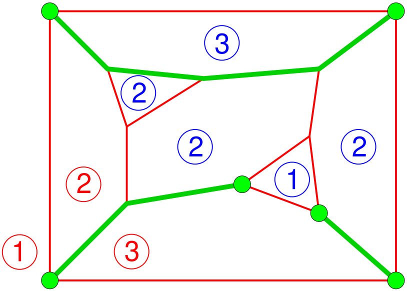


Unit vector games

Let $b_1, \ldots, b_n \in \mathbb{R}^m$, $\quad B = [b_1 \cdots b_n]$

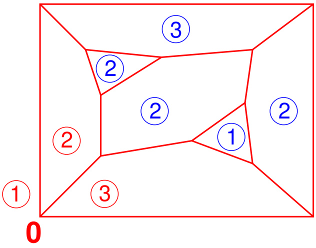
$\ell(1), \ldots, \ell(n) \in \{1, \ldots, m\}$ be labels

$P = \{x \in \mathbb{R}^m \mid x \geq 0,\ B^\top x \leq 1\}$

with labels of P for binding inequalities:

label i : $\quad x_i \geq 0 \quad (1 \leq i \leq m)$

label $\ell(j)$: $\quad b_j^\top x \leq 1 \quad (1 \leq j \leq n)$



Theorem $\quad x \neq 0$ completely labeled vertex of $P \quad \Leftrightarrow$

(x, y) Nash equilibrium of (U, B) where $U = [e_{\ell(1)} \cdots e_{\ell(n)}]$

e_i = ith unit vector in $\mathbb{R}^m$

Summary

Nash equilibria of bimatrix games

are completely labeled vertices of facet-labeled polytopes $\boldsymbol{P}$

(assuming there is one completely labeled vertex $\boldsymbol{x} = \mathbf{0}$ of $\boldsymbol{P}$ whose incident facet inequalities can w.l.o.g. be written as $\boldsymbol{x} \geq \mathbf{0}$, which is not a NE but the artificial equilibrium).

For generic games (simple polytopes), the number of completely labeled vertices is **even**, and hence the number of NE is odd.

Evenness = Parity Argument, complexity class **PPAD**.

Degeneracy resolution

Integer pivoting

Degeneracy

In pivoting, **degeneracy** means at least one **zero** basic variable in a basic feasible solution

$\Rightarrow$ additional **labels** as binding inequalities (not just the nonbasic variables)

occurs when **leaving variable not unique**

Example: z_2 enters:

$$\begin{aligned} s_1 &= 1 & & - 3z_2 \\ z_1 &= \tfrac{1}{3} + \tfrac{2}{3}s_2 & & - z_2 \end{aligned}$$

Degeneracy

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Apply to general system $Ax = b, \quad x \geq 0$ written as $A_B x_B + A_N x_N = b$ with basic columns B, nonbasic columns N

Lexicographic degeneracy resolution

$$Ax = b$$

Lexicographic degeneracy resolution

$$
\begin{aligned}
Ax &= b \\
A_B x_B + A_N x_N &= b
\end{aligned}
$$

Lexicographic degeneracy resolution

$$
\begin{aligned}
\boldsymbol{Ax} &= \boldsymbol{b} \\
\boldsymbol{A_B x_B} + \boldsymbol{A_N x_N} &= \boldsymbol{b} \\
\boldsymbol{A_B x_B} &= \boldsymbol{b} - \boldsymbol{A_N x_N}
\end{aligned}
$$

Lexicographic degeneracy resolution

$$\begin{aligned}
Ax &= b \\
A_B x_B + A_N x_N &= b \\
A_B x_B &= b - A_N x_N
\end{aligned}$$

dictionary $\boxed{x_B = A_B^{-1} b - A_B^{-1} A_N x_N}$

Lexicographic degeneracy resolution

perturb $\boldsymbol{b}$ to $\boldsymbol{b} + \vec{\varepsilon}_*$ with small $\varepsilon > 0$, $\quad \vec{\varepsilon} = (1, \varepsilon, \varepsilon^2, \ldots \varepsilon^m)^\top$

$$
\begin{array}{lclcl}
\boldsymbol{A}\boldsymbol{x} & = & \boldsymbol{b} & & \\
\boldsymbol{A}_{\boldsymbol{B}}\boldsymbol{x}_{\boldsymbol{B}} + \boldsymbol{A}_{\boldsymbol{N}}\boldsymbol{x}_{\boldsymbol{N}} & = & \boldsymbol{b} & & \\
\boldsymbol{A}_{\boldsymbol{B}}\boldsymbol{x}_{\boldsymbol{B}} & = & \boldsymbol{b} & - & \boldsymbol{A}_{\boldsymbol{N}}\boldsymbol{x}_{\boldsymbol{N}}
\end{array}
$$

dictionary $\boxed{\boldsymbol{x}_{\boldsymbol{B}} = \boldsymbol{A}_{\boldsymbol{B}}^{-1}\boldsymbol{b} - \boldsymbol{A}_{\boldsymbol{B}}^{-1}\boldsymbol{A}_{\boldsymbol{N}}\boldsymbol{x}_{\boldsymbol{N}}}$

Lexicographic degeneracy resolution

perturb $\boldsymbol{b}$ to $\boldsymbol{b} + \vec{\varepsilon}_*$ with small $\varepsilon > 0$, $\quad \vec{\varepsilon} = (1, \varepsilon, \varepsilon^2, \ldots \varepsilon^m)^\top$

$$
\begin{aligned}
\boldsymbol{Ax} &= [\,\boldsymbol{b} \mid \boldsymbol{I}\,]\,\vec{\varepsilon} \\
\boldsymbol{A_B x_B} + \boldsymbol{A_N x_N} &= [\,\boldsymbol{b} \mid \boldsymbol{I}\,]\,\vec{\varepsilon} \\
\boldsymbol{A_B x_B} &= [\,\boldsymbol{b} \mid \boldsymbol{I}\,]\,\vec{\varepsilon} - \boldsymbol{A_N x_N}
\end{aligned}
$$

dictionary $\boxed{\boldsymbol{x_B} = \boldsymbol{A_B^{-1}}[\,\boldsymbol{b} \mid \boldsymbol{I}\,]\,\vec{\varepsilon} - \boldsymbol{A_B^{-1} A_N x_N}}$

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perturb $\boldsymbol{b}$ to $\boldsymbol{b} + \vec{\varepsilon}_*$ with small $\varepsilon > 0$, $\quad \vec{\varepsilon} = (1, \varepsilon, \varepsilon^2, \ldots \varepsilon^m)^\top$

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\boldsymbol{Ax} &= [\,\boldsymbol{b} \mid \boldsymbol{I}\,]\,\vec{\varepsilon} \\
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\boldsymbol{A_B x_B} &= [\,\boldsymbol{b} \mid \boldsymbol{I}\,]\,\vec{\varepsilon} - \boldsymbol{A_N x_N} \\
\boldsymbol{x_B} &= \boldsymbol{A_B^{-1}}[\,\boldsymbol{b} \mid \boldsymbol{I}\,]\,\vec{\varepsilon} - \boldsymbol{A_B^{-1} A_N x_N} \\
\boldsymbol{x_B} &= [\,\boldsymbol{A_B^{-1} b} \mid \boldsymbol{A_B^{-1}}\,]\,\vec{\varepsilon} - \boldsymbol{A_B^{-1} A_N x_N}
\end{aligned}
$$

Lexicographic degeneracy resolution

perturb $\boldsymbol{b}$ to $\boldsymbol{b} + \vec{\varepsilon}_*$ with small $\varepsilon > 0$, $\quad \vec{\varepsilon} = (1, \varepsilon, \varepsilon^2, \ldots \varepsilon^m)^\top$

$$
\begin{aligned}
Ax &= [\, b \mid I \,]\, \vec{\varepsilon} \\
A_B x_B + A_N x_N &= [\, b \mid I \,]\, \vec{\varepsilon} \\
A_B x_B &= [\, b \mid I \,]\, \vec{\varepsilon} - A_N x_N \\
x_B &= A_B^{-1} [\, b \mid I \,]\, \vec{\varepsilon} - A_B^{-1} A_N x_N \\
x_B &= [\, A_B^{-1} b \mid A_B^{-1} \,]\, \vec{\varepsilon} - A_B^{-1} A_N x_N
\end{aligned}
$$

nondegeneracy $\Leftrightarrow$ $x_B > 0$ for small $\varepsilon > 0$ $\Leftrightarrow$ $[\, A_B^{-1} b \mid A_B^{-1} \,]$ **lexico-positive** (first nonzero element in each row is > 0).

Lexicographic degeneracy resolution

perturb $\boldsymbol{b}$ to $\boldsymbol{b} + \vec{\varepsilon}_*$ with small $\varepsilon > 0$, $\quad \vec{\varepsilon} = (1, \varepsilon, \varepsilon^2, \ldots \varepsilon^m)^\top$

$$
\begin{aligned}
\boldsymbol{A}\boldsymbol{x} &= [\, \boldsymbol{b} \mid \boldsymbol{I} \,]\, \vec{\varepsilon} \\
\boldsymbol{A}_B \boldsymbol{x}_B + \boldsymbol{A}_N \boldsymbol{x}_N &= [\, \boldsymbol{b} \mid \boldsymbol{I} \,]\, \vec{\varepsilon} \\
\boldsymbol{A}_B \boldsymbol{x}_B &= [\, \boldsymbol{b} \mid \boldsymbol{I} \,]\, \vec{\varepsilon} - \boldsymbol{A}_N \boldsymbol{x}_N \\
\boldsymbol{x}_B &= \boldsymbol{A}_B^{-1} [\, \boldsymbol{b} \mid \boldsymbol{I} \,]\, \vec{\varepsilon} - \boldsymbol{A}_B^{-1} \boldsymbol{A}_N \boldsymbol{x}_N \\
\boldsymbol{x}_B &= [\, \boldsymbol{A}_B^{-1} \boldsymbol{b} \mid \boldsymbol{A}_B^{-1} \,]\, \vec{\varepsilon} - \boldsymbol{A}_B^{-1} \boldsymbol{A}_N \boldsymbol{x}_N
\end{aligned}
$$

nondegeneracy $\Leftrightarrow$ $\boldsymbol{x}_B > \boldsymbol{0}$ for small $\varepsilon > 0$ $\Leftrightarrow$ $[\, \boldsymbol{A}_B^{-1} \boldsymbol{b} \mid \boldsymbol{A}_B^{-1} \,]$ **lexico-positive** (first nonzero element in each row is > 0).

Example:

$$
\begin{bmatrix} 1 & -9 & 4 & 0 \\ 0 & 3 & -100 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix} \vec{\varepsilon} = \begin{bmatrix} 1 - 9\varepsilon + 4\varepsilon^2 \\ 3\varepsilon - 100\varepsilon^2 + 2\varepsilon^3 \\ 5\varepsilon^3 \end{bmatrix}
$$

Integer pivoting for $Cz + s = 1$

(basic columns in red)	z_1	z_2	s_1	s_2	RHS
	4	**3**	**1**	**0**	**1**
z_1 enters, s_2 leaves	$\boxed{\mathbf{7}}$	**2**	**0**	**1**	**1**

Integer pivoting for $Cz + s = 1$

(basic columns in red)	z_1	z_2	s_1	s_2	RHS	
	4	**3**	**1**	**0**	**1**	$\times$ **7**
z_1 enters, s_2 leaves	$\boxed{7}$	**2**	**0**	**1**	**1**	
	28	**21**	**7**	**0**	**7**	
	7	**2**	**0**	**1**	**1**	

Integer pivoting for $Cz + s = 1$

(basic columns in red)	z_1	z_2	s_1	s_2	RHS	
	4	**3**	**1**	**0**	**1**	$\times$ **7**
z_1 enters, s_2 leaves	$\boxed{\mathbf{7}}$	**2**	**0**	**1**	**1**	
	28	**21**	**7**	**0**	**7**	
	7	**2**	**0**	**1**	**1**	
z_2 enters, s_1 leaves	**0**	$\boxed{\mathbf{13}}$	**7**	**−4**	**3**	
	7	**2**	**0**	**1**	**1**	

Integer pivoting for $Cz + s = 1$

(basic columns in red)	z_1	z_2	s_1	s_2	RHS	
	4	3	1	0	1	$\times$ 7
z_1 enters, s_2 leaves	$\boxed{7}$	2	0	1	1	
	28	21	7	0	7	
	7	2	0	1	1	
z_2 enters, s_1 leaves	0	$\boxed{13}$	7	-4	3	
	7	2	0	1	1	$\times$ 13
	0	13	7	-4	3	
	91	26	0	13	13	

Integer pivoting for $Cz + s = 1$

(basic columns in red)	z_1	z_2	s_1	s_2	RHS	
	4	3	1	0	1	$\times$ 7
z_1 enters, s_2 leaves	$\boxed{7}$	2	0	1	1	
	28	21	7	0	7	
	7	2	0	1	1	
z_2 enters, s_1 leaves	0	$\boxed{13}$	7	-4	3	
	7	2	0	1	1	$\times$ 13
	0	13	7	-4	3	
	91	26	0	13	13	
	0	13	7	-4	3	
(numbers grow)	91	0	-14	21	7	

Integer pivoting for $Cz + s = 1$

(basic columns in red)	z_1	z_2	s_1	s_2	RHS	
	4	**3**	**1**	**0**	**1**	$\times$ **7**
z_1 enters, s_2 leaves	$\boxed{7}$	**2**	**0**	**1**	**1**	
	28	**21**	**7**	**0**	**7**	
	7	**2**	**0**	**1**	**1**	
z_2 enters, s_1 leaves	**0**	$\boxed{13}$	**7**	**−4**	**3**	
	7	**2**	**0**	**1**	**1**	$\times$ **13**
	0	**13**	**7**	**−4**	**3**	
	91	**26**	**0**	**13**	**13**	
	0	**13**	**7**	**−4**	**3**	
(numbers grow)	**91**	**0**	**−14**	**21**	**7**	/ **7**
	0	**13**	**7**	**−4**	**3**	
	13	**0**	**−2**	**3**	**1**	