Extensive Games and the Sequence Form

Bernhard von Stengel

Department of Mathematics
London School of Economics

Overview

- Linear programming and zero-sum games

- Extensive games
  - perfect recall and the sequence form
  - computing equilibria with the sequence form

Linear programming duality

Primal and dual linear programs

Primal LP:

$$\begin{aligned} \textbf{maximize} \quad & \boldsymbol{c}^\top \boldsymbol{y} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{y} \le \boldsymbol{b}\,, \\ & \boldsymbol{y} \ge \boldsymbol{0}\,. \end{aligned}$$

Dual LP:

$$\begin{aligned} \textbf{minimize} \quad & \boldsymbol{x}^\top \boldsymbol{b} \\ \text{subject to} \quad & \boldsymbol{x} \ge \boldsymbol{0}\,, \\ & \boldsymbol{x}^\top \boldsymbol{A} \ge \boldsymbol{c}^\top. \end{aligned}$$

Primal and dual linear programs

Primal LP:

$$\begin{aligned} \textbf{maximize} \quad & \boldsymbol{c}^\top \boldsymbol{y} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{y} \le \boldsymbol{b}, \\ & \boldsymbol{y} \ge \boldsymbol{0}. \end{aligned}$$

Dual LP:

$$\begin{aligned} \textbf{minimize} \quad & \boldsymbol{x}^\top \boldsymbol{b} \\ \text{subject to} \quad & \boldsymbol{x} \ge \boldsymbol{0}, \\ & \boldsymbol{x}^\top \boldsymbol{A} \ge \boldsymbol{c}^\top. \end{aligned}$$

Weak LP duality: For any **feasible** primal $\boldsymbol{y}$, dual $\boldsymbol{x}$:

$$\boldsymbol{c}^\top \boldsymbol{y} \le \boldsymbol{x}^\top \boldsymbol{b}$$

Primal and dual linear programs

Primal LP:

$$\begin{aligned} \textbf{maximize} \quad & \boldsymbol{c}^\top \boldsymbol{y} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{y} \le \boldsymbol{b}\,, \\ & \boldsymbol{y} \ge \boldsymbol{0}\,. \end{aligned}$$

Dual LP:

$$\begin{aligned} \textbf{minimize} \quad & \boldsymbol{x}^\top \boldsymbol{b} \\ \text{subject to} \quad & \boldsymbol{x} \ge \boldsymbol{0}\,, \\ & \boldsymbol{x}^\top \boldsymbol{A} \ge \boldsymbol{c}^\top. \end{aligned}$$

Weak LP duality: For any **feasible** primal $\boldsymbol{y}$, dual $\boldsymbol{x}$:

$$(\boldsymbol{c}^\top)\boldsymbol{y} \le (\boldsymbol{x}^\top \boldsymbol{A})\boldsymbol{y} = \boldsymbol{x}^\top(\boldsymbol{A}\boldsymbol{y}) \le \boldsymbol{x}^\top(\boldsymbol{b})$$

Primal and dual linear programs

Primal LP:

$$\begin{aligned} \textbf{maximize} \quad & \boldsymbol{c}^\top \boldsymbol{y} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{y} \leq \boldsymbol{b}, \\ & \boldsymbol{y} \geq \boldsymbol{0}. \end{aligned}$$

Dual LP:

$$\begin{aligned} \textbf{minimize} \quad & \boldsymbol{x}^\top \boldsymbol{b} \\ \text{subject to} \quad & \boldsymbol{x} \geq \boldsymbol{0}, \\ & \boldsymbol{x}^\top \boldsymbol{A} \geq \boldsymbol{c}^\top. \end{aligned}$$

Weak LP duality: For any **feasible** primal $\boldsymbol{y}$, dual $\boldsymbol{x}$:

$$\boldsymbol{c}^\top \boldsymbol{y} \leq \boldsymbol{x}^\top \boldsymbol{b}$$

So $\boxed{\boldsymbol{c}^\top \boldsymbol{y} = \boldsymbol{x}^\top \boldsymbol{b}} \Rightarrow$ $\boldsymbol{y}$ optimal for primal LP, $\boldsymbol{x}$ optimal for dual LP.

Primal and dual linear programs

Primal LP:

$$\begin{aligned} \textbf{maximize} \quad & \boldsymbol{c}^\top \boldsymbol{y} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{y} \le \boldsymbol{b}\,, \\ & \boldsymbol{y} \ge \boldsymbol{0}\,. \end{aligned}$$

Dual LP:

$$\begin{aligned} \textbf{minimize} \quad & \boldsymbol{x}^\top \boldsymbol{b} \\ \text{subject to} \quad & \boldsymbol{x} \ge \boldsymbol{0}\,, \\ & \boldsymbol{x}^\top \boldsymbol{A} \ge \boldsymbol{c}^\top. \end{aligned}$$

Weak LP duality: For any **feasible** primal $\boldsymbol{y}$, dual $\boldsymbol{x}$:

$$\boldsymbol{c}^\top \boldsymbol{y} \le \boldsymbol{x}^\top \boldsymbol{b}$$

So $\boxed{\boldsymbol{c}^\top \boldsymbol{y} = \boldsymbol{x}^\top \boldsymbol{b}} \Rightarrow \boldsymbol{y}$ optimal for primal LP, $\boldsymbol{x}$ optimal for dual LP.

Strong LP duality: If both primal and dual LP are feasible, then they have (optimal) solutions $\boldsymbol{y}$ and $\boldsymbol{x}$ with $\boldsymbol{c}^\top \boldsymbol{y} = \boldsymbol{x}^\top \boldsymbol{b}$.

Primal and dual linear programs

Primal LP:

$$\begin{aligned} \textbf{maximize} \quad & \boldsymbol{c}^\top \boldsymbol{y} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{y} \le \boldsymbol{b}\,, \\ & \boldsymbol{y} \ge \boldsymbol{0}\,. \end{aligned}$$

Dual LP:

$$\begin{aligned} \textbf{minimize} \quad & \boldsymbol{x}^\top \boldsymbol{b} \\ \text{subject to} \quad & \boldsymbol{x} \ge \boldsymbol{0}\,, \\ & \boldsymbol{x}^\top \boldsymbol{A} \ge \boldsymbol{c}^\top. \end{aligned}$$

Weak LP duality: For any **feasible** primal $\boldsymbol{y}$, dual $\boldsymbol{x}$:

$$\boldsymbol{c}^\top \boldsymbol{y} \le \boldsymbol{x}^\top \boldsymbol{b}$$

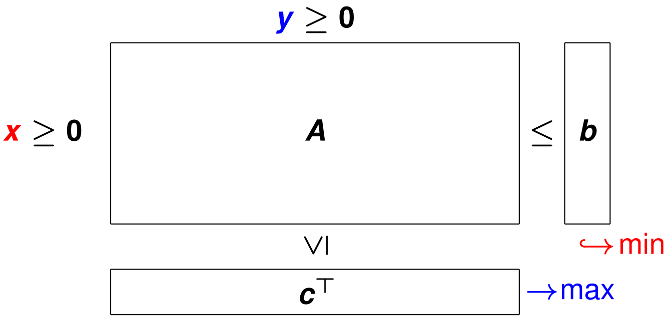
So $\boxed{\boldsymbol{c}^\top \boldsymbol{y} = \boldsymbol{x}^\top \boldsymbol{b}}$ $\Rightarrow$ $\boldsymbol{y}$ optimal for primal LP, $\boldsymbol{x}$ optimal for dual LP.

Strong LP duality: If both primal and dual LP are feasible, then they have (optimal) solutions $\boldsymbol{y}$ and $\boldsymbol{x}$ with $\boldsymbol{c}^\top \boldsymbol{y} \ge \boldsymbol{x}^\top \boldsymbol{b}$.

Tucker diagram

Primal LP: **maximize** $\boldsymbol{c}^\top \boldsymbol{y}$ subject to $A\boldsymbol{y} \le \boldsymbol{b}, \quad \boldsymbol{y} \ge \mathbf{0}.$

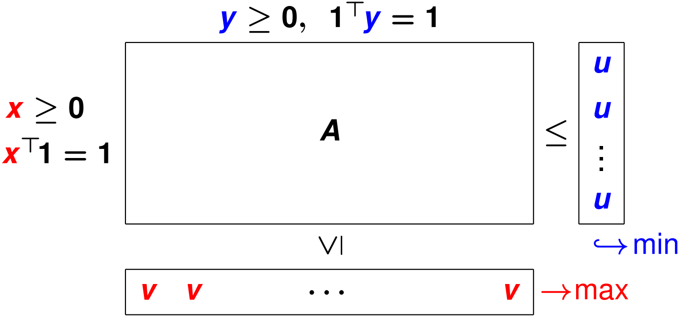
Dual LP: **minimize** $\boldsymbol{x}^\top \boldsymbol{b}$ subject to $\boldsymbol{x}^\top A \ge \boldsymbol{c}^\top, \quad \boldsymbol{x} \ge \mathbf{0}.$



Zero-sum game $(A, -A)$ written as general LP

Minimizer: **minimize** u subject to $Ay \leq \mathbf{1}u$, $y \in Y$.

Maximizer: **maximize** v subject to $x^\top A \geq v\mathbf{1}^\top$, $x \in X$.



Zero-sum game $(A, -A)$ written as general LP

Minimizer: **minimize** u subject to $Ay \le \mathbf{1}u, \quad y \in Y$.

Maximizer: **maximize** v subject to $x^\top A \ge v\mathbf{1}^\top, \quad x \in X$.

$$
\begin{array}{c|ccc|c|c|c}
 & & y \ge 0 & & u & & \\
\hline
 & & & & -1 & & 0 \\
x \ge 0 & & A & & \vdots & \le & \vdots \\
 & & & & -1 & & 0 \\
\hline
v & -1 & \cdots & -1 & 0 & = & -1 \\
\hline
 & & \geq & & \| & & \hookrightarrow \text{min} \\
 & 0 & \cdots & 0 & -1 & \to \text{max} &
\end{array}
$$

Simpler LP with positive payoffs

$$\boldsymbol{x}^\top(\boldsymbol{A} + \mathbf{1}\alpha\mathbf{1}^\top)\boldsymbol{y} \;=\; \boldsymbol{x}^\top\boldsymbol{A}\boldsymbol{y} + \boldsymbol{x}^\top\mathbf{1}\alpha\mathbf{1}^\top\boldsymbol{y} \;=\; \boldsymbol{x}^\top\boldsymbol{A}\boldsymbol{y} + \alpha$$

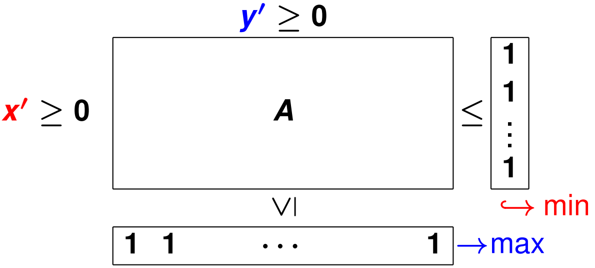
Simpler LP with positive payoffs

$$x^\top(A + \mathbf{1}\alpha\mathbf{1}^\top)y \;=\; x^\top A y + x^\top \mathbf{1}\alpha\mathbf{1}^\top y \;=\; x^\top A y + \alpha$$

$\Rightarrow$ w.l.o.g. $A > \mathbf{0}$, min-max cost $u > 0$, max-min payoff $v > 0$, replace y by $y' = y\frac{1}{u}$, and x by $x' = x\frac{1}{v}$,

Minimizer: **maximize** $\mathbf{1}^\top y'$ $(= \frac{1}{u})$ subject to $Ay' \leq \mathbf{1}$,

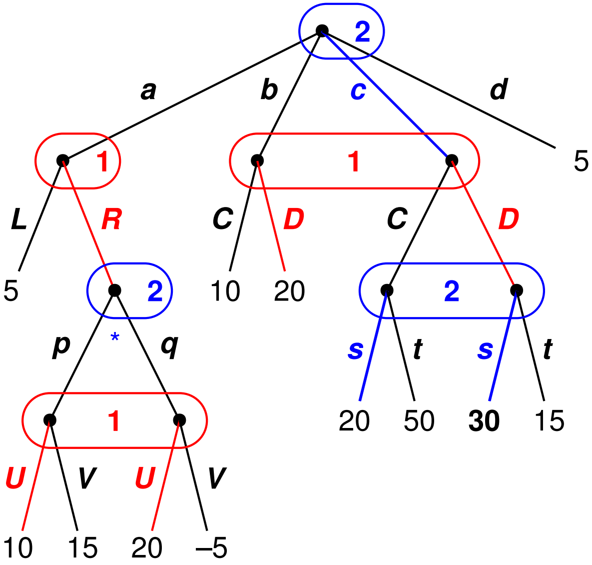
Maximizer: **minimize** $\mathbf{1}^\top x'$ $(= \frac{1}{v})$ subject to $x'^\top A \geq \mathbf{1}^\top$



Extensive Games,

Sequence Form

Game tree (game in extensive form)



Strategic (or normal) form

Strategy of a player:
specifies a move for **every** information set of that player.

	a, p, s	*a, p, t*	*a, q, s*	*a, q, t*	*b, p, s*	*b, p, t*	*b, q, s*	*b, q, t*	*c, p, s*	*c, p, t*	*c, q, s*	*c, q, t*	*d, p, s*	*d, p, t*	*d, q, s*	*d, q, t*
L*, *U*, *C	5	5	5	5	10	10	10	10	20	50	20	50	5	5	5	5
L*, *V*, *C	5	5	5	5	10	10	10	10	20	50	20	50	5	5	5	5
L*, *U*, *D	5	5	5	5	20	20	20	20	30	15	30	15	5	5	5	5
L*, *V*, *D	5	5	5	5	20	20	20	20	30	15	30	15	5	5	5	5
R*, *U*, *C	10	10	20	20	10	10	10	10	20	50	20	50	5	5	5	5
R*, *U*, *D	10	10	20	20	20	20	20	20	30	15	30	15	5	5	5	5
R*, *V*, *C	20	20	–5	–5	10	10	10	10	20	50	20	50	5	5	5	5
R*, *V*, *D	10	10	20	20	20	20	20	20	30	15	30	15	5	5	5	5

Reduced strategic form

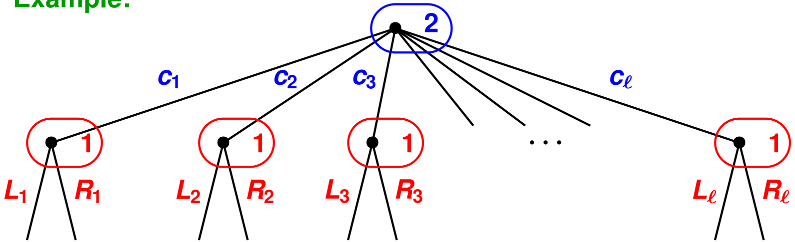
Reduced strategy of a player:
specifies a move for every information set of that player,
except for those information sets unreachable due to an **own** earlier move (where we write $*$ instead of a move).

	$a, p, *$	$a, q, *$	$b, *, *$	$c, *, s$	$c, *, t$	$d, *, *$
$L, *, C$	5	5	10	20	50	5
$L, *, D$	5	5	20	30	15	5
R, U, C	10	20	10	20	50	5
R, U, D	10	20	20	30	15	5
R, V, C	15	–5	10	20	50	5
R, V, D	15	–5	20	30	15	5

Exponential blowup of strategic form

number of pure strategies typically
exponential in number of information sets.

Example:



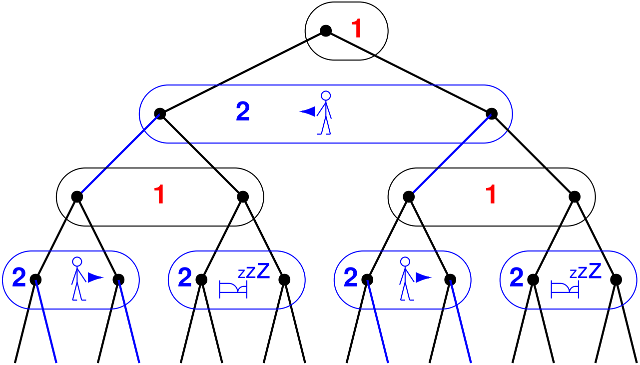
number of information sets $=$ ℓ
number of pure strategies $=$ 2^ℓ

Example [Kuhn]: simplified poker game,

number of information sets $=$ **13**
number of pure strategies $=$ **8192**

Exponential blowup of reduced strategic form

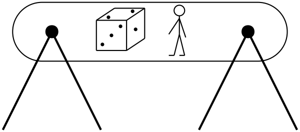
Example: Game with (1) **bounded** number of moves per node, (2) no **subgames** (otherwise simplify by solving subgames first).



This tree with n nodes: $\approx 2^{\sqrt{n}/2}$ strategies per player, reduced strategic form still (sub-)**exponential** in **tree** size.

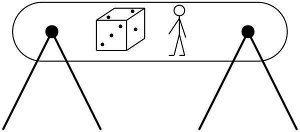
Use behavior strategies

Behavior strategy = **local** randomization



Use behavior strategies

Behavior strategy = **local** randomization

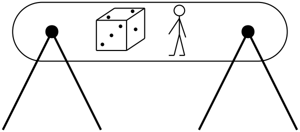


Mixed strategy too redundant, use behavior strategy instead:

- only one variable per **move**:
  player 1 chooses L with probability X_L
  player 1 chooses R with probability X_R . . .
  player 2 chooses a with probability Y_a . . .
- expected payoff =
  $5\, Y_a X_L + 10\, Y_a X_R Y_p X_U + 15\, Y_a X_R Y_p X_V + \cdots$

Use behavior strategies

Behavior strategy = **local** randomization



Mixed strategy too redundant, use behavior strategy instead:

- only one variable per **move**:
  player 1 chooses L with probability X_L
  player 1 chooses R with probability X_R . . .
  player 2 chooses a with probability Y_a . . .

- expected payoff =
  $5\ Y_a X_L + 10\ Y_a X_R Y_p X_U + 15\ Y_a X_R Y_p X_V + \cdots$

- problem: **nonlinear**!

Variable transformation

For each **sequence** σ of moves of player 1
introduce new variable x_σ

- new variables replace products:
  if $\sigma = PQRS$ then $x_\sigma = X_P X_Q X_R X_S$

- Example:

$$
\begin{aligned}
x_L &= X_L \\
x_{RU} &= X_R X_U \\
&\dots \\
y_a &= Y_a \\
y_{ap} &= Y_a Y_p \\
&\dots
\end{aligned}
$$

- expected payoff $= \quad 5\, x_L\, y_a + 10\, x_{RU}\, y_{ap} + 15\, x_{RV}\, y_{ap} + \cdots$
  is **linear** in variables of one player.

New paradigm: Sequences instead of pure strategies

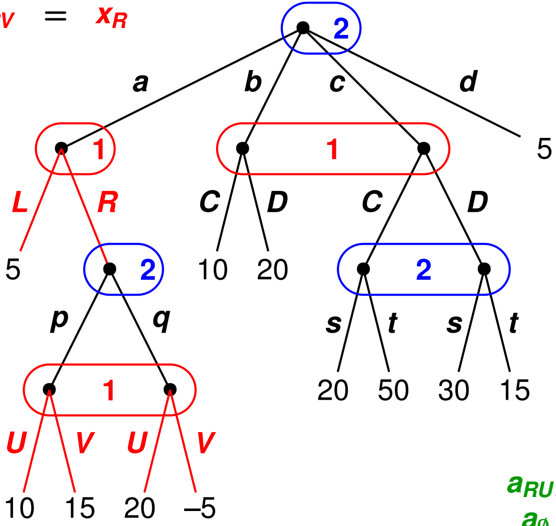
Before:

pure strategy	i
probability	x_i
mixed strategy	x
characterized by	$\mathbf{1}^\top x = 1$
expected payoff	$x^\top A y$

After:

sequence	σ
realization probability	x_σ
realization plan	x
characterized by	$Ex = e$
expected payoff	$x^\top A y$

$$x_\emptyset = 1$$
$$x_L + x_R = x_\emptyset$$
$$x_{RU} + x_{RV} = x_R$$



$$a_{RU,ap} = 10$$
$$a_{\emptyset,d} = 5$$
$$a_{RU,b} = 0$$

Realization plans

Realization plan $\boldsymbol{x} = (x_\emptyset, x_L, x_R, x_C, x_D, x_{RU}, x_{RV})$

(= vector of realization probabilities)

characterized by $\boldsymbol{x} \geq \mathbf{0}$ and **linear** equalities

$$
\begin{array}{rcl}
x_\emptyset &=& 1 \\
x_\emptyset &=& x_L + x_R \\
x_\emptyset &=& x_C + x_D \\
x_R &=& x_{RU} + x_{RV}
\end{array}
$$

written as $\boldsymbol{E}\boldsymbol{x} = \boldsymbol{e}$ with

$$
\boldsymbol{E} = \begin{bmatrix} 1 & & & & & & \\ -1 & 1 & 1 & & & & \\ -1 & & & 1 & 1 & & \\ & & -1 & & & 1 & 1 \end{bmatrix}, \qquad \boldsymbol{e} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

The sequence form

Payoff matrix A

	$\emptyset$	a	b	c	d	ap	aq	bs	bt
$\emptyset$					5				
L		5							
R									
RU						10	20		
RV						15	–5		
C			10					20	50
D			20					30	15

expected payoff $x^\top A y$,

rows played with x subject to $x \geq \mathbf{0}$, $Ex = e$,

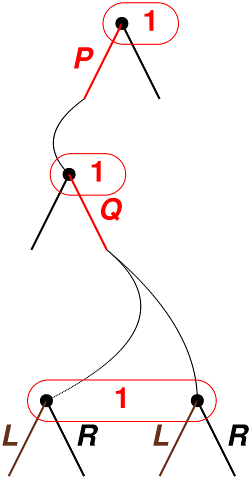
columns played with y subject to $y \geq \mathbf{0}$, $Fy = f$.

How to play

Given: realization plan $\boldsymbol{x}$ with $\boldsymbol{Ex} = \boldsymbol{e}$.

Move $\boldsymbol{L}$ is last move of **unique** sequence,
say $\boldsymbol{PQL}$, where $\quad x_{PQL} + x_{PQR} = x_{PQ}$.

$$\Rightarrow \quad \text{behavior-probability}(L) = \frac{x_{PQL}}{x_{PQ}}.$$



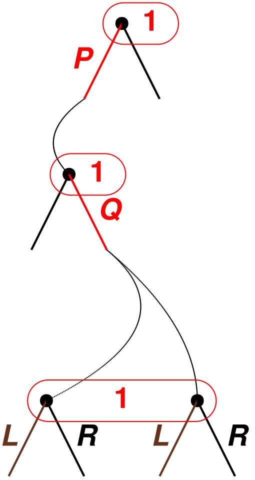
How to play

Given: realization plan x with $Ex = e$.

Move L is last move of **unique** sequence, say PQL, where $x_{PQL} + x_{PQR} = x_{PQ}$.

$$\Rightarrow \quad \text{behavior-probability}(L) = \frac{x_{PQL}}{x_{PQ}} .$$

Required assumption of **perfect recall** [Kuhn 1953, Selten 1975]:
Each node in an information set is preceded by same sequence, here PQ, of the player's **own** earlier moves.



Solving the Sequence Form:

Constrained Games

Constrained games

Polyhedrally constrained game:

Player 1's strategy set

$$X = \{x \in \mathbb{R}^m \mid Ex = e,\ x \geq 0\}$$

e.g. $E = [1\ 1 \cdots 1]$, $e = 1$: strategy **simplex**

Constrained games

Polyhedrally constrained game:

Player 1's strategy set

$$X = \{x \in \mathbb{R}^m \mid Ex = e,\ x \geq 0\}$$

e.g. $E = [1\ 1 \cdots 1]$, $e = 1$: strategy **simplex**

Player 2's strategy set

$$Y = \{y \in \mathbb{R}^n \mid Fy = f,\ y \geq 0\}$$

Constrained games

Polyhedrally constrained game:

Player 1's strategy set

$$X = \{x \in \mathbb{R}^m \mid Ex = e,\ x \geq \mathbf{0}\}$$

e.g. $E = [1\ 1 \cdots 1]$, $e = 1$: strategy **simplex**

Player 2's strategy set

$$Y = \{y \in \mathbb{R}^n \mid Fy = f,\ y \geq \mathbf{0}\}$$

payoff matrices A, B,

expected payoffs $x^\top Ay$, $x^\top By$ for $(x, y) \in X \times Y$.

Best responses in a constrained game

$\boldsymbol{x}$ in $\boldsymbol{X}$ best response against $\boldsymbol{y}$ in $\boldsymbol{Y}$: solves primal LP

$$\begin{array}{ll} \text{maximize} & \boldsymbol{x}^\top(\boldsymbol{A}\boldsymbol{y}) \\ \text{subject to} & \boldsymbol{E}\boldsymbol{x} = \boldsymbol{e} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

Best responses in a constrained game

$\boldsymbol{x}$ in $\boldsymbol{X}$ best response against $\boldsymbol{y}$ in $\boldsymbol{Y}$: solves primal LP

$$\begin{array}{ll} \text{maximize} & \boldsymbol{x}^\top(\boldsymbol{A}\boldsymbol{y}) \\ \text{subject to} & \boldsymbol{E}\boldsymbol{x} = \boldsymbol{e} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

Dual LP (with same value, = best response payoff to player 1):

$$\begin{array}{ll} \text{minimize} & \boldsymbol{e}^\top\boldsymbol{u} \\ \text{subject to} & \boldsymbol{E}^\top\boldsymbol{u} \geq \boldsymbol{A}\boldsymbol{y} \end{array}$$

Best responses in a constrained game

$\boldsymbol{x}$ in $\boldsymbol{X}$ best response against $\boldsymbol{y}$ in $\boldsymbol{Y}$: solves primal LP

$$\begin{aligned} \text{maximize} \quad & \boldsymbol{x}^\top(\boldsymbol{A}\boldsymbol{y}) \\ \text{subject to} \quad & \boldsymbol{E}\boldsymbol{x} = \boldsymbol{e} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

Dual LP (with same value, = best response payoff to player 1):

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{e}^\top\boldsymbol{u} \\ \text{subject to} \quad & \boldsymbol{E}^\top\boldsymbol{u} \geq \boldsymbol{A}\boldsymbol{y} \end{aligned}$$

$\boldsymbol{x}, \boldsymbol{u}$ optimal $\Leftrightarrow$ complementary slackness:

$$\boldsymbol{x}^\top(\boldsymbol{E}^\top\boldsymbol{u} - \boldsymbol{A}\boldsymbol{y}) = \boldsymbol{0}$$

Best responses in a constrained game

$\boldsymbol{x}$ in $\boldsymbol{X}$ best response against $\boldsymbol{y}$ in $\boldsymbol{Y}$: solves primal LP

$$\begin{array}{ll} \text{maximize} & \boldsymbol{x}^\top(\boldsymbol{A}\boldsymbol{y}) \\ \text{subject to} & \boldsymbol{E}\boldsymbol{x} = \boldsymbol{e} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

Dual LP (with same value, = best response payoff to player 1):

$$\begin{array}{ll} \text{minimize} & \boldsymbol{e}^\top \boldsymbol{u} \\ \text{subject to} & \boldsymbol{E}^\top \boldsymbol{u} \geq \boldsymbol{A}\boldsymbol{y} \end{array}$$

$\boldsymbol{x}, \boldsymbol{u}$ optimal $\Leftrightarrow$ complementary slackness:

$$\boldsymbol{x}^\top(\boldsymbol{E}^\top \boldsymbol{u} - \boldsymbol{A}\boldsymbol{y}) = \boldsymbol{0}$$

= best-response condition (only pure best responses can have positive probability)

Constrained zero-sum games

[Charnes 1953]   Let $\boldsymbol{B} = -\boldsymbol{A}$.

For $\boldsymbol{y} \in \boldsymbol{Y}$, best-response payoff to player 1 = value of LP

$$\begin{array}{ll} \text{maximize} & \boldsymbol{x}^\top(\boldsymbol{A}\boldsymbol{y}) \\ \text{subject to} & \boldsymbol{E}\boldsymbol{x} = \boldsymbol{e} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

equals value of dual LP

$$\begin{array}{ll} \text{minimize} & \boldsymbol{e}^\top \boldsymbol{u} \\ \text{subject to} & \boldsymbol{E}^\top \boldsymbol{u} \geq \boldsymbol{A}\boldsymbol{y} \end{array}$$

Constrained zero-sum games

[Charnes 1953] Let $\boldsymbol{B} = -\boldsymbol{A}$.

For $\boldsymbol{y} \in \boldsymbol{Y}$, best-response payoff to player 1 = value of LP

$$\begin{array}{ll} \text{maximize} & \boldsymbol{x}^\top(\boldsymbol{A}\boldsymbol{y}) \\ \text{subject to} & \boldsymbol{E}\boldsymbol{x} = \boldsymbol{e} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

equals value of dual LP

$$\begin{array}{ll} \text{minimize} & \boldsymbol{e}^\top \boldsymbol{u} \\ \text{subject to} & \boldsymbol{E}^\top \boldsymbol{u} \geq \boldsymbol{A}\boldsymbol{y} \end{array}$$

which is also **minimized** by player 2 for $\boldsymbol{y} \in \boldsymbol{Y}$, i.e. as solution to

$$\begin{array}{ll} \text{minimize} & \boldsymbol{e}^\top \boldsymbol{u} \\ \text{subject to} & \boldsymbol{E}^\top \boldsymbol{u} - \boldsymbol{A}\boldsymbol{y} \geq \boldsymbol{0} \\ & \boldsymbol{F}\boldsymbol{y} = \boldsymbol{f} \\ & \boldsymbol{y} \geq \boldsymbol{0}. \end{array}$$

Example

1) Best-response LP

$$\max_{x} \quad x^\top (Ay)$$

$$\text{subject to} \quad Ex = e$$

$$x \geq 0$$

$$\geq 0 \quad \begin{array}{c} x_\emptyset \\ x_L \\ x_R \\ x_C \\ x_D \end{array} \begin{array}{|ccc|} \hline 1 & -1 & -1 \\ & 1 & \\ & 1 & \\ & & 1 \\ & & 1 \\ \hline \end{array} \; \Bigg| \; \begin{array}{|c|} \hline 0 \\ 2 \\ 2 \\ 1 \\ 0 \\ \hline \end{array} \downarrow \max$$

$$\begin{array}{|ccc|} \hline 1 & 0 & 0 \\ \hline \end{array} \;(=)$$

2) **dual LP**

$$\min_{u} \quad e^\top u$$

$$\text{subject to} \quad E^\top u \geq Ay$$

$$\begin{array}{ccc} u_0 & u_1 & u_2 \end{array}$$

$$\begin{array}{|ccc|} \hline 1 & -1 & -1 \\ & 1 & \\ & 1 & \\ & & 1 \\ & & 1 \\ \hline \end{array} \geq \begin{array}{|c|} \hline 0 \\ 2 \\ 2 \\ 1 \\ 0 \\ \hline \end{array}$$

$$\begin{array}{|ccc|} \hline 1 & 0 & 0 \\ \hline \end{array} \to \min$$

2) **dual LP**

$$\min_{\mathbf{u}} \quad e^\top \mathbf{u}$$
$$\text{subject to} \quad E^\top \mathbf{u} \geq A\mathbf{y}$$

$$
\begin{array}{ccc}
u_0 & u_1 & u_2 \\
\hline
1 & -1 & -1 \\
 & 1 & \\
 & 1 & \\
 & & 1 \\
 & & 1
\end{array}
\geq
\begin{array}{c}
0 \\ 2 \\ 2 \\ 1 \\ 0
\end{array}
$$

$$
\begin{array}{ccc}
1 & 0 & 0
\end{array}
\rightarrow \min
$$

3) Treat $\mathbf{y}$ as a variable:

$$\min_{\mathbf{u},\,\mathbf{y}} \quad e^\top \mathbf{u}$$
$$\text{subject to} \quad E^\top \mathbf{u} \geq A\mathbf{y}$$
$$F\mathbf{y} = f$$
$$\mathbf{y} \geq 0$$

$$
\begin{array}{ccc}
u_0 & u_1 & u_2 \\
\hline
1 & -1 & -1 \\
 & 1 & \\
 & 1 & \\
 & & 1 \\
 & & 1
\end{array}
\geq
\begin{array}{cccc}
y_\emptyset & y_a & y_b & y_c \geq 0 \\
\hline
 & & & \\
 & 6 & 0 & \\
 & 2 & 4 & \\
 & & & 3 \\
 & & & 0 \\
\hline
1 & & & \\
-1 & 1 & 1 & 1
\end{array}
\quad
\begin{array}{c}
= \begin{array}{c} 1 \\ 0 \end{array}
\end{array}
$$

$$
\begin{array}{ccc}
1 & 0 & 0
\end{array}
\rightarrow \min
$$

Linear size instead of exponential size

Input: 2-player game tree with perfect recall.

Theorem [Romanovskii 1961], [von Stengel 1996]
A zero-sum game is solved via an LP of linear size:

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{e}^\top \boldsymbol{u} \\ \text{subject to} \quad & \boldsymbol{E}^\top \boldsymbol{u} - \boldsymbol{A}\boldsymbol{y} \geq \boldsymbol{0} \\ & \boldsymbol{F}\boldsymbol{y} = \boldsymbol{f} \\ & \boldsymbol{y} \geq \boldsymbol{0}. \end{aligned}$$

Linear size instead of exponential size

Input: 2-player game tree with perfect recall.

Theorem [Romanovskii 1961], [von Stengel 1996]
A zero-sum game is solved via an LP of linear size:

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{e}^\top \boldsymbol{u} \\ \text{subject to} \quad & \boldsymbol{E}^\top \boldsymbol{u} - \boldsymbol{A}\boldsymbol{y} \geq \boldsymbol{0} \\ & \boldsymbol{F}\boldsymbol{y} = \boldsymbol{f} \\ & \boldsymbol{y} \geq \boldsymbol{0}. \end{aligned}$$

Theorem [Koller / Megiddo / von Stengel 1996]
A non-zero-sum game is solved via an LCP of linear size. One equilibrium is found by Lemke's algorithm.

Linear size instead of exponential size

Input: 2-player game tree with perfect recall.

Theorem [Romanovskii 1961], [von Stengel 1996]
A zero-sum game is solved via an LP of linear size:

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{e}^\top \boldsymbol{u} \\ \text{subject to} \quad & \boldsymbol{E}^\top \boldsymbol{u} - \boldsymbol{A}\boldsymbol{y} \geq \boldsymbol{0} \\ & \boldsymbol{F}\boldsymbol{y} = \boldsymbol{f} \\ & \boldsymbol{y} \geq \boldsymbol{0}. \end{aligned}$$

Theorem [Koller / Megiddo / von Stengel 1996]
A non-zero-sum game is solved via an LCP of linear size. One equilibrium is found by Lemke's algorithm.

[von Stengel / Elzen / Talman, *Econometrica* 2002]
This algorithm mimicks the **Harsanyi-Selten tracing procedure** and finds a normal-form perfect equilibrium.
(Allows variation of **starting vector** or **prior**.)

LCP – Lemke's algorithm

Consider a **prior** $(\bar{x}, \bar{y})$, and a new variable z_0 in the system

$$
\begin{array}{rcl}
 & & \begin{array}{rcccl} E x & + & e\, z_0 & = & e \\ F y & + & f\, z_0 & = & f \end{array} \\
r & = & E^\top u - A y - A\bar{y}\, z_0 \geq 0 \\
s & = & F^\top v - B^\top x - B^\top \bar{x}\, z_0 \geq 0 \\
 & & x,\ y, \quad z_0 \geq 0
\end{array}
$$

Equilibrium condition $x^\top r = 0, \quad y^\top s = 0, \quad [\, z_0 = 0 \,].$

Initial solution $z_0 = 1$, $x = 0$, $y = 0$.

Complementary pivoting:

$x_\sigma \leftrightarrow r_\sigma$, $y_\tau \leftrightarrow s_\tau$, until z_0 leaves the basis.