PPAD

Bernhard von Stengel

Department of Mathematics
London School of Economics

Plan

- Lemke-Howson paths have a **direction**
  - prove via signs of determinants
  - index of an equilibrium
- Finding one Nash equilibrium of a bimatrix game is PPAD-complete
  - but seems fast in practice

PPAD = Polynomial Parity Argument with Direction

use signs of determinants

Equilibria of symmetric and bimatrix games

For $d \times d$ matrix C, consider polytope

$$P = \{ z \in \mathbb{R}^d \mid -z \leq \mathbf{0}, \quad Cz \leq \mathbf{1} \}$$

with $2d$ inequalities labeled ①, . . . , ⓓ, ①, . . . , ⓓ when **tight**.

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with $2d$ inequalities labeled ①, ..., ⓓ, ①, ..., ⓓ when **tight**.

Completely labeled $z \neq \mathbf{0}$ (scaled as probability vector)

$\Leftrightarrow$ Nash equilibrium (z, z) of game $(C, C^\top)$

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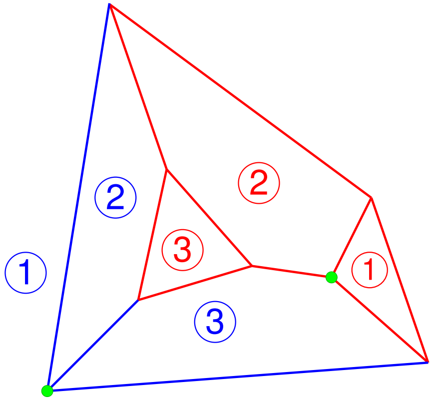
bimatrix game (A, B): $\quad C = \begin{pmatrix} 0 & A \\ B^\top & 0 \end{pmatrix}, \quad z = (x, y)$:

Completely labeled $(x, y) \neq (\mathbf{0}, \mathbf{0})$

$\Leftrightarrow$ Nash equilibrium (x, y) of game (A, B)

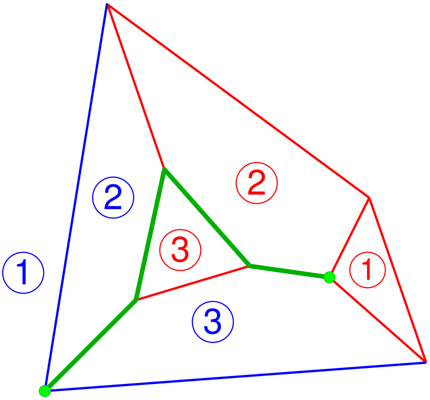
Path of "almost completely labeled" edges

$P = \{ z \in \mathbb{R}^3 \mid -z \leq 0, \;\; Cz \leq 1 \}$, two compl. labeled vertices



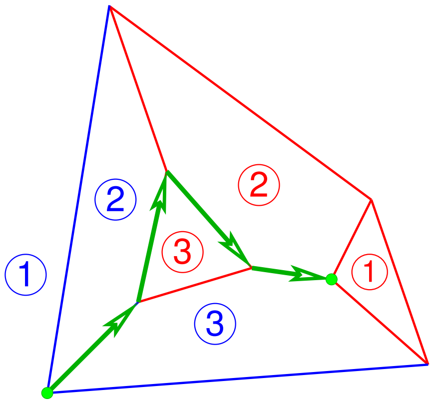
Path of "almost completely labeled" edges

path of edges with labels ②, ③ (label ① missing)



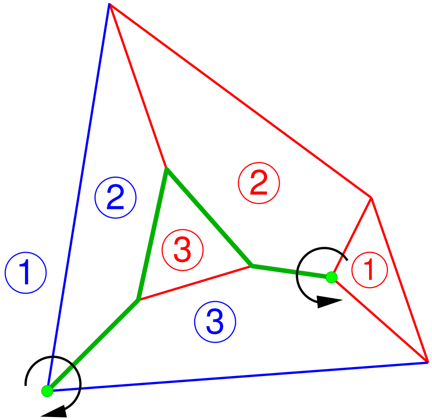
Path of "almost completely labeled" edges

orientation of edges: ② on left, ③ on right



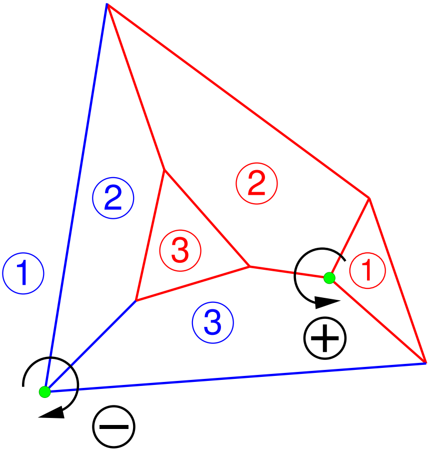
Path of “almost completely labeled” edges

opposite orientation (“sign”) of endpoints



Path of “almost completely labeled” edges

equilibrium **sign** ⊖ or ⊕ does not depend on path



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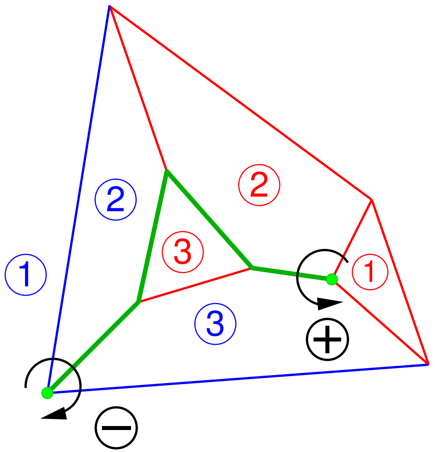

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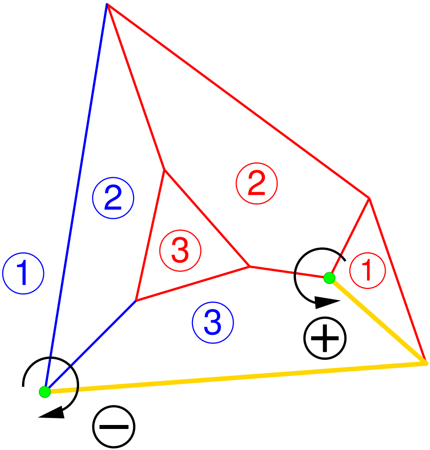

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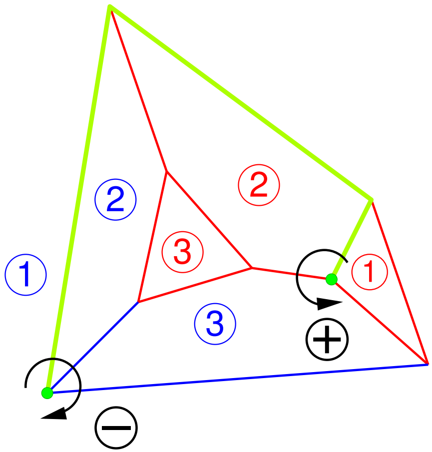

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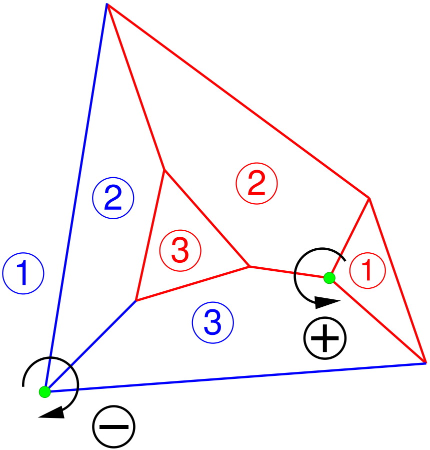

Completely labeled points come in pairs

Theorem [Parity Argument]

Let P be a labeled polytope.

Then P has an **even** number of completely labeled vertices.

Completely labeled points come in pairs of opposite sign

Theorem [Parity Argument with Direction]

Let $\boldsymbol{P}$ be a labeled polytope.

Then $\boldsymbol{P}$ has an **even** number of completely labeled vertices.
Half of these have **sign** $\ominus$, half have sign $\oplus$.

Completely labeled points come in pairs of opposite sign

Theorem [Parity Argument with Direction]

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Half of these have **sign** ⊖, half have sign ⊕.

---

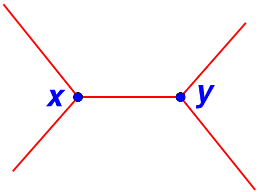
sign of completely labeled $\boldsymbol{x}$ is **sign of determinant** of the matrix of facet normal vectors in order of their labels: if (e.g.) facet $\boldsymbol{a_i}^\top \boldsymbol{x} = \beta_i$ has label $\boldsymbol{i} =$ ①, ②, ..., ⓓ, then

$$\boldsymbol{sign}(\boldsymbol{x}) = \boldsymbol{sign}\ |a_1\ a_2 \cdots a_d|$$

Pivoting changes signs

Lemma

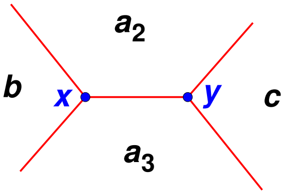
Let $x, y \in \mathbb{R}^d$ be adjacent vertices of a simple polytope $P \subset \mathbb{R}^d$



Pivoting changes signs

Lemma

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ be adjacent vertices of a simple polytope $\boldsymbol{P} \subset \mathbb{R}^d$ with facet normals $\boldsymbol{b}, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_d$ for $\boldsymbol{x}$ and $\boldsymbol{c}, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_d$ for $\boldsymbol{y}$.



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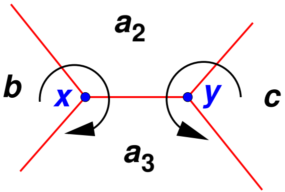
Then $|\boldsymbol{b}\ \boldsymbol{a}_2 \cdots \boldsymbol{a}_d|$ and $|\boldsymbol{c}\ \boldsymbol{a}_2 \cdots \boldsymbol{a}_d|$ have opposite sign.



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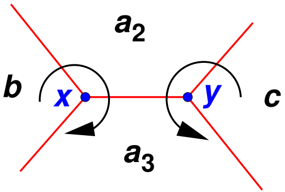
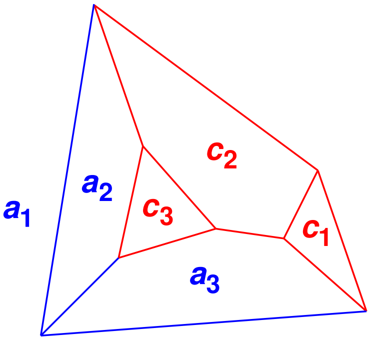


Proof is short, see B. von Stengel (2021), Finding Nash equilibria of two-player games. arXiv:2102.04580.

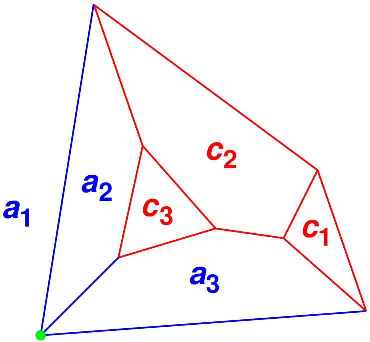
General Parity Argument with Direction

Facet normal vectors a_1 a_2 a_3 c_1 c_2 c_3, labels **1 2 3 1 2 3**



General Parity Argument with Direction

Start at a_1 a_2 a_3, sign ⊖

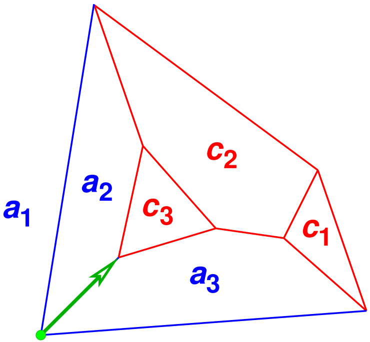


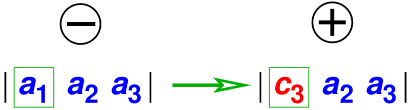
⊖

$|\boxed{a_1}\ a_2\ a_3|$

General Parity Argument with Direction

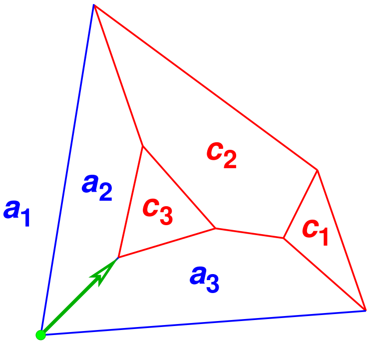
Start at a_1 a_2 a_3, sign ⊖, label **1** missing, $a_1 \rightarrow c_3$ gives sign ⊕





General Parity Argument with Direction

Switch columns c_3 and a_3 in determinant: back to sign ⊖



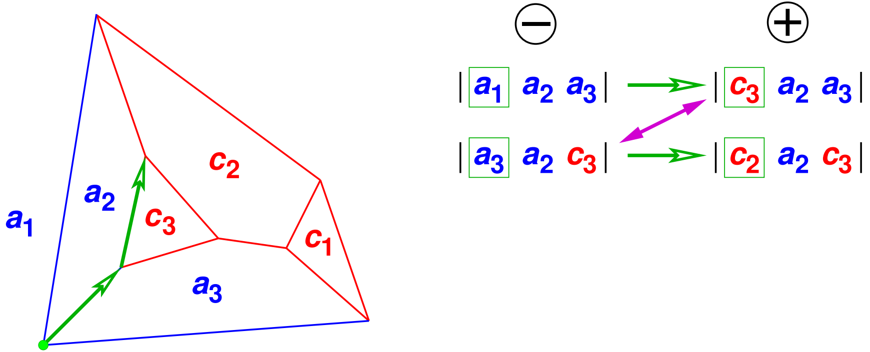
⊖ ⊕

$|a_1\ a_2\ a_3| \rightarrow |c_3\ a_2\ a_3|$

$|a_3\ a_2\ c_3|$

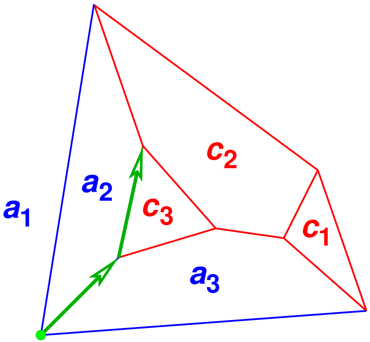
General Parity Argument with Direction

next pivot $a_3 \to c_2$ gives sign $\oplus$



General Parity Argument with Direction

Switch columns c_2 and a_2 in determinant: back to sign $\ominus$



$\ominus$	$\oplus$
$\lvert\, a_1 \; a_2 \; a_3 \,\rvert$	$\lvert\, c_3 \; a_2 \; a_3 \,\rvert$
$\lvert\, a_3 \; a_2 \; c_3 \,\rvert$	$\lvert\, c_2 \; a_2 \; c_3 \,\rvert$
$\lvert\, a_2 \; c_2 \; c_3 \,\rvert$	

General Parity Argument with Direction

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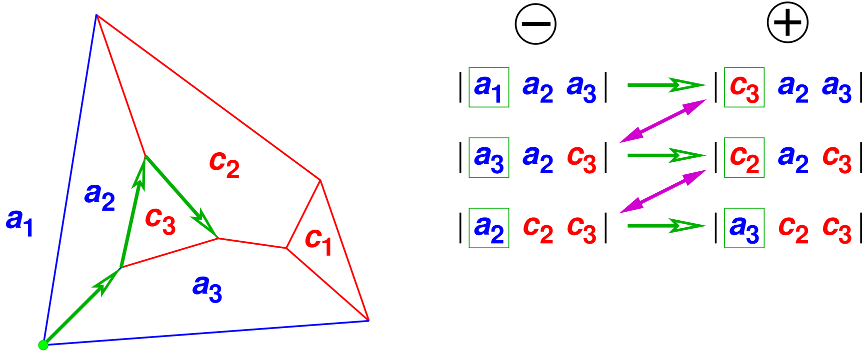

General Parity Argument with Direction

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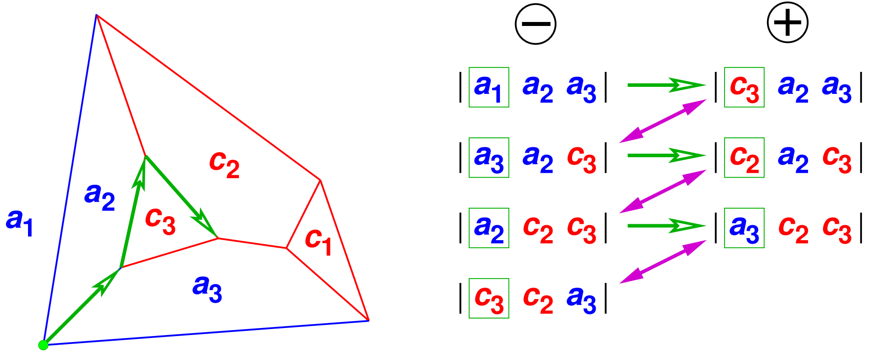
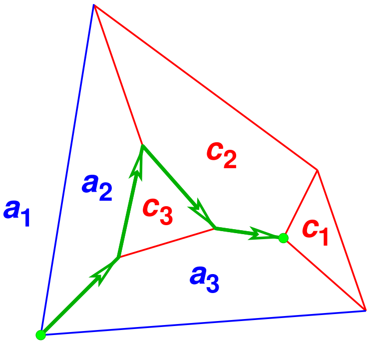
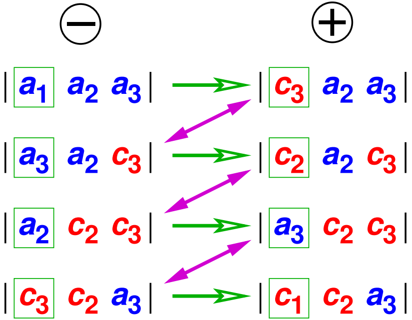


General Parity Argument with Direction

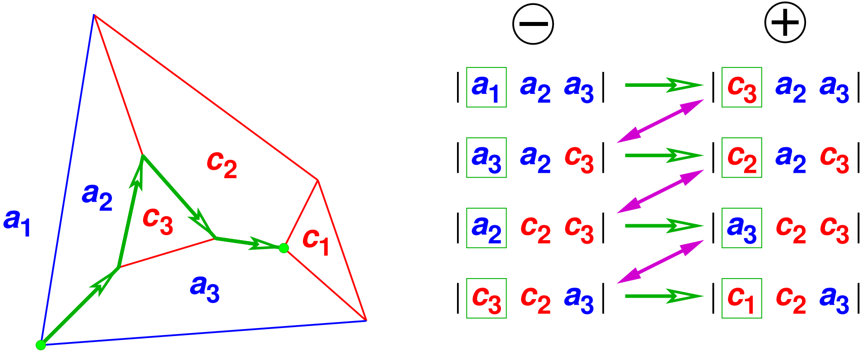
Last pivot $c_3 \to c_1$ gives sign ⊕, opposite to starting sign ⊖.





General Parity Argument with Direction

Only need: sign-switching of **pivots** and **column exchanges**



A more abstract example

$\ominus$

$| \boxed{a_1}\ a_2\ a_3\ a_4\ a_5 |$

A more abstract example

$$\ominus \qquad\qquad\qquad\qquad\qquad \oplus$$

$$|\boxed{a_1}\ a_2\ a_3\ a_4\ a_5| \longrightarrow |\boxed{c_3}\ a_2\ \underline{a_3}\ a_4\ a_5|$$

A more abstract example

$\ominus$ $\oplus$

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$$|\boxed{a_3}\ a_2\ c_3\ a_4\ a_5|$$

A more abstract example

$\ominus$ $\qquad\qquad\qquad\qquad$ $\oplus$

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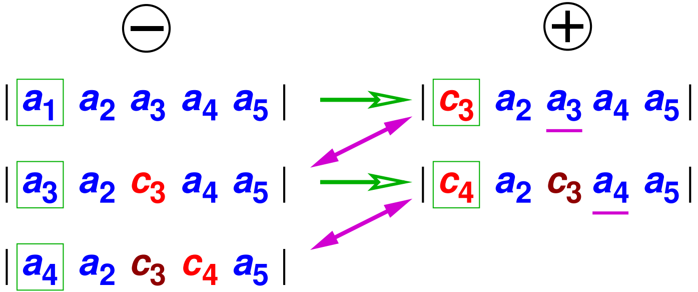
$$\left| \boxed{a_3}\ a_2\ c_3\ a_4\ a_5 \right| \longrightarrow \left| \boxed{c_4}\ a_2\ c_3\ \underline{a_4}\ a_5 \right|$$

A more abstract example



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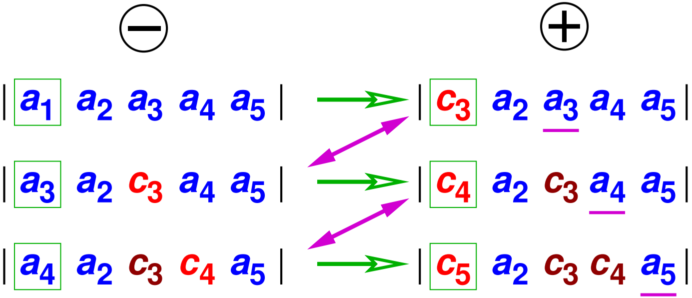

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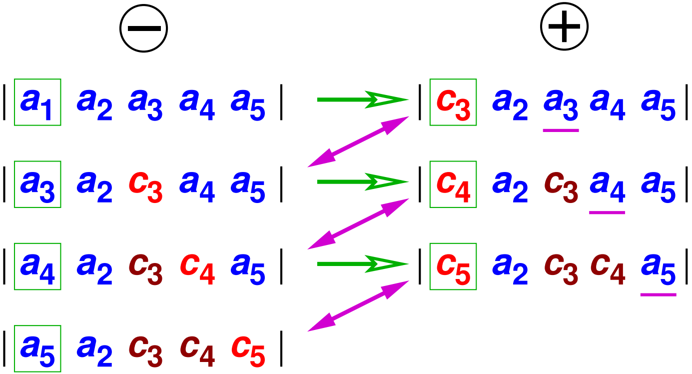

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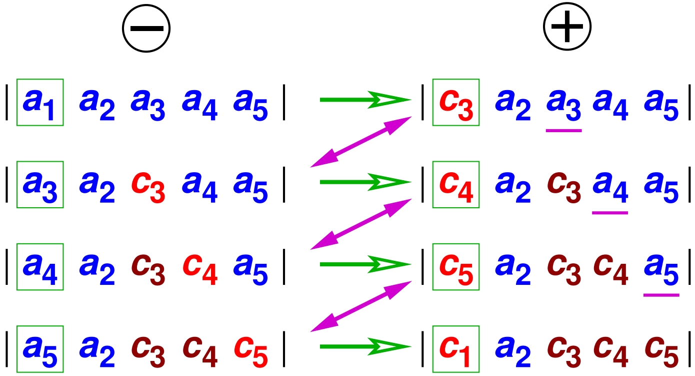

Sign vs. index of an equilibrium

Index of an equilibrium

Theorem [Shapley 1974]

A nondegenerate bimatrix game (A, B) has an odd number of equilibria, one more of index $\oplus$ than of index $\ominus$.

Index of an equilibrium

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A nondegenerate bimatrix game (A, B) has an odd number of equilibria, one more of index ⊕ than of index ⊖.

[*Proof:* Endpoints of pivoting paths have opposite index ⊖ and ⊕.]

Index of an equilibrium

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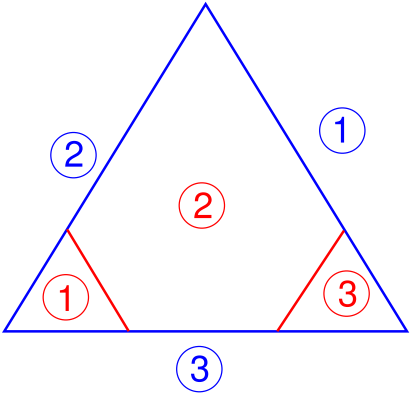
[*Proof:* Endpoints of pivoting paths have opposite index ⊖ and ⊕.]

Equilibria of index ⊕ include every

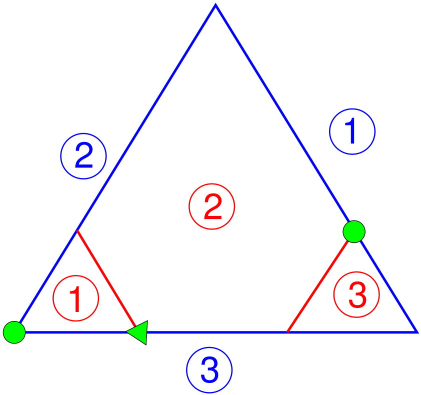
- pure-strategy equilibrium
- unique equilibrium
- **dynamically stable** equilibrium [Hofbauer 2003]

Dynamically stable equilibrium: needs index ⊕



	①	②	③
①	**3**	**0**	**0**
②	**2**	**2**	**2**
③	**0**	**3**	**0**

Dynamically stable equilibrium: needs index ⊕



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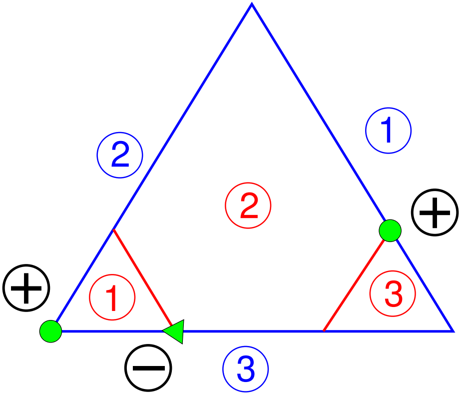
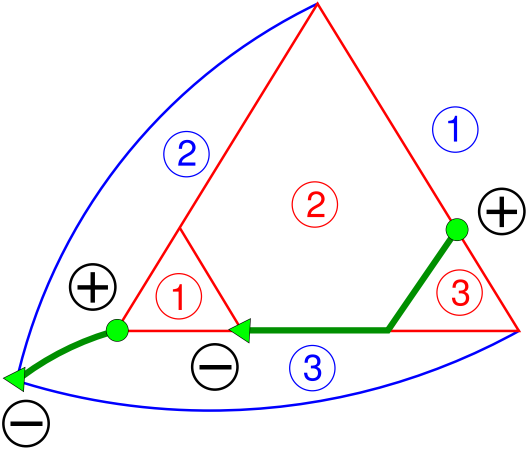


	1	2	3
1	**3**	**0**	**0**
2	**2**	**2**	**2**
3	**0**	**3**	**0**

Dynamically stable equilibrium: needs index ⊕



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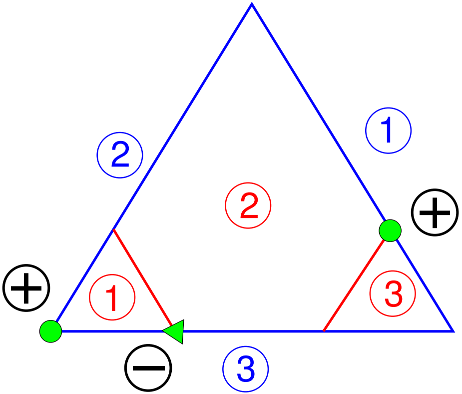

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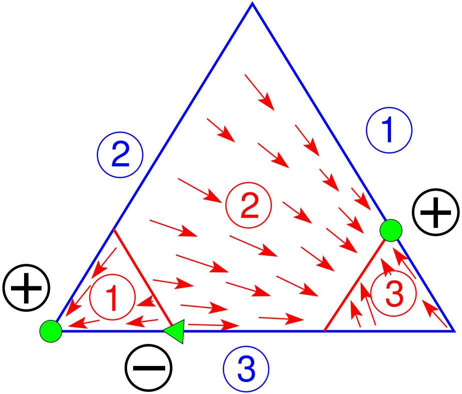


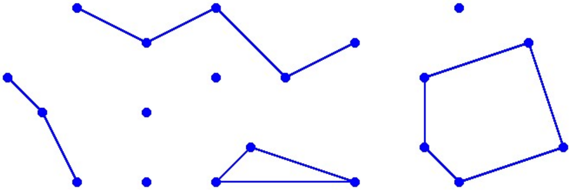
	1	2	3
1	**3**	**0**	**0**
2	**2**	**2**	**2**
3	**0**	**3**	**0**

PPAD-completeness

The Parity Argument (PA)

Given: Implicit graph G of degree at most 2 (every node has at most 2 neighbors).

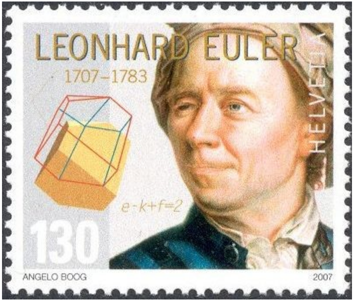
Then G is a collection of paths and cycles:

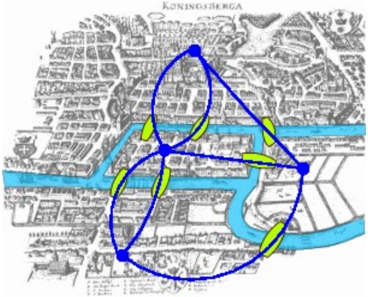


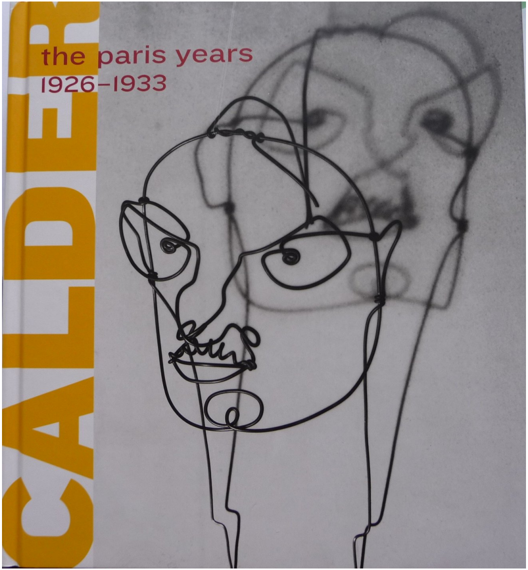
The number of degree-1 nodes (endpoints of paths) is **even**.

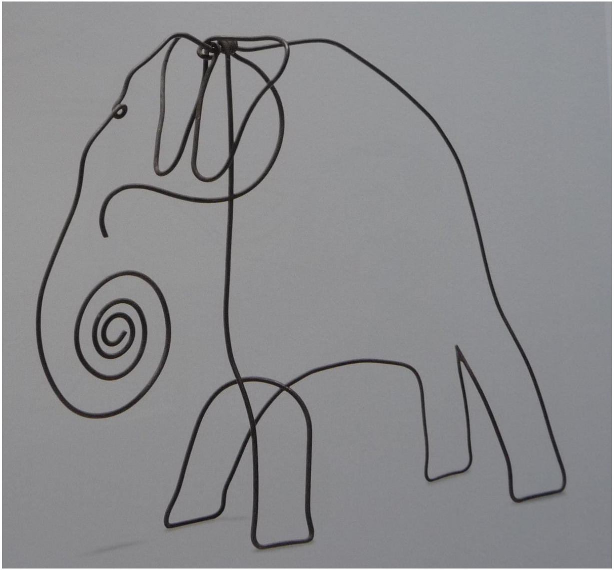
More generally (Euler)

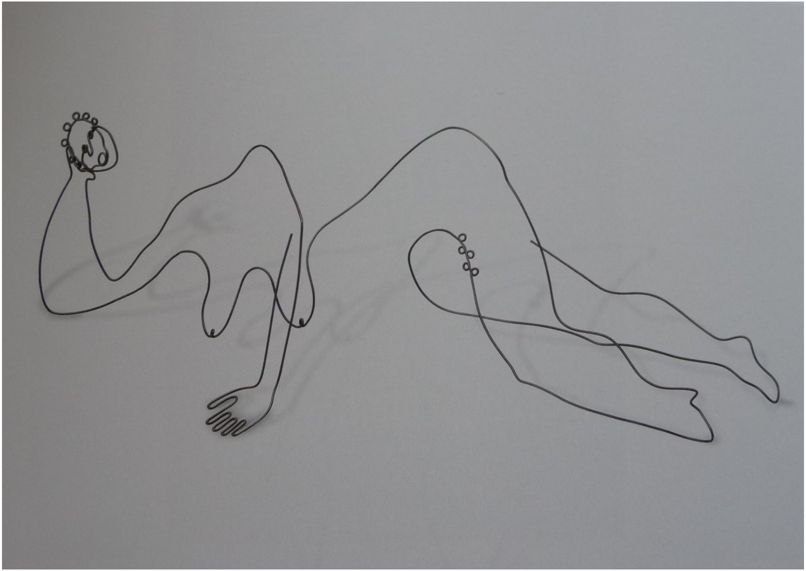
The number of odd-degree nodes of a graph is even:

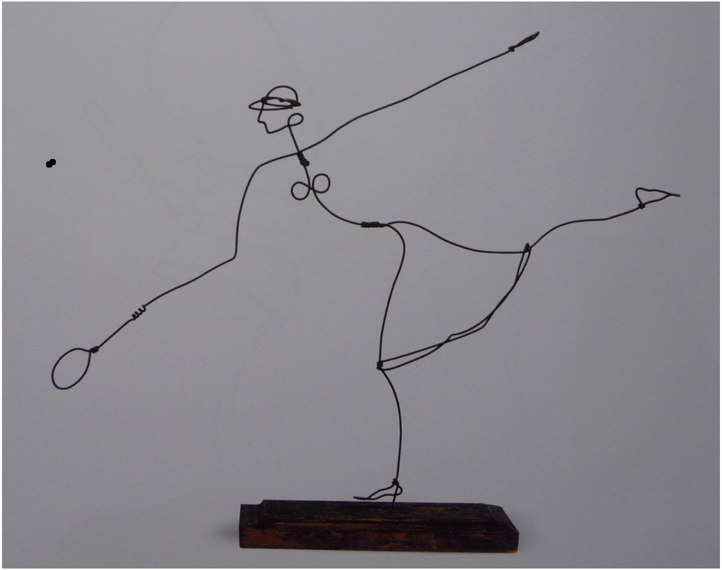


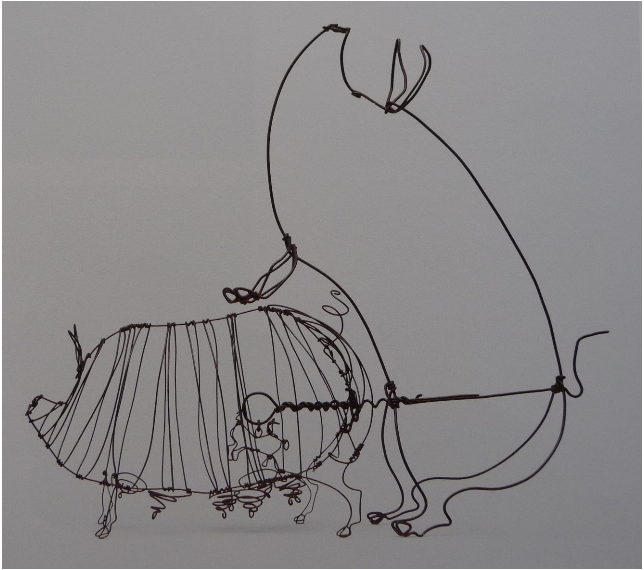



CALDER
the paris years
1926–1933



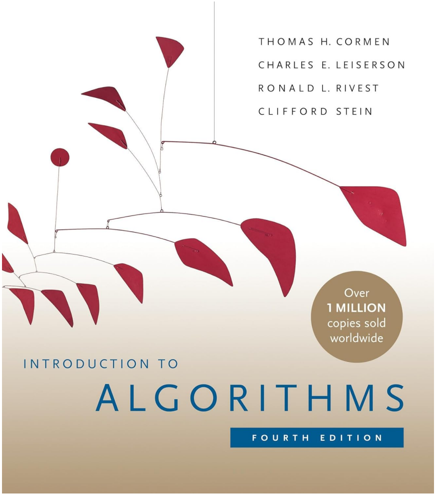






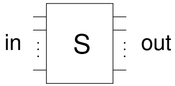
Schweinkram (filth)!

The computational complexity view



Implicit graph via Boolean circuits

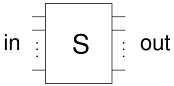
Successor circuit



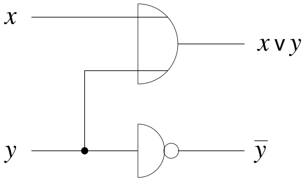
x,y	S(x,y)
00	01
01	10
10	11
11	10

Implicit graph via Boolean circuits

Successor
circuit

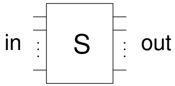


x,y	S(x,y)
00	01
01	10
10	11
11	10



Implicit graph via Boolean circuits

Successor circuit



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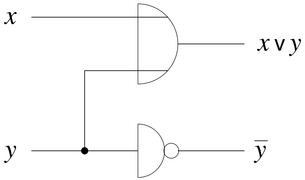

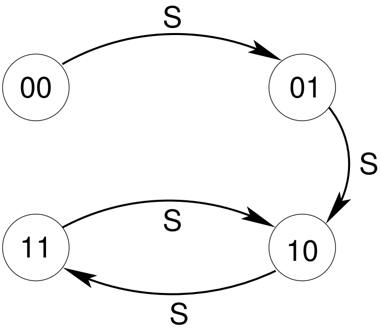

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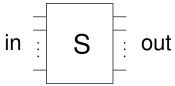

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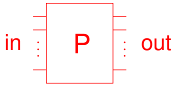
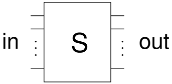
Predecessor circuit



x,y	P(x,y)
00	11
01	00
10	01
11	00

Implicit graph via Boolean circuits

Successor circuit



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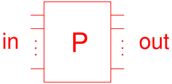

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01	00
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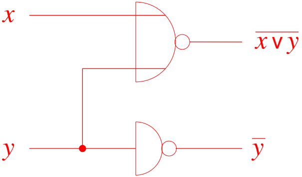

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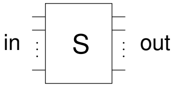

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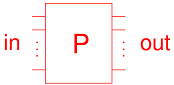

x,y	P(x,y)
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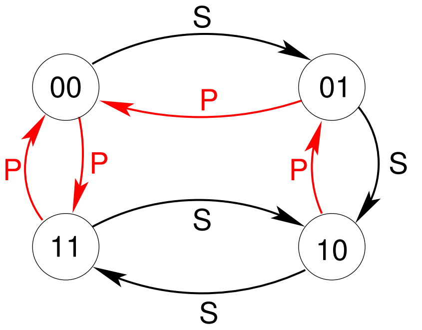
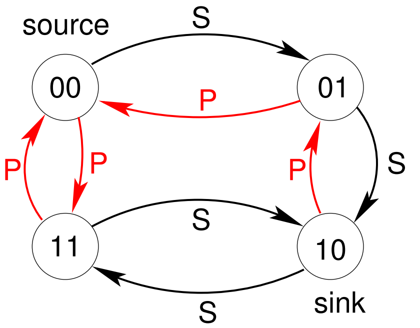


Sources and Sinks

u predecessor of v
$\Longleftrightarrow$ v successor of u
$\Longleftrightarrow$ v = S(u), u = P(v)



u source $\Longleftrightarrow$ $S(P(u)) \neq u$

v sink $\Longleftrightarrow$ $P(S(v)) \neq v$

Sources and Sinks

u predecessor of v
$\Leftrightarrow$ v successor of u
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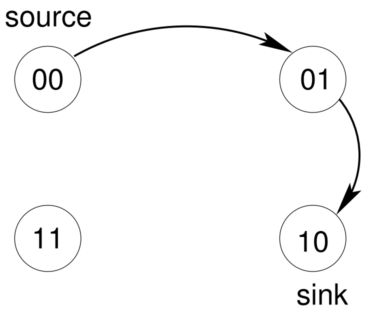

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v sink $\Leftrightarrow$ $P(S(v)) \neq v$

The problem End–Of–the–Line (EOL)

Input:

circuits S, P: $2^n \rightarrow 2^n$

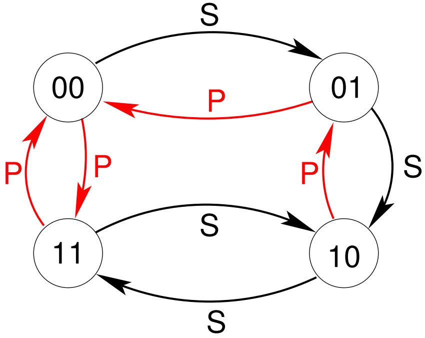
polynomial size in n

source 0^n

Output:

Any sink, or

source other than 0^n



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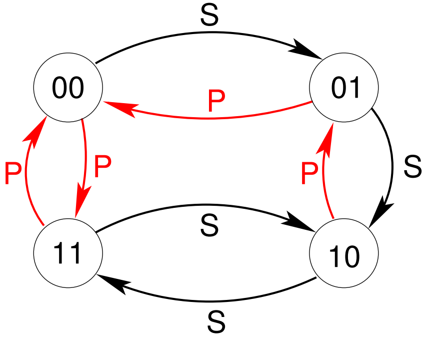

PPAD = any instances of EOL

"polynomial parity argument with direction"

[PaPADimitriou 1994]

PPAD-completeness

A computational problem is **PPAD-complete** if EOL can be reduced to it.

[Chen & Deng 2005]:
2-NASH is PPAD-complete.

Problem 2-NASH:

Input: 2-player game (A,B) in strategic form with integer payoffs.

Output: One Nash equilibrium of (A,B).

Don’t be fooled:

2-NASH is tractable in practice

just like the simplex algorithm for LP

Comments on PPAD-completeness and proof

- Many path-following problems are PPAD-complete
  - Sperner
  - Scarf's Lemma (market equilibria)
- Classic problem: **3D Brouwer** (discretized fixed points)
  - End-of-Line reduces to Brouwer [huge blowup]
  - encode Brouwer fixed points as Nash equilibria
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- Lemke's algorithm with random starting points
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- **In progress (and stuck):** Better PPAD-completeness proof?
  - complementary paths on polytopes for **invertible** circuits to encode End-of-Line?
  - encode sinks/sources as Nash equilibria