# Correlated Equilibria – Existence via LP Duality

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# Plan

- Correlated equilibria of $\boldsymbol{n}$-player games
- Existence proof without existence of Nash equilibria

# Correlated equilibria

# Nash equilibria

| I \ II | left | right |
|---|---|---|
| Top | 4 (I), 4 (II) | 1 (I), 5 (II) |
| Bottom | 5 (I), 1 (II) | 0 (I), 0 (II) |

play

| | |
|---|---|
| 0 | 0 |
| 1 | 0 |

pay: 5 (I), 1 (II)

play

| | |
|---|---|
| 0 | 1 |
| 0 | 0 |

pay: 1 (I), 5 (II)

play

| | |
|---|---|
| 1/4 | 1/4 |
| 1/4 | 1/4 |

pay: 2.5 (I), 2.5 (II)

# Correlated equilibria

| I \ II | left | right |
|---|---|---|
| Top | 4, 4 | 1, 5 |
| Bottom | 5, 1 | 0, 0 |

(In each cell, the first number is for player I (red) and the second is for player II (blue).)

**play**

| | |
|---|---|
| 0 | 1/2 |
| 1/2 | 0 |

**pay**: I: 3, II: 3

**play**

| | |
|---|---|
| 1/3 | 1/3 |
| 1/3 | 0 |

**pay**: I: 3 1/3, II: 3 1/3

**play**

| | |
|---|---|
| 0 | 1/3 |
| 1/3 | 1/3 |

**pay**: I: 2, II: 2

# Incentive constraints

| I \ II | left | right |
|---|---|---|
| Top | 4, 4 | 1, 5 |
| Bottom | 5, 1 | 0, 0 |

play

| | |
|---|---|
| a | b |
| c | d |

$$a + b + c + d = 1$$

$$a, b, c, d \geq 0$$

$$4a + 1b \geq 5a + 0b$$

$$5c + 0d \geq 4c + 1d$$

$$\Leftrightarrow \quad b \geq a, \quad c \geq d$$

$$4a + 1c \geq 5a + 0c$$

$$5b + 0d \geq 4b + 1d$$

$$\Leftrightarrow \quad c \geq a, \quad b \geq d$$

# Linear incentive constraints!

**set** of correlated equilibria

= polytope, defined by linear incentive constraints that compare any two strategies of a player

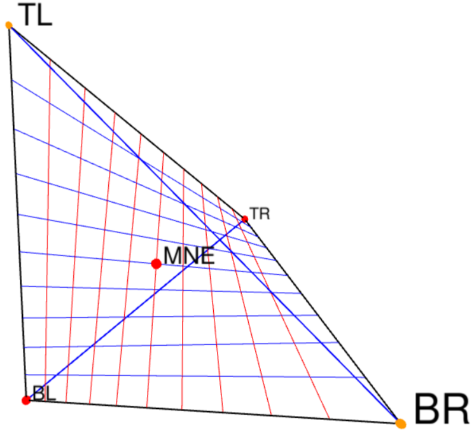
- variables = probabilities for strategy profiles

- holds for any number of players

- find easily CE with maximum payoff(-sum)

# The correlated-equilibrium polytope

| I \ II | L | R |
|---|---|---|
| T | 4, 4 | 1, 5 |
| B | 5, 1 | 0, 0 |




http://www.maths.lse.ac.uk/Personal/stengel/05.html

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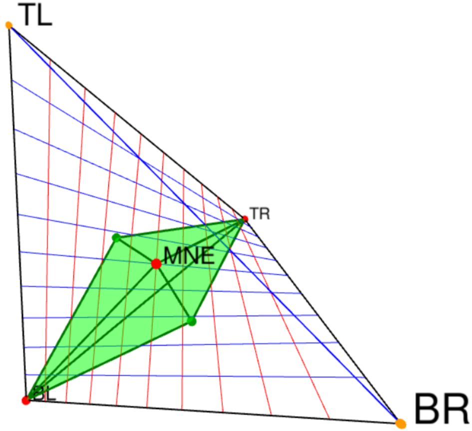


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# The correlated-equilibrium polytope

| I \ II | *L* | *R* |
|---|---|---|
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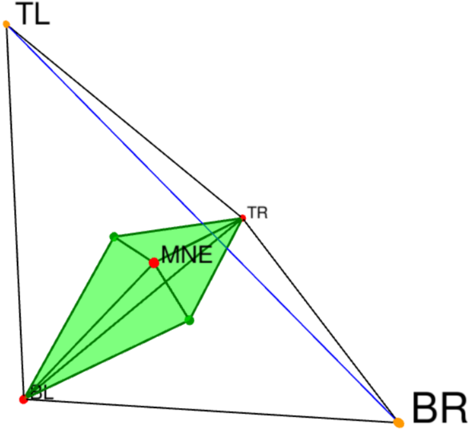


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# Existence proof

# Correlated equilibrium and strategies

player $i$, pure strategy set $S_i$, strategy profiles $S = S_i \times S_{-i}$,

$u^i(a, s_{-i})$ = payoff to player $i$ for $a \in S_i$.

**Incentive constraints** for CE distribution $z$ on $S$ :

for all players $i$ and all $a, b \in S_i$ :

$$\sum_{s_{-i} \in S_{-i}} z(a, s_{-i}) \left[ u^i(a, s_{-i}) - u^i(b, s_{-i}) \right] \geq 0$$

# LP existence proof for CE

[Hart/Schmeidler 1989; Nau/McCardle 1990]

Existence of CE via LP duality

Auxiliary game:

| | | |
|---|---|---|
| **Row** | chooses $s \in S$ | mixes with $\boxed{z(s)}$ |
| **Col** | chooses player $i$ and $a, b \in S_i$ | mixes with $y^i_{a\,b}$ |
| payoff to **Row** | $= u^i(a, s_{-i}) - u^i(b, s_{-i})$ | if $s = (a, s_{-i})$, |
| | $= \mathbf{0}$ | otherwise. |

Payoff matrix $U$, expected payoffs $z^\top U y$.

Auxiliary game has value $\mathbf{0}$ $\Leftrightarrow$ CE $z$ exists.

# Expected payoff in auxiliary game

$\boldsymbol{z}^\top \boldsymbol{U}\boldsymbol{y} =$

$$\sum_{i} \sum_{s_{-i} \in S_{-i}} \sum_{a,b \in S_i} z(a, s_{-i}) \left[ u^i(a, s_{-i}) - u^i(b, s_{-i}) \right] y^i_{a\,b}$$

to show value **0**

it suffices: $\forall \boldsymbol{y}\ \exists$ pure $\boldsymbol{s}$: $(\boldsymbol{U}\boldsymbol{y})_{\boldsymbol{s}} \geq \mathbf{0}$

can show: $\forall \boldsymbol{y}\ \exists$ product distribution $\boldsymbol{x}$ : $\boldsymbol{x}^\top \boldsymbol{U}\boldsymbol{y} = \mathbf{0}$

$$x(s) = \prod_i x^i(s_i).$$

# Use product distribution

$$
\begin{aligned}
& \boldsymbol{x}^\top U \boldsymbol{y} \\
&= \sum_i \sum_{s_{-i} \in S_{-i}} \sum_{a,b \in S_i} x^i(a) x(s_{-i}) \left[ u^i(a, s_{-i}) - u^i(b, s_{-i}) \right] y^i_{a\,b} \\
&= \sum_i \sum_{s_{-i} \in S_{-i}} x(s_{-i}) \cdot \square
\end{aligned}
$$

$$
\square = \boxed{\left[ \sum_{a,b \in S_i} x^i(a) u^i(a, s_{-i}) y^i_{a\,b} - \sum_{a,b \in S_i} x^i(a) u^i(b, s_{-i}) y^i_{a\,b} \right]}
$$

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\square = \boxed{\left[ \sum_{a,b \in S_i} x^i(a) u^i(a, s_{-i}) y^i_{a\,b} - \sum_{c,b \in S_i} x^i(c) u^i(b, s_{-i}) y^i_{c\,b} \right]}
$$

# Use product distribution

$$x^\top U y$$

$$= \sum_i \sum_{s_{-i} \in S_{-i}} \sum_{a,b \in S_i} x^i(a) x(s_{-i}) \left[ u^i(a, s_{-i}) - u^i(b, s_{-i}) \right] y^i_{a\,b}$$

$$= \sum_i \sum_{s_{-i} \in S_{-i}} x(s_{-i}) \cdot \boxed{\phantom{X}}$$

$$\boxed{\phantom{X}} = \boxed{\left[ \sum_{a,b \in S_i} x^i(a) u^i(a, s_{-i}) y^i_{a\,b} - \sum_{c,a \in S_i} x^i(c) u^i(a, s_{-i}) y^i_{c\,a} \right]}$$

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set [ ] to zero for suitable $x^i(a)$ depending on $y^i_{a\,b}$

# Neutralizing deviation plans

**Lemma** [Hart/Schmeidler 1989]
(Lemma 12.10 in *Game Theory Basics*)

$\forall y_{a\,b} \geq \mathbf{0} \;\; \exists$ probabilities $x(a)$ [which give $x^\top U y = 0$]

$$\forall a \in S_i \qquad x(a) \sum_{b \in S_i} y_{a\,b} = \sum_{c \in S_i} x(c) y_{c\,a}$$

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$$\forall a \in S_i \qquad x(a) \sum_{b \in S_i} y_{ab} = \sum_{c \in S_i} x(c) y_{ca}$$

**Interpretation** (for each original player $i$):

Increase w.l.o.g. diagonal elements $y_{aa}$. Adversary's $y$ is a Markov chain, a "deviation plan" that says how to deviate from $a$ to $b$.

Then $x$ is a stationary distribution that stays invariant under that Markov chain, so the adversary **gains nothing** with $y$.

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Then $x$ is a stationary distribution that stays invariant under that Markov chain, so the adversary **gains nothing** with $y$.

$\Rightarrow$ Auxiliary game has value $\mathbf{0}$, CE exists!

# CE for compactly specified games

**Example**

Anonymous game with many players, same actions, and payoffs specified by number of other players choosing an action.

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Anonymous game with many players, same actions, and payoffs specified by number of other players choosing an action.

**General** compactly specified game = game in strategic form with

- polynomial number of players
- polynomial number of actions per player
- polynomial-time evaluation of payoffs for product profiles $\boldsymbol{x}$

$\Rightarrow$ **Theorem** [Papadimitriou/Roughgarden 2008]:
Can find **one** CE in polynomial time.

# **Algorithm** [Papadimitriou/Roughgarden 2008]

- iterate ellipsoid algorithm to find deviation plans $\boldsymbol{y} = (\boldsymbol{y}^{i}_{a,b})$
- in each iteration, neutralize via behavior profile $\boldsymbol{x}$ to get $\boldsymbol{x}^\top \boldsymbol{U}\boldsymbol{y} = \mathbf{0}$
- derandomize $\boldsymbol{x}$ to pure profile $\boldsymbol{s}$ with payoff $(\boldsymbol{U}\boldsymbol{y})_{\boldsymbol{s}} \geq \mathbf{0}$ to **Row** [Jiang/Leyton-Brown 2010]
- infeasibility after polynomially many iterations

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$\Rightarrow$ polynomially many rows $\overline{\boldsymbol{U}}$ of $\boldsymbol{U}$ suffice to solve $\overline{\boldsymbol{z}}^\top \overline{\boldsymbol{U}} \geq \mathbf{0}$

$\Rightarrow$ one CE $\overline{\boldsymbol{z}}$ found in polynomial time

# Open problem

**Given:** Extensive game with perfect recall

**Want:** Find **one** CE for the strategic form in polynomial time.

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Alternative approach [von Stengel/Forges 2008]:
EFCE = Extensive-Form Correlated Equilibrium

recommending (and comparing / learning) **moves** rather than strategies.