Prophet Inequalities

Part 3: Online combinatorial auctions and balanced prices

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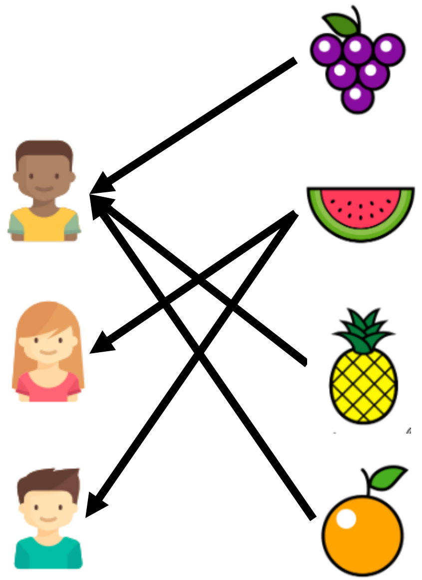
ADFOCS 2024 Summer School
August 2024

Online Combinatorial Auctions

The basic set-up:

- n buyers with valuation functions $v_i \sim \mathcal{D}_i, v_i: 2^{[m]} \to \mathbb{R}_{\geq 0}$ arriving one-by-one
- m items



n bidders m items

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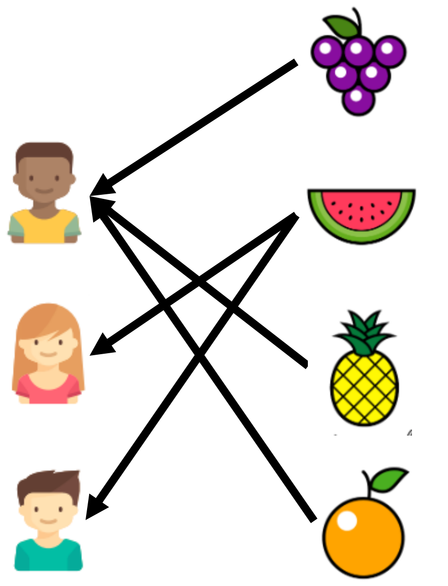
- Immediately and irrevocably assign a subset X_i of the (not yet) allocated items $[m] \setminus (\cup_{i'<i} X_i)$



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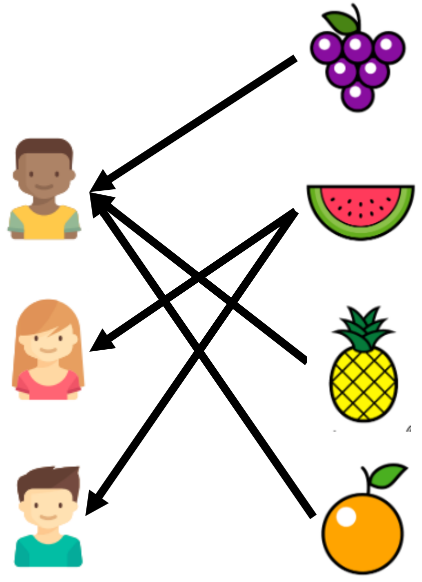


Goal: Maximize $\mathbb{E}[\sum_i v_i(X_i)]$ (a.k.a. "expected welfare")

Benchmark ("prophet"): $\mathbb{E}[\sum_i v_i(\underbrace{OPT_i(\boldsymbol{v})}))]$

= items buyer i receives in optimal allocation

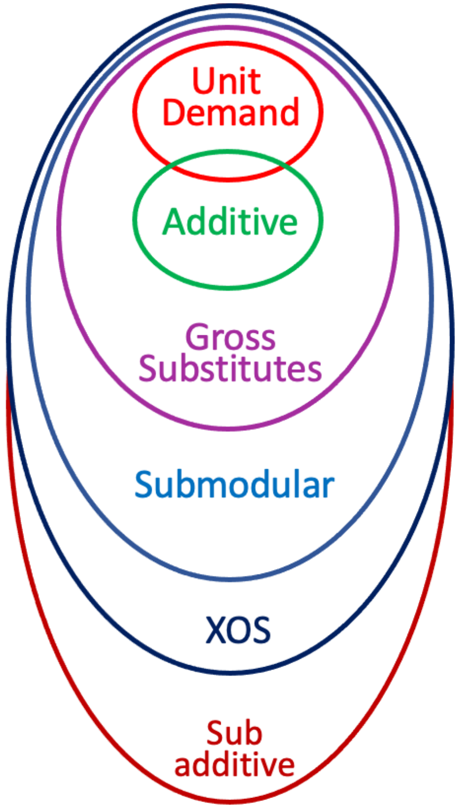
Hierarchy of Valuations

We will always assume monotonicity:

- Valuation function $v_i: 2^{[m]} \to \mathbb{R}_{\geq 0}$ is monotone if
  - $v_i(S) \leq v_i(T)$ for $S \subseteq T$

We will also impose some structure, e.g.,

- Valuation function v_i is unit demand if
  - $v_i(S) = \max_{j \in S} v_{ij}$
- Valuation function v_i is subadditive if
  - $v_i(S \cup T) \leq v_i(S) + v_i(T)$



"hierarchy of complement-free valuations"

[Lehman Lehman Nisan 2006]

Posted-Price Mechanism

Particularly desirable solution:

- Post (static, anonymous) item prices p_j for $j \in [m]$
- Buyer i buys set of still available items X_i that maximizes

$$u_i(X_i, \boldsymbol{p}) = v_i(X_i) - \sum_{j \in X_i} p_j$$

buyer i's value for set X_i

sum of the prices of the items in X_i

(is simple and has nice economic properties)

\$4 \$2 \$3

High-Level Intuition

Prices serve two purposes:

- They should be high enough
  - This is to ensure that items are protected from being snapped away by low-value buyers
- They should be low enough
  - This is to ensure that high-value buyers, when they come along, actually buy these items

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$\Longrightarrow$ we want prices to "balance" these two forces

Plan for Part 3

- Alternative "economic" proof of classic single-choice prophet inequalities via "balanced prices"
- The balanced prices framework and its main extension theorem
  - Proof for known valuations that extends to Bayesian setting
  - Simplifies problem to the problem of finding balanced prices for known valuations
- In particular: Factor 2 prophet inequality / posted-price mechanism for XOS combinatorial auctions

Outline Other Parts

Part 1: Introduction

Part 2: Online matching and contention resolution

Part 3: Online combinatorial auctions and balanced prices

Part 4: Data-driven prophet inequalities

Recall: The Classic Prophet Inequality

The Problem

- Given known distributions $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n$ over (non-negative) values:
  - A gambler gets to see realizations $v_i \sim \mathcal{D}_i$ one-by-one, and needs to immediately and irrevocable decide whether to accept v_i
  - The prophet sees the entire sequence of values $v_1, v_2, \ldots, v_n$ at once, and can simply choose the maximum value
- **Question:** What's the worst-case gap between $\mathbb{E}[\text{value accepted by gambler}]$ and $\mathbb{E}[\text{value accepted by prophet}]$?

$=: \mathbb{E}[ALG]$

$= \mathbb{E}[\max_i v_i]$

Prophet Inequality

Theorem [Samuel-Cahn '84]

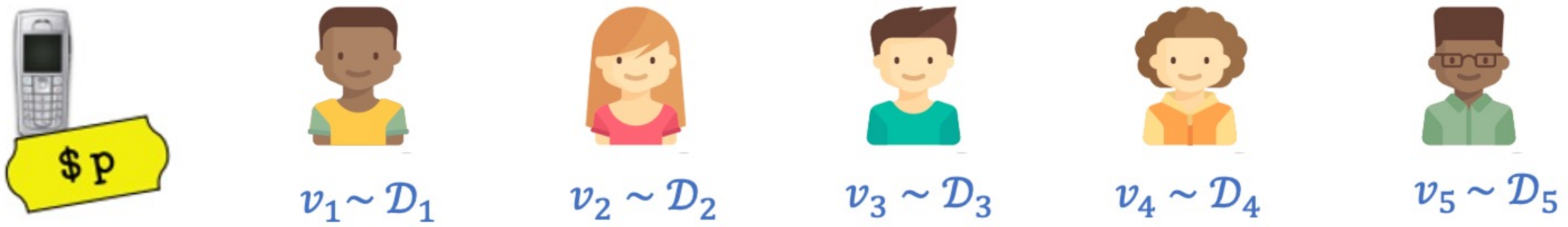
For all distributions $\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_n$, there is a threshold algorithm ALG_τ such that $\mathbb{E}[ALG_\tau] \geq \frac{1}{2}\, \mathbb{E}[\max_i v_i]$.

Threshold algorithm: set threshold/price τ, accept first $v_i \geq \tau$

Samuel-Cahn (from Gil Kalai's Blog)

Proof via Balanced Prices

Economic Interpretation



- There are n buyers with values $v_i \sim \mathcal{D}_i$, and a single item with price p
- Buyer i has a utility of $v_i - p$ for buying the item
  - If the item is still available when it's buyer i's turn, she will buy if $v_i - p \geq 0$

Bottom line: One-to-one correspondence between online algorithm with threshold $\tau = p$ and rational economic desicisions of the buyers

Economic Terminology

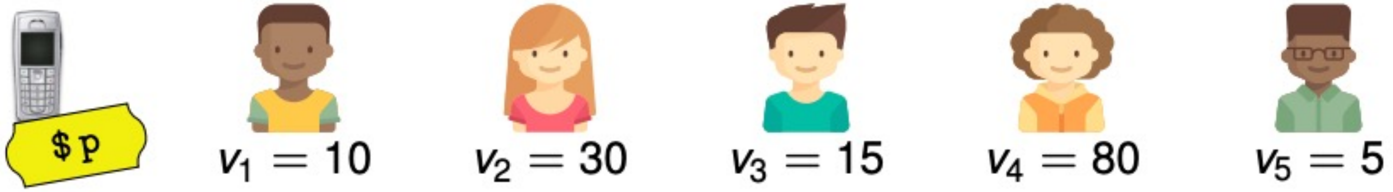
For (fixed) values $\boldsymbol{v} = (v_1, \dots, v_n)$:

- We will write $\text{utility}_i(\boldsymbol{v})$ (or $u_i(\boldsymbol{v})$ for shory) for buyer i's utility
- We will write $\text{revenue}(\boldsymbol{v})$ for the revenue
  - The revenue is p if the item is sold, 0 otherwise
- We will write $\text{welfare}(\boldsymbol{v})$ for the welfare
  - This is the value v_i of the buyer that buys the item (0 if the item is not sold)
- Note that: $\text{welfare}(\boldsymbol{v}) = \sum_i \text{utility}_i(\boldsymbol{v}) + \text{revenue}(\boldsymbol{v})$

Our goal: Want to show that there exists a price p such that

$$\mathbb{E}[\text{welfare}(v)] \geq \frac{1}{2}\, \mathbb{E}[\max_i v_i\,]$$

An Argument for Known Valuations



Price $p = \frac{1}{2} \max_i v_i$ is "balanced":

Let $v_{i^\star} = \max_i v_i$.

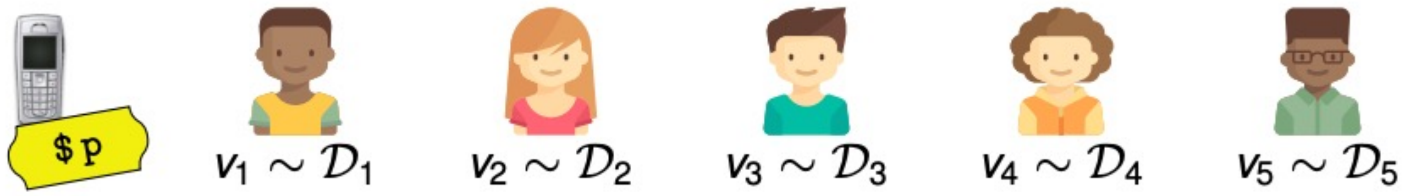
- **Case 1:** Some buyer $i' < i^\star$ buys the item:
  - $\Longrightarrow \quad \text{revenue}(\boldsymbol{v}) \geq p \geq \frac{1}{2} \max_i v_i$
- **Case 2:** No buyer $i' < i^\star$ buys the item:
  - $\Longrightarrow \quad \sum_i \text{utility}_\text{i}(\boldsymbol{v}) \geq u_{i^\star}(\boldsymbol{v}) \geq v_{i^\star} - p = \frac{1}{2} \max_i v_i$

In either case:

$$\text{welfare}(\boldsymbol{v}) = \sum_i \text{utility}_\text{i}(\boldsymbol{v}) + \text{revenue}(\boldsymbol{v}) \geq 1/2 \cdot \max_i v_i$$

Q.E.D.
(w/ compl. info)

Extension to Bayesian Setting



Let $\widehat{\boldsymbol{v}} \sim \mathcal{D}$ denote an independent sample

Consider price $p = \mathbb{E}\left[p^{\widehat{\boldsymbol{v}}}\right]$, where $p^{\widehat{\boldsymbol{v}}} = \frac{1}{2} \cdot \max_i \widehat{v}_i$

Define $\mathrm{SOLD}_i(\boldsymbol{v}) \coloneqq$ item is sold to buyers $1, \ldots, i$ when values are $\boldsymbol{v}$

Define $OPT(\boldsymbol{v}) \coloneqq$ bidder that receives the item in the optimal (welfare-maximizing) allocation for values $\boldsymbol{v}$

Extension to Bayesian Setting

To establish a bound on the expected welfare, we will again establish bounds on the expected revenue and the expected sum of utilities.

Revenue:

$$
\begin{aligned}
\mathbb{E}[\text{revenue}(\boldsymbol{v})] &= \mathbb{E}[p \cdot 1_{\text{SOLD}_n(\boldsymbol{v})}] \\
&= p \cdot \mathbb{E}[1_{\text{SOLD}_n(\boldsymbol{v})}] \\
&= \frac{1}{2}\,\mathbb{E}[\max_i \hat{v}_i] \cdot \mathbb{E}[1_{\text{SOLD}_n(\boldsymbol{v})}] \\
&= \frac{1}{2}\,\mathbb{E}[\max_i v_i] \cdot \mathbb{E}[1_{\text{SOLD}_n(\boldsymbol{v})}]
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- To bound the sum of utilities, first consider some buyer i
- Buyer i can draw an independent sample $\boldsymbol{v}_{-i}^{(i)} \sim \mathcal{D}_{-i}$ and buy if
    - buyer i gets the item in the optimal allocation for $(\boldsymbol{v}_i, \boldsymbol{v}_{-i}^{(i)})$ and
    - the item has not been sold to buyers $1, \ldots, i-1$

Extension to Bayesian Setting

a.k.a. "hallucination trick"

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$$\Longrightarrow \mathbb{E}[u_i(\boldsymbol{v})] \geq \mathbb{E}[(v_i - p) \cdot 1_{OPT(\boldsymbol{v}_i, \boldsymbol{v}_{-i}^{(i)}) = i} \cdot 1_{\neg \text{SOLD}_{i-1}(v)}]$$

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only depends on v_i and $\boldsymbol{v}_{-i}^{(i)}$

only depends on $v_1, \ldots, v_{i-1}$

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since $\boldsymbol{v}_{-i}^{(i)}$ and $\boldsymbol{v}_{-i}$ are identically distributed

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$$\geq \mathbb{E}[1_{\neg \text{SOLD}_n(v)}]$$

since $\neg \text{SOLD}_n(v) \Longrightarrow \neg \text{SOLD}_{i-1}(v)$

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Extension to Bayesian Setting

- Summing over all buyers $i \in [n]$, we thus obtain

$$\mathbb{E}[\sum_i u_i(\boldsymbol{v})] \geq \sum_i \mathbb{E}[(v_i - p) \cdot 1_{OPT(\boldsymbol{v})=i}] \cdot \mathbb{E}[1_{\neg \text{SOLD}_n(\boldsymbol{v})}]$$

$$= (\mathbb{E}[\max_i v_\text{i}] - p) \cdot \mathbb{E}[1_{\neg \text{SOLD}_n(\boldsymbol{v})}]$$

$$= \frac{1}{2} \mathbb{E}[\max_i v_\text{i}] \cdot \mathbb{E}[1_{\neg \text{SOLD}_n(\boldsymbol{v})}]$$

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Putting everything together:

$$\mathbb{E}[\text{welfare}(\boldsymbol{v})] = \mathbb{E}[\sum_i u_i(\boldsymbol{v})] + \mathbb{E}[\text{revenue}(\boldsymbol{v})]$$

$$\geq \frac{1}{2} \mathbb{E}[\max_i v_i] \cdot \underbrace{(\mathbb{E}[1_{\neg \text{SOLD}_n(v)}] + \mathbb{E}[1_{\text{SOLD}_n(v)}])}_{=1}$$

Q.E.D.

Prophet Inequalities via Balanced Prices

[Weinberg Kleinberg 2012, Feldman Gravin Lucier 2015,
Dütting Feldman Kesselheim Lucier 2017]

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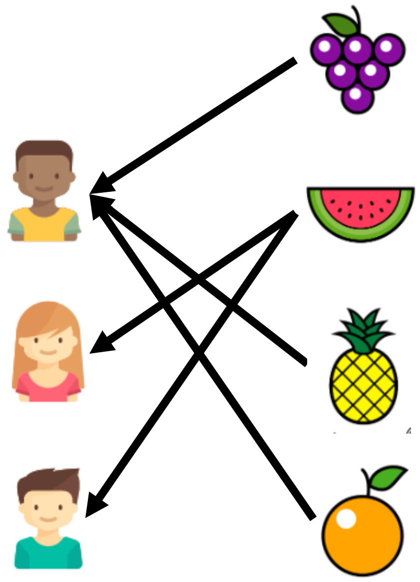

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XOS Valuations

Definition. A valuation function $v_i: 2^{[m]} \to \mathbb{R}_{\geq 0}$ is fractionally subadditive (XOS) if there are $v_{ij}^{\ell} \in \mathbb{R}_{\geq 0}$ such that

$$v_i(S) = \max_{\ell} \sum_{j \in S} v_{ij}^{\ell} \quad .$$

XOS Valuations

Definition. A valuation function $v_i: 2^{[m]} \to \mathbb{R}_{\geq 0}$ is fractionally subadditive (XOS) if there exist additive functions v_i^ℓ with $\ell \in [k]$ such that

$$v_i(S) = \max_\ell v_i^\ell(S) = \max_\ell \sum_{j \in S} v_{ij}^\ell$$

Examples:

- Additive: $v_i(S) = \sum_{j \in S} v_{ij}$
- Unit demand: $v_i(S) = \max_{j \in S} v_{ij}$
- Budget additive: $v_i(S) = \min\{\sum_{j \in S} v_{ij}, B\}$
- Submodular: $v_i(S \cup \{j\}) - v_i(S) \geq v_i(T \cup \{j\}) - v_i(T)$ for $S \subseteq T$

The FGL 15 Result

Definition. [Feldman Gravin Lucier 2015]
For any distributions $\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_n$ over XOS valuation functions, there exist (static, anonymous) item prices such that for the resulting allocation $X_1, \dots, X_n$:

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Main technique: Balanced prices.

Balanced Prices

Definition. [Dütting Feldman Kesselheim Lucier 2017]
A valuation function $v_i: 2^{[m]} \rightarrow \mathbb{R}_{\geq 0}$ admits balanced prices if for every set of items $U \subseteq [m]$ there exist item prices p_j for $j \in U$ such that for all $T \subseteq U$:

(1) $\sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus \mathrm{T})$

(2) $\sum_{j \in U \setminus \mathrm{T}} p_j \leq v_i(U \setminus \mathrm{T})$

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Known fact (implicit in [FGL 15]): XOS valuation functions admit balanced prices.

(See exercise!)

Examples: Balanced Prices

Two conditions:

(1) $\sum_{j\in T} p_j \geq v_i(U) - v_i(U\backslash \mathrm{T})$ $(\forall T \subseteq U)$ and

(2) $\sum_{j\in S} p_j \leq v_i(S)$ $(\forall S \subseteq U)$

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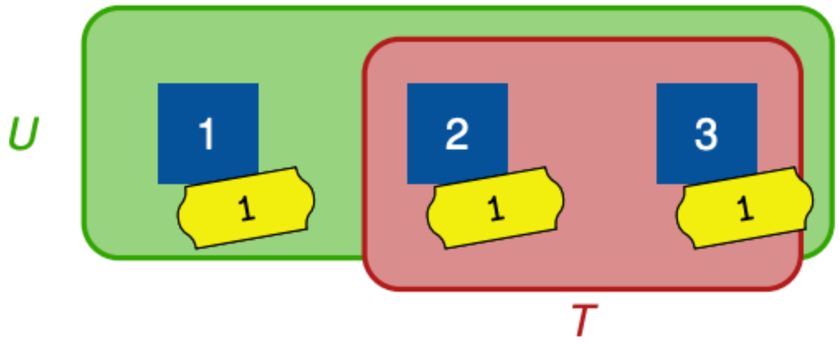

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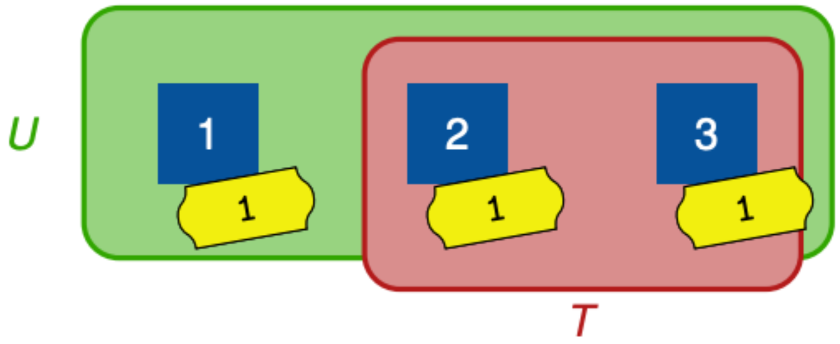

Example 1: Additive

$v_i(S) = |S|$

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Example 2: Unit demand

$$v_i(S) = \begin{cases} 0 \ \text{ if } S = \emptyset \\ 1 \ \text{ if } S \neq \emptyset \end{cases}$$

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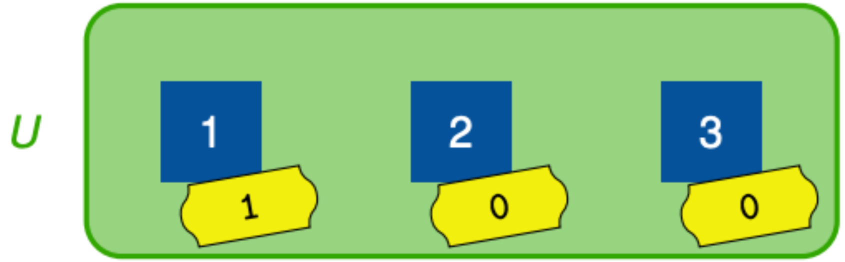

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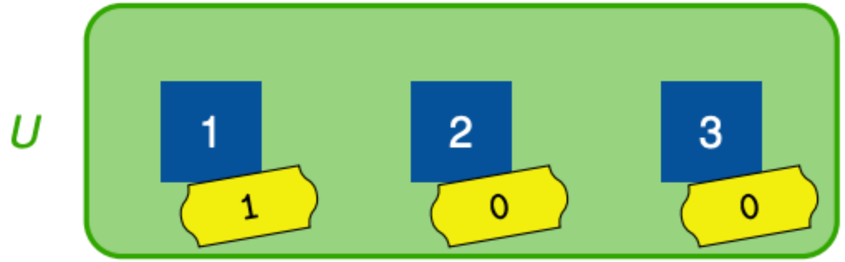

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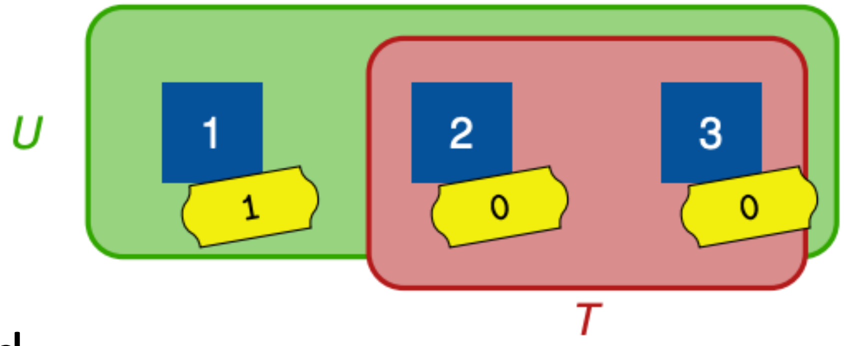

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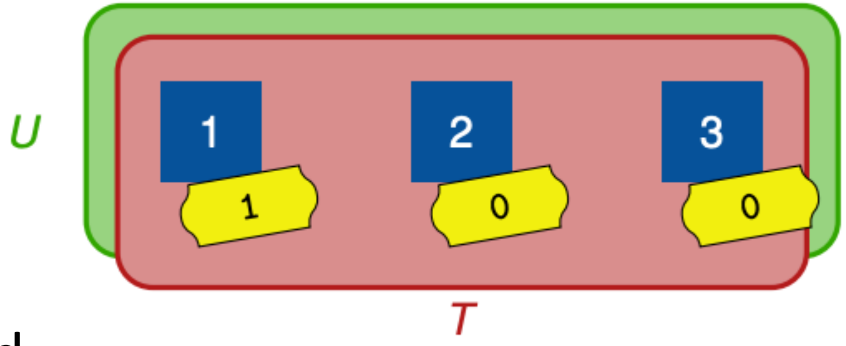

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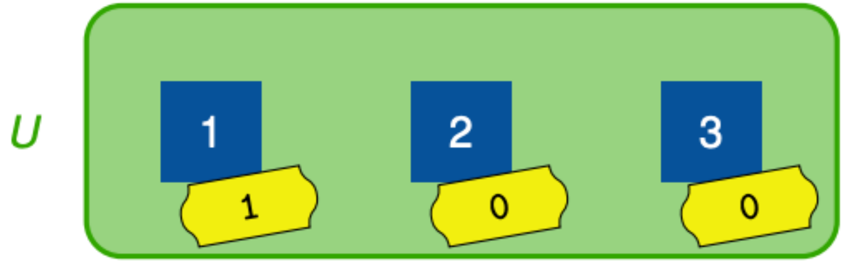

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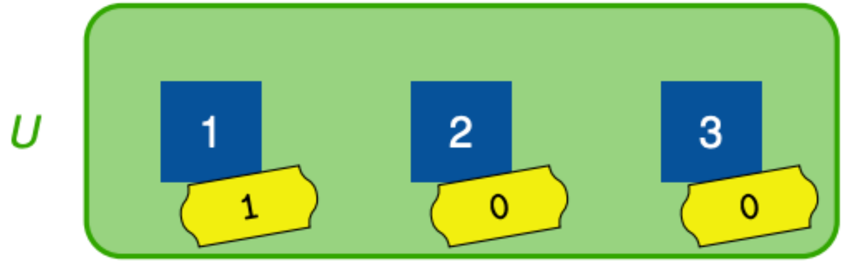

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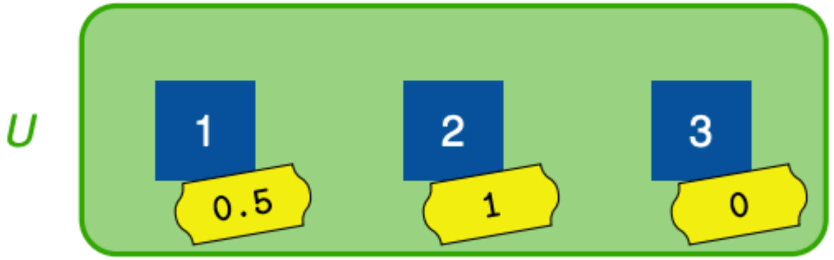

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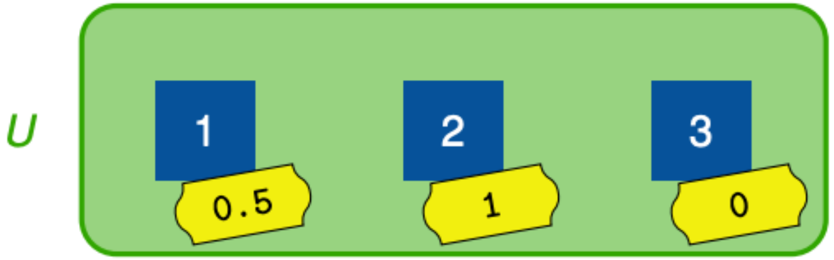

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Main Theorem

Theorem. [Dütting Feldman Kesselheim Lucier 2017]
If a class of valuations admits balanced prices, then for any distributions $\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_n$ there exist (static, anonymous) item prices such that for the resulting allocation $X_1, \dots, X_n$:

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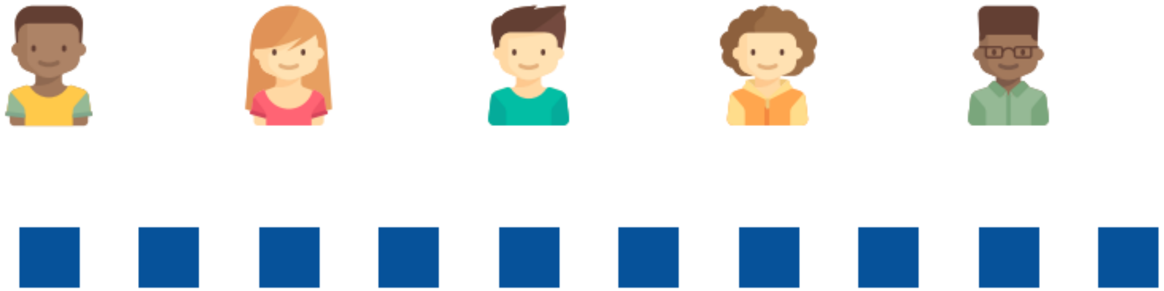
Up next: How we set prices & the argument for complete information.

How we Set the Prices



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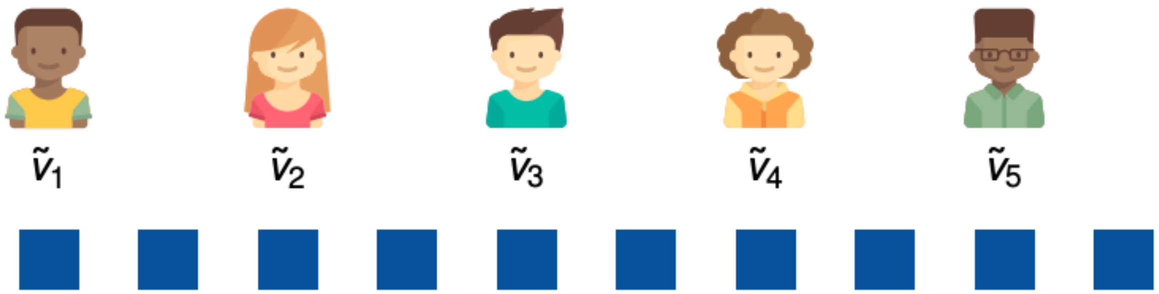

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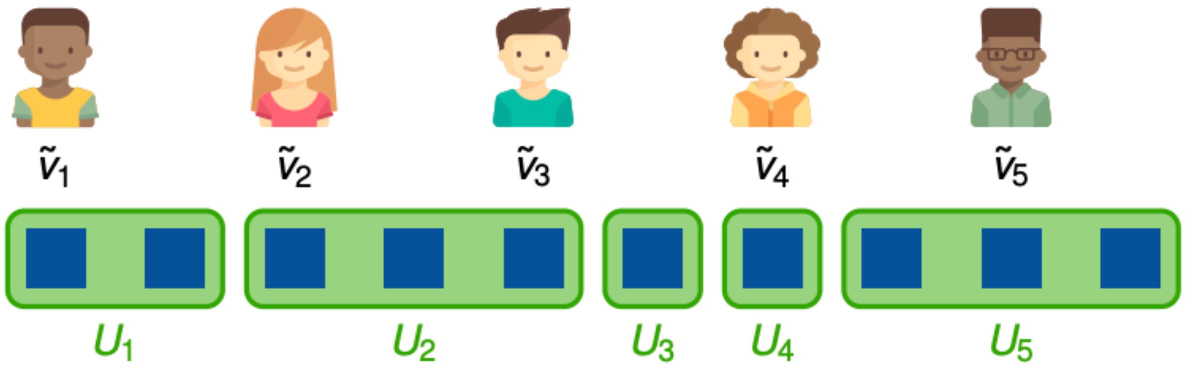

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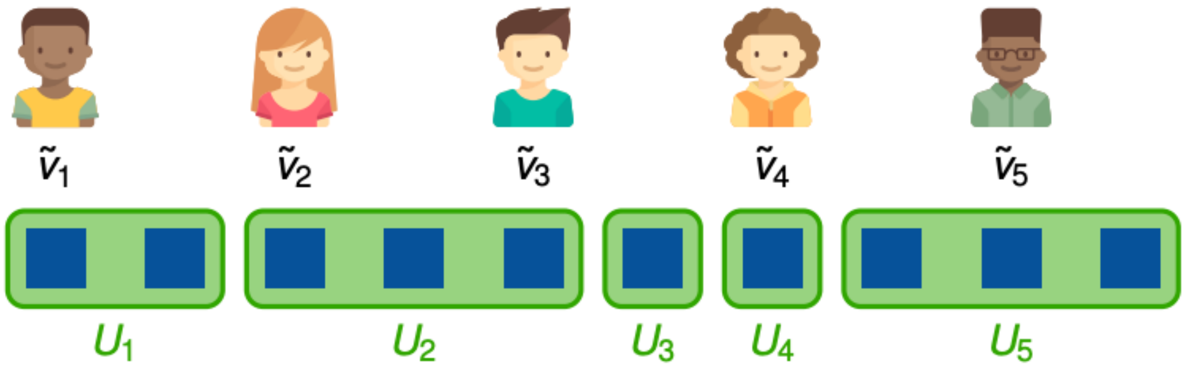

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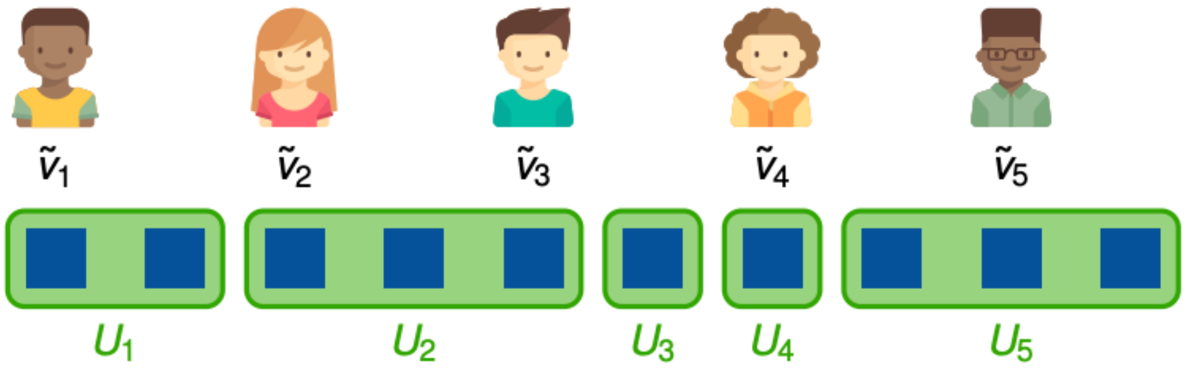

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Price for item j: $\bar{p}_j \;= \frac{1}{2} \cdot \; \mathbb{E}_{\hat{\boldsymbol{v}} \sim \mathcal{D}}[p_j^{\hat{\boldsymbol{v}}}]$.

Proof of Factor 2

(Complete Information)

Let $U_i = \{j \mid i \text{ gets } j \text{ in } OPT(\boldsymbol{v})\}$ (for all $i \in [n]$)

Set price $\bar{p}_j = \frac{p_j}{2}$ for $j \in U_i$. (p_j = balanced price for v_i, U_i)

Let $T_i = \{j \mid j \in U_i \text{ sold to } i' \neq i\}$.

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(1) $\sum_{j \in T_i} p_j \geq v_i(U_i) - v_i(U_i \setminus T_i)$

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Then, for the allocation $X_1, \ldots, X_n$, we have:

$$u_i(X_i, \overline{\boldsymbol{p}}) + \sum_{j \in T_i} \overline{p}_j$$

$$\geq \left(v_i(U_i \setminus T_i) - \sum_{j \in U_i \setminus T_i} \overline{p}_j\right) + \sum_{j \in T_i} \overline{p}_j$$

$$\geq \left(v_i(U_i \setminus T_i) - \frac{1}{2} v_i(U_i \setminus T_i)\right) + \frac{1}{2}\left(v_i(U_i) - v_i(U_i \setminus T_i)\right)$$

$$= \frac{1}{2} v_i(U_i)$$

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Then, for the allocation $X_1, \dots, X_n$, we have:

$$\begin{aligned}
\sum_i v_i(X_i) &\geq \sum_i (u_i(X_i, \overline{\boldsymbol{p}}) + \sum_{j \in T_i} \overline{p}_j) \\
&\geq \sum_i [(v_i(U_i \setminus \mathrm{T}_i) - \sum_{j \in U_i \setminus T_i} \overline{p}_j) + \sum_{j \in T_i} \overline{p}_j] \\
&\geq \sum_i [(v_i(U_i \setminus \mathrm{T}_i) - \frac{1}{2} v_i(U_i \setminus \mathrm{T}_i)) + \frac{1}{2}(v_i(U_i) - v_i(U_i \setminus T_i))] \\
&= \sum_i \frac{1}{2} v_i(U_i)
\end{aligned}$$

Balancedness:

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Q.E.D.

(w/ compl. info)

Discussion

- Reduces the problem to finding balanced prices for fixed valuations
  - Often much easier to think about this complete information problem

- The result can be generalized/strengthened in two ways:
  - Prices may be adaptive (required for constant-factor for matroids) [Feldman Svensson Zenklusen 2021]
  - Inequalities can be relaxed

- Captures several known proofs such as [Feldman Gravin Lucier 2015] and [Kleinberg Weinberg 2012] (and leads to new results)

Further Results

- Prices can be computed in poly-time via LP-relaxation (rather than integral optimum) [Dütting Feldman Kesselheim Lucier 2017]
- Techniques also applicable for revenue maximization [Cai Zhao 2017]
- For subadditive combinatorial auctions this approach is limited to $\Omega(\log m)$ approximation, but $O(\log \log m)$ possible via relaxation of balancedness [Dütting Kesselheim Lucier 2020]

The $O(\log \log m)$ bound is attained by (static/anonym.) item prices.

Beyond Balanced Prices

Subadditive Buyers

Theorem [Correa Cristi 2023]

For subadditive combinatorial auctions, there exists an $O(1)$-competitive online algorithm against the prophet benchmark.

- For subadditive buyers “simultaneous first-price item auctions” have a constant Price of Anarchy (with respect to Bayes-Nash equilibria) [Feldman Fu Gravin Lucier 2013]

- Can view proof as reduction to constant Price of Anarchy of ”simultaneous all-pay item auctions with random reserves”

Open question: Via (static/anonym.) pricing?

Cf. reduction in [Banihashem et al. ‘24] (but adaptive, bundle prices)

Summary

- Alternative "economic" proof of classic single-choice prophet inequalities via "balanced prices"
- The balanced prices framework and its main extension theorem
  - Proof for known valuations that extends to Bayesian setting
  - Simplifies problem to the problem of finding balanced prices for known valuations
- In particular: Factor 2 prophet inequality / posted-price mechanism for XOS combinatorial auctions

Thanks! Coffee!

Additional Slides

Balanced Prices for XOS Valuations

Lemma. For XOS valuation v_i and set U the following prices p_j for $j \in U$ are balanced:

- let v_i^ℓ be such that $v_i(U) = \sum_{j \in U} v_{ij}^\ell$
- set $p_j = v_{ij}^\ell$

(v_i^ℓ is also known as the "additive supporting function" of v_i on set U)

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Proof: Exercise!