max planck institut
informatik

Fair Division

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SIC Saarland Informatics Campus

ADFOCS
2024

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Divide **items** among agents in a **fair** manner.

Spectrum of the Problems

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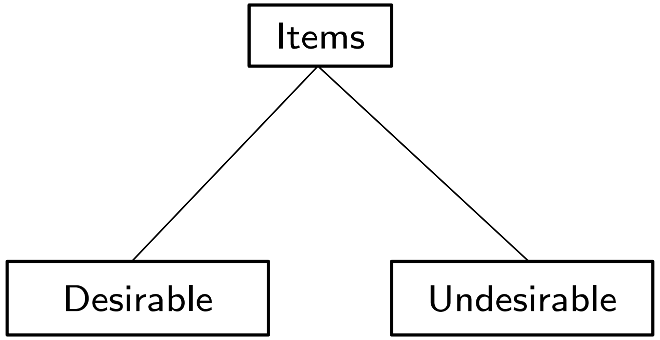

Spectrum of the Problems

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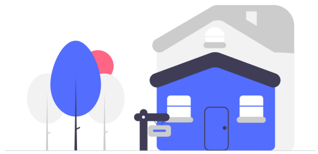
Items

- Desirable
- Undesirable

Divorce settlements



Household chores

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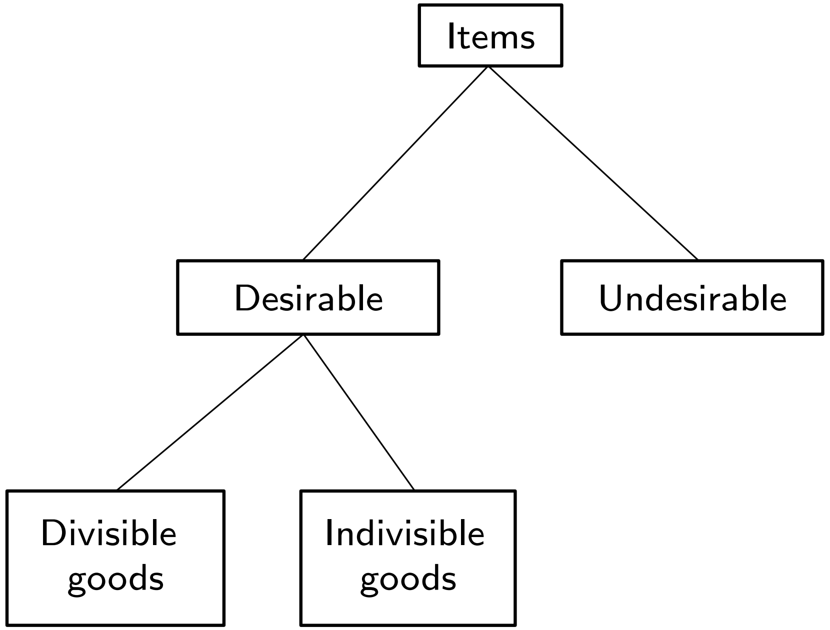

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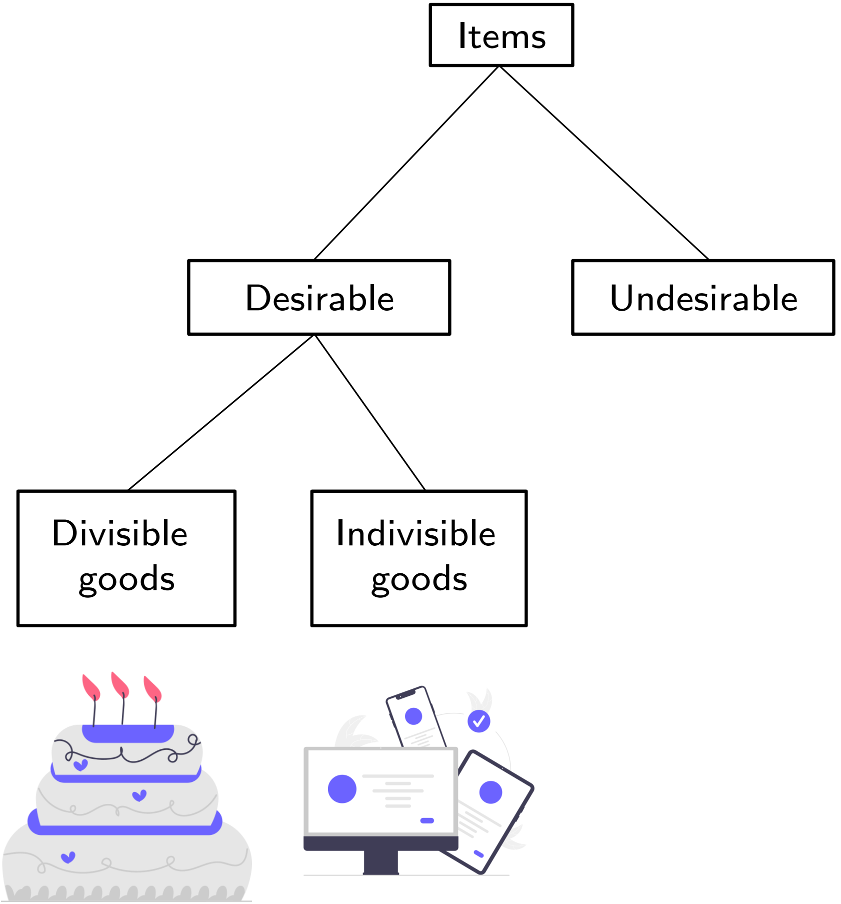

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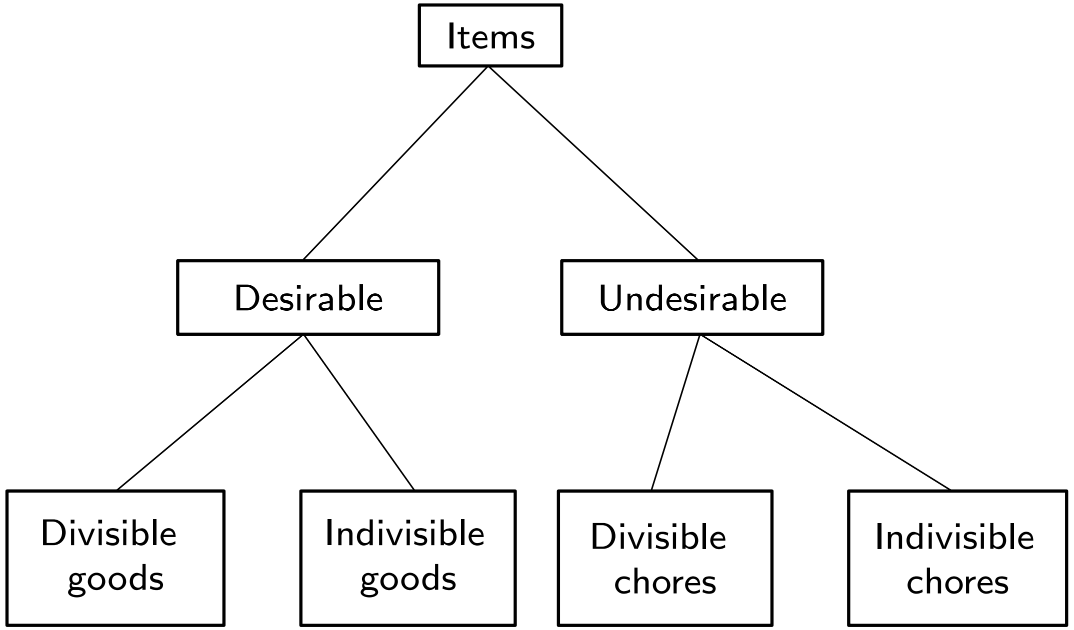

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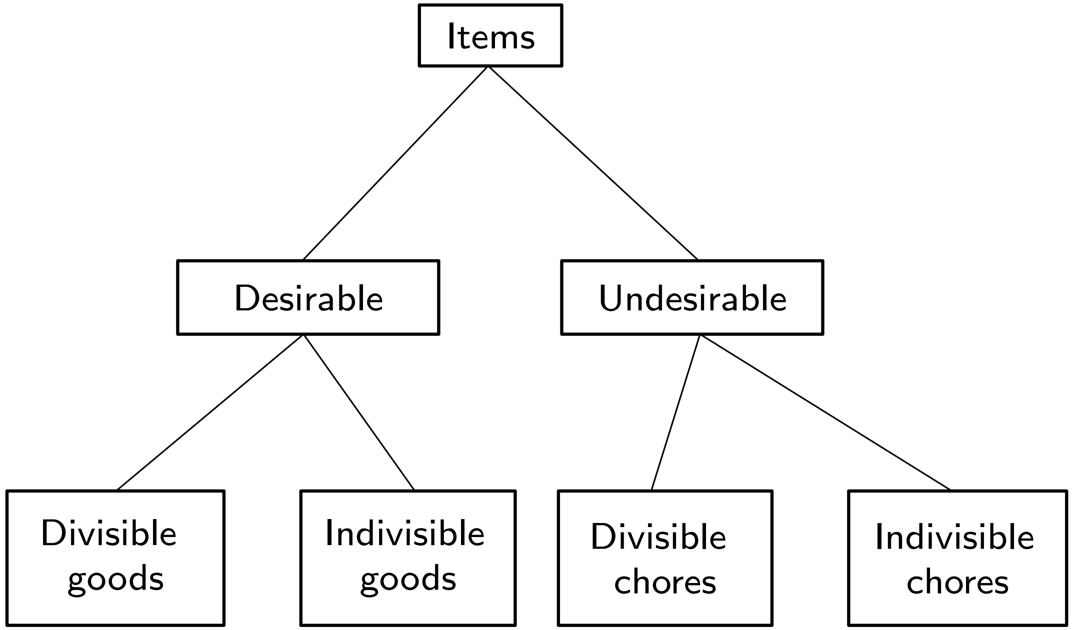

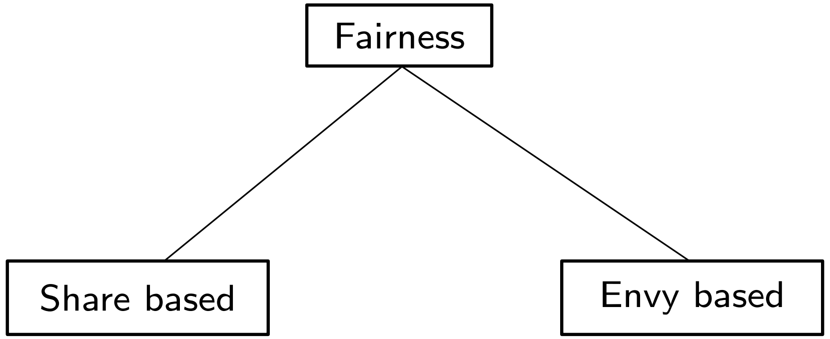

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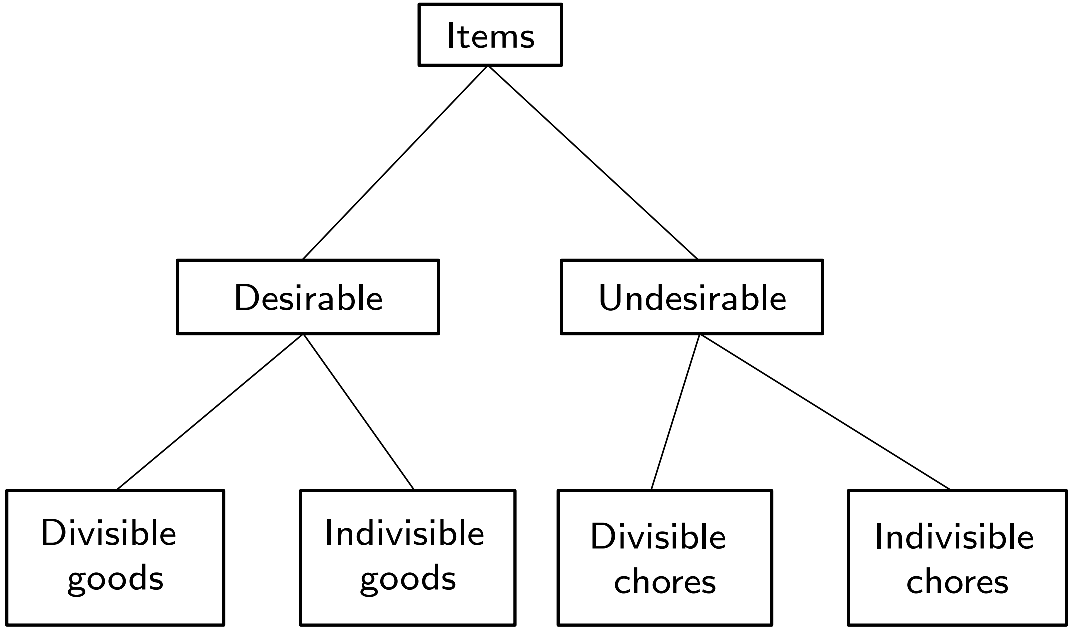

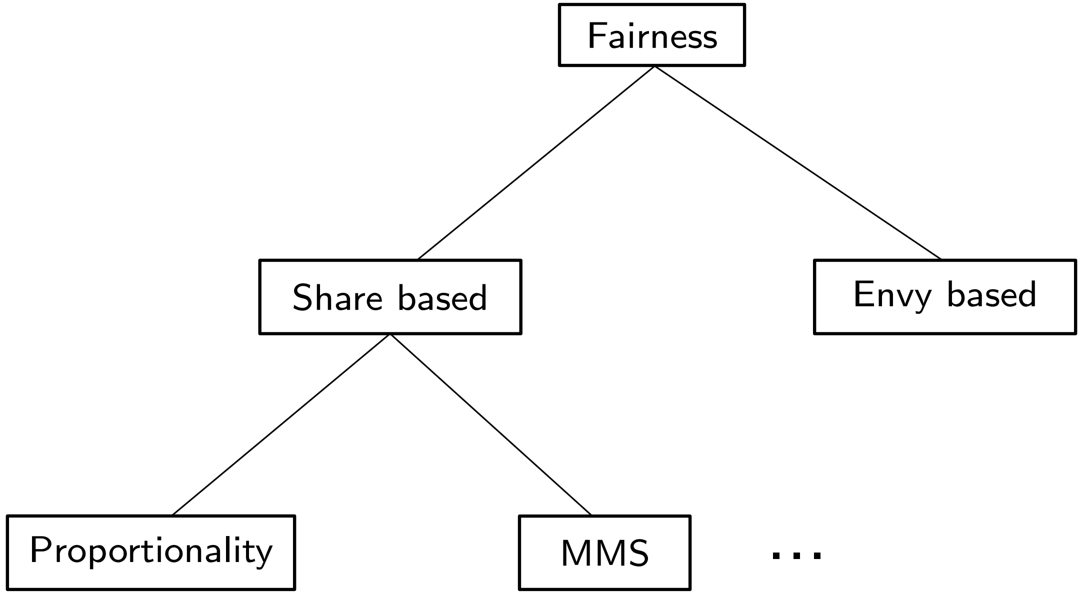

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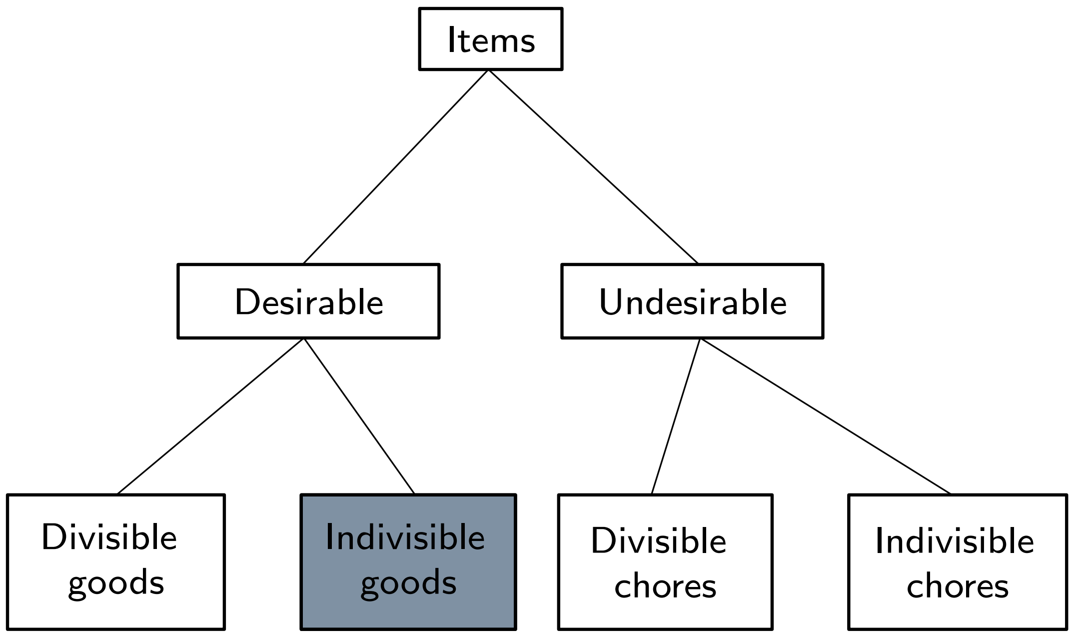

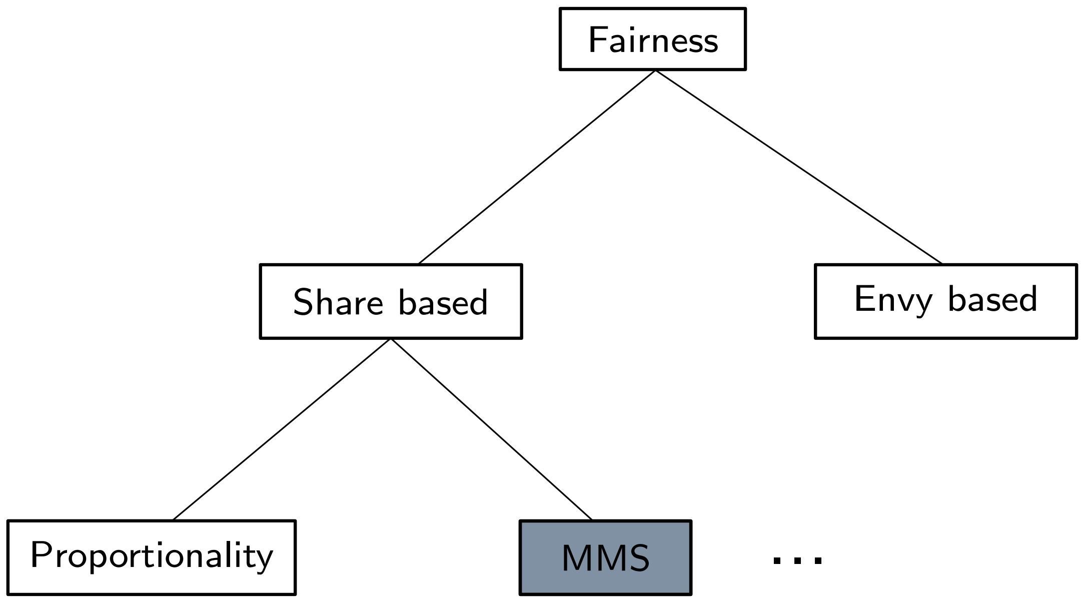

Problem Definition

Given: $\mathcal{I} = (N, M, V)$

- N: set of n agents
- M: set of m indivisible goods
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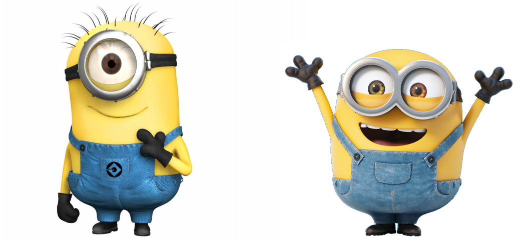

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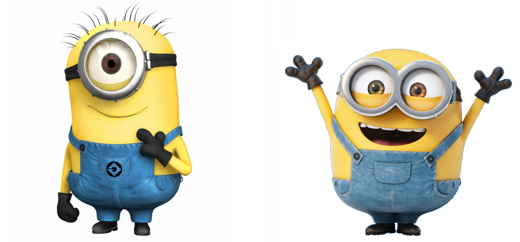

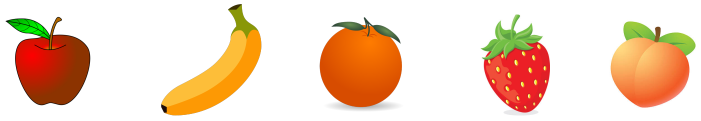

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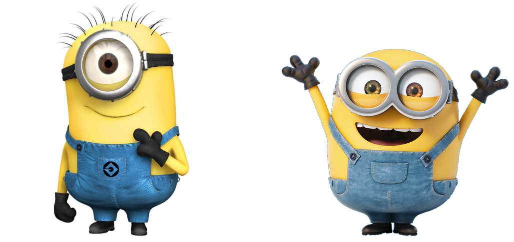

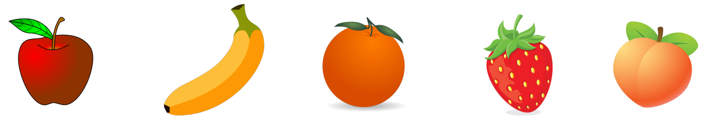

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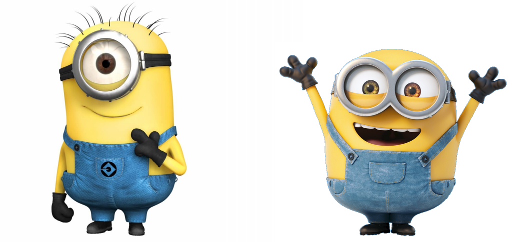

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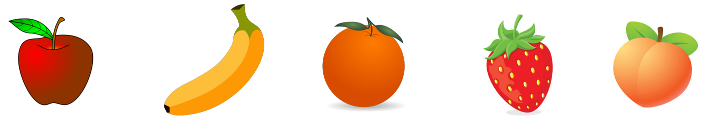

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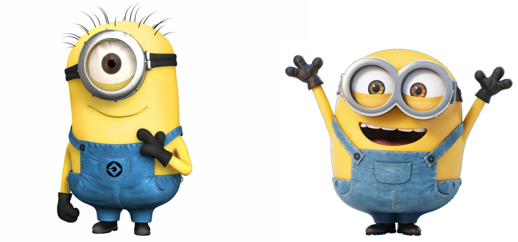

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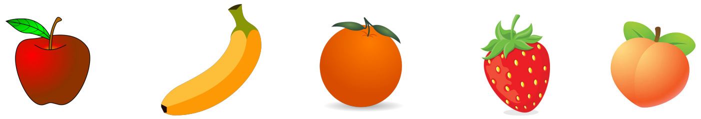

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A partition $X = (X_1, X_2, \ldots, X_n)$ of M

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Share based notions: $v_i(X_i) \geq t_i$ for all agents i

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	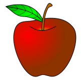	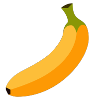			
	2	2	3	4	3
	1	5	2	3	2

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	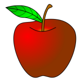	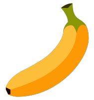			
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	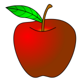	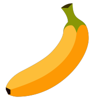			
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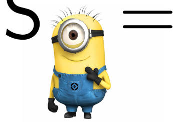
MMS $= 6$

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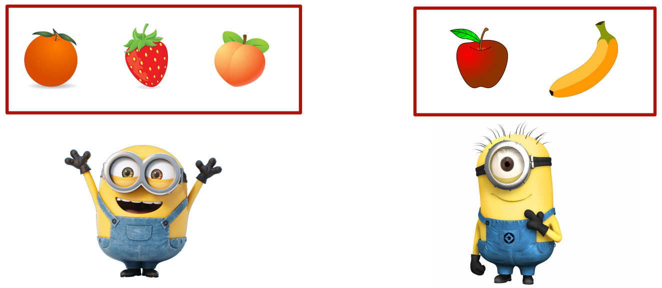
MMS = 6

MMS allocation:

10 6



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Related Work: Lower Bounds

- 2011: MMS is introduced [Budish'11]
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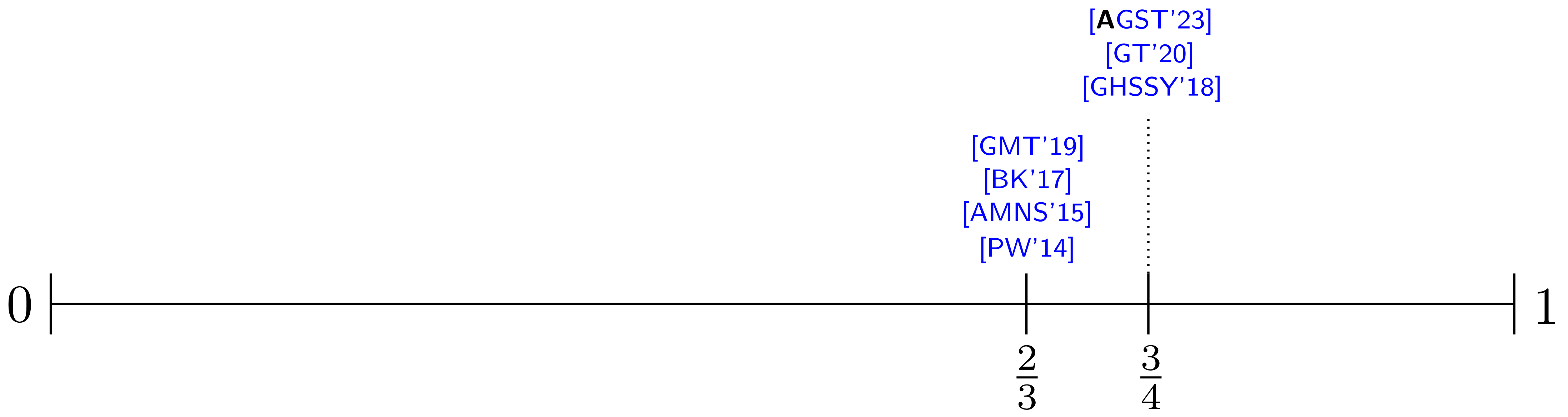

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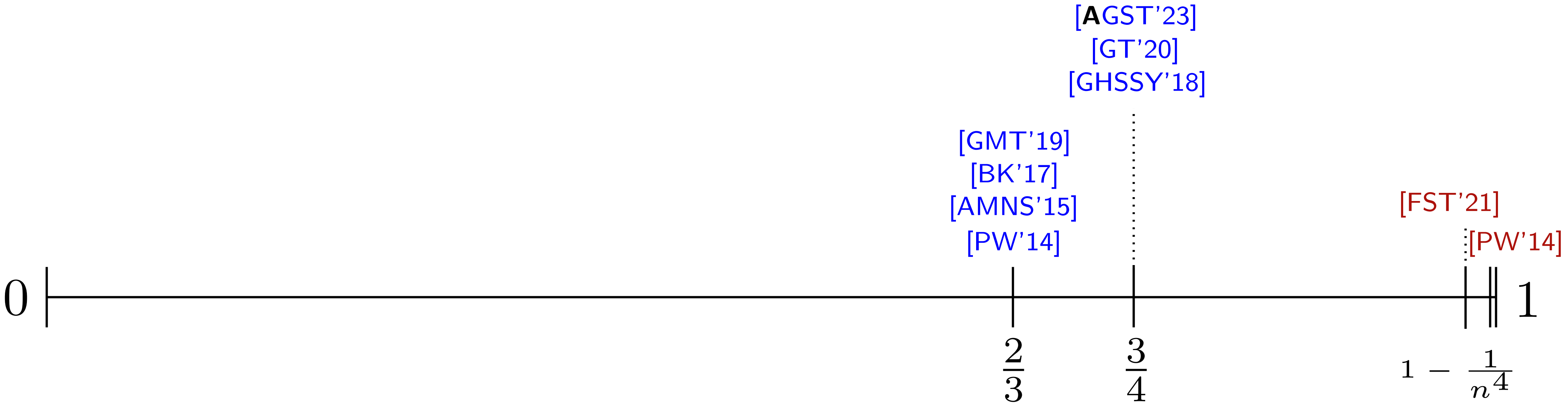

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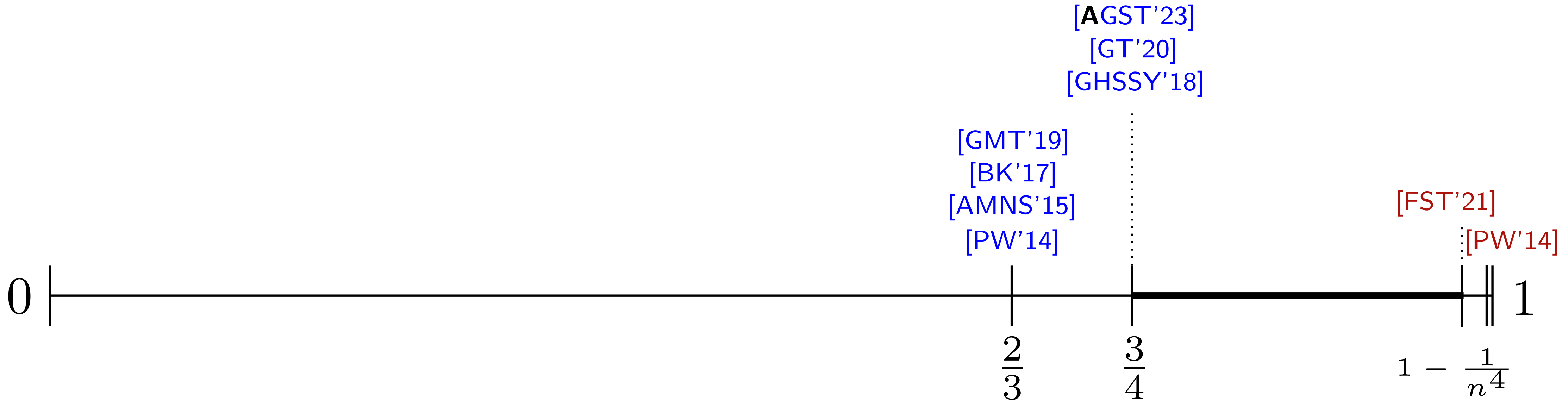
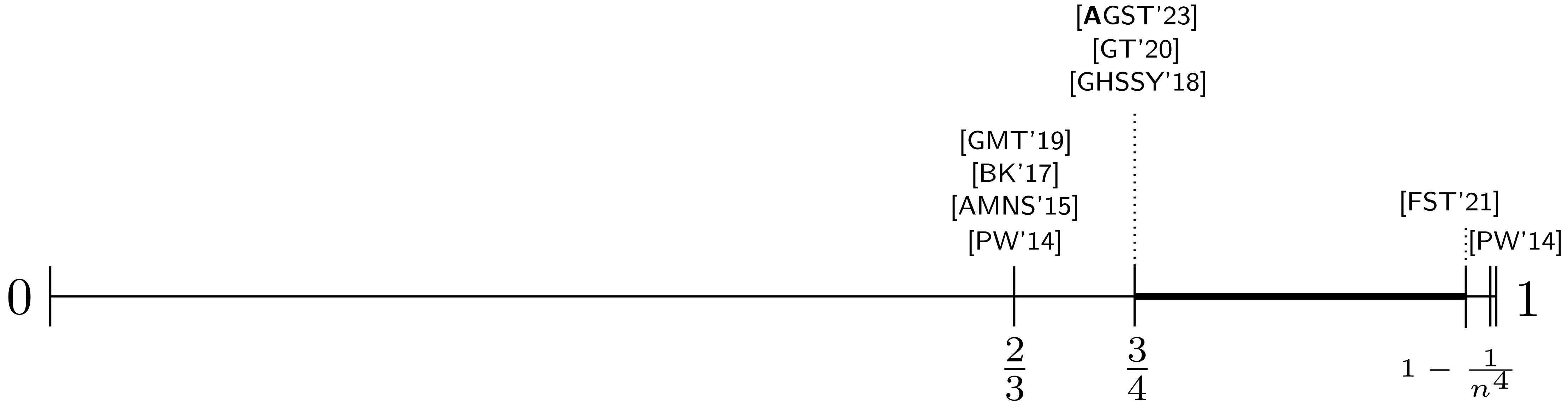


Our Result



[AGST'23]
[GT'20]
[GHSSY'18]
[GMT'19]
[BK'17]
[AMNS'15]
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[FST'21]
[PW'14]
0
1
$\frac{2}{3}$
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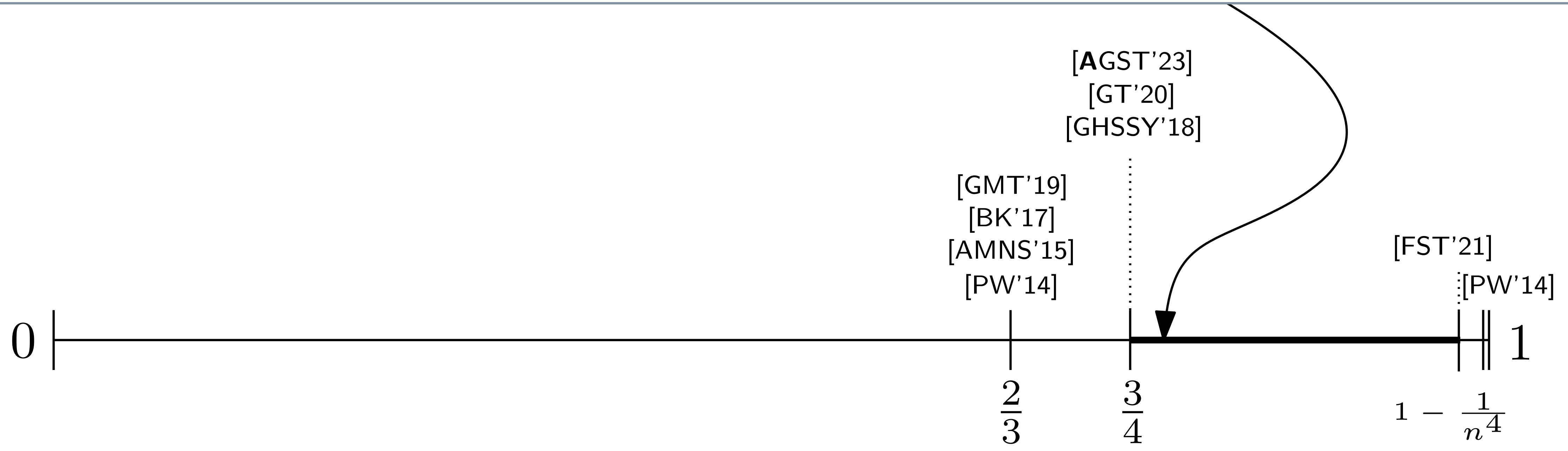
Theorem [A., Garg SODA'24]

When agents have additive valuations, there always exists a $(3/4 + \epsilon)$-MMS allocation for $\epsilon \approx 0.0007$.



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