**Auction Design: Max Revenue**

**How to sell a used car?**
- negotiate
- market research
- listed price (to see what it's worth)
- run an auction
- advertise

**Given:**
- 1 item
- n buyers, \( v_i \sim F_i \)
- Sell item to max revenue

**Mechanism:**

- Bidders w/ values \( v_i \) submit \( x_i \)
- Mechanism allocates item
- Payments

**Bayesian Nash Equil (BNE)**
- Strategies: \( \{v_i\} \rightarrow \{x_i\} \)
- Common prior \( v_i \sim F_i \)
- Outcomes \( x_i, F_i(v_i) \)
- Interim outcomes \( x_i(v_i) = E_{F_i}\left[ x_i | v_i \right] \)
- Interim utility \( u_i(v_i) = v_i x_i(v_i) - p_i(v_i) \)

**Example A: 2nd price auction**
2 bidders \( v_i \sim U[0,1] \)

Mechanism: solicit bids \( b_i \)
- if \( b_i > b_j \), \( v_i \) wins and pays \( b_j \)
- else \( b_j > b_i \), \( v_j \) wins

Equilibrium: what is \( v_i \)'s best response to \( b_j \)?

\[
\begin{align*}
& b_i > b_j : v_i \leftarrow b_j \\
& b_i < b_j : v_i \leftarrow b_i \\
& \text{assume } v_i > b_i \\
& \text{assume } v_j < b_j
\end{align*}
\]

* A truthful dominant strategy equil.

**Example B: 1st price auction**

Mechanism: solicit bids \( b_i \)
- if \( b_i > b_j \), \( v_i \) wins and pays \( b_j \)
- else \( b_j > b_i \), \( v_j \) wins

Equilibrium: Guess and check, \( v_i = v_f \)
- if \( \text{I bid } b \), \( F_i[v_i] = E_{F_i}\left[ v_i | b > v_i \right] = F_i[b > v_i] = P_i[v_i < b] = 2b - v_i > v_f \)
- best response: given \( v_f \), pick \( b^* = \arg \max (v_f - b^*\) \)

\[
\begin{align*}
& b^* = \arg \max (v_f - b^*) \\
& = \arg \max (2b^* (v_f - b^*)) \\
& = v_f/2
\end{align*}
\]

**Question:** \( E_{v_i, v_f} [\text{Rev(A)}] = 1/3 \equiv E_{v_i, v_f} [\text{Rev(B)}] = 1/3 \)

**Optimal Revenue? Example C.**

2nd price auction w/reserve \( r > v \): if higher bid \( > r \), win + pay \( max(r, 2^{nd} \text{ highest bid}) \)

Revenue: label bidders \( t_i \) s.t. \( u_i > v_2 \)

\[
\begin{align*}
\text{case 1: } & r \geq v_i > v_2 \quad r^2 \\
\text{case 2: } & v_i > v_2 \geq r \quad \left(1-r^2 \right) \\
\text{case 3: } & v_i \geq r > v_2 \quad 2(1-r)r
\end{align*}
\]

\[
\begin{align*}
\text{optimal at } r &= 1/2 \\
\text{rev} &= \frac{5}{12}
\end{align*}
\]

Bayesian assumption: \( v_i \sim F_i \), \( F_i \) is common knowledge.

Example. 1 buyer, \( v_i \sim U[0,1] \)

Optimal posted price?

\[
\begin{align*}
\text{rev}(p) &= p \cdot F_i[v_i \geq p] = p \cdot (1-p) \\
\text{or}\text{p}^* &= \arg \max p(1-p) = \arg \max (p - p^2) \\
&= \frac{1}{2} \\
\text{rev} &= p \cdot (1-p) = 1/4
\end{align*}
\]

**def BNE:** \( v_x \) is an \( x \)

\[ v_i x_i(v_i) - p_i(v_i) \geq v_i x_i(2) - p_i(2) \]

(assume \( sc \) is empty)
A mechanism is direct if \( b_i = \text{vals}_i \). The Revelation Principle: Any outcome \((x_i, p_i)\) implemented by some mechanism in an equilibrium can be implemented by an incentive-compatible direct mechanism.

**Proof (Sketch):** Given a mechanism, strategies \( \xi \), direct mechanism inputs \( v + \text{feats} = \xi(v) \) to original mechanism.

**Characterization Theorem:** \((x_i, p_i)\) are the BNE of a mechanism if and only if:

1. **Monotonicity:** \( x_i(v_i) \) monotone non-decreasing.
2. **Payment Identity:** \( p_i(v_i) = v_i x_i(v_i) - \sum_{i'} x_i(v_i')d_{i'} \)

**Proof:** (i) \( \Rightarrow \) BNE:

\[
\begin{align*}
\text{Surplus: } & v x_i(v_i) \\
\text{Payment: } & by \xi \\
\text{Utility: } & surplus - payment
\end{align*}
\]

**Q:** Could agent with value \( v \) benefit by impersonating an agent of value \( v' \)?

**BNE \( \Rightarrow \) (i) \& (ii):** Follow incentive constraints

\[
\begin{align*}
\forall v \in (v(v_i') - p(v_i') \geq v x_i(v_i') - p(v_i')) \\
\forall v(x(v_i') - p(v_i') \geq v_i x_i(v_i') - p(v(v_i))
\end{align*}
\]

**Consequence (Revenue Equivalence):**

Auctions w/same alloc in BNE have the same revenue.

**Example:** 1st price auction: 2 bidders, \( v_i \in [0, 1] \)

- Guess \(\xi(v)\) is monotone in \( v \) \( \Rightarrow \) same alloc as 2nd price auction.
- \( p(v) = p[v \text{ wins in 2nd price}] \)
  \( = E[2nd \text{ price payment} | v] \)
  \( = E[v \text{ wins in } 2nd \text{ price}] \times E[2nd \text{ highest value} | v \text{ is highest}] \)
  \( \Rightarrow \xi(v) = E[2nd \text{ highest value} | v \text{ is highest}] = \frac{v}{2} \)

Since \( \frac{v}{2} \) is monotone in \( v \), it must be a BNE.
\[ \text{Since } \frac{v}{2} \text{ is monotone in } v, \text{ it must be a BNE.} \]

This time: optimizing BNE, Myerson's virtual val

Recall Characterization Thm.

\[ (x,p) \text{ implementable in BNE of some mech.} \]

\[ \Downarrow \]

monotonicity \( x_i(v_i) \) monotone non-decreasing
payment identity \( p_i(v_i) = v_i x_i(v_i) - \int_0^{v_i} x_i(z) dz + p_o \)

Lemma: [Myerson '81]

\[ E [p_i(v_i)] = E [\Phi_i(v_i) x_i(v_i)] \]

where \( \Phi_i(v_i) = v_i - \frac{1-F(v_i)}{f(v_i)} \) is the virtual value.

Approach:
- calculate virtual values \( \Phi_i \)
- choose \( x \) to max \( E [\Phi_i(v_i) x_i(v_i)] \)
- check that \( x \) is monotone
- use payment identity to calc. \( p \)

Example \( A^1 \) buyer, \( v \sim U[0,1] \)

\[ \Phi(v) = v - \frac{1-v}{1} = 2v-1 \]

\[ \begin{aligned}
\text{Example } E \text{ n buyers, } v_i \sim U[0,1] \\
\Phi_i(v_i) &= 2v_i - 1 \\
\det & \max_x \sum_i \Phi_i(v_i) x_i(v_i) \}
\end{aligned} \]

allocate to highest \( v_i \) if \( v_i \geq 1/2 \)

\[ \Rightarrow 2^\text{nd price auction} \]

w/ reserve \( = 1/2 \)

Pf. (of Myerson's Lemma)

\[ E[p_i(v)] = E[v x_i(v) - \int_0^{v_i} x_i(z) dz] \]

\[ = \int_0^1 \left( v x(v) - \int_0^v x_i(z) dz \right) f(v) dv \]

\[ = \int_0^1 v x(v) f(v) dv - \int_0^1 \int_0^v \frac{1}{2} x_i(z) dz f(v) dv \]

recall integration by parts: \( \int_a^b h dz g = [h a - b] - \int_a^b g dh \)

\[ = \int_0^1 v x(v) f(v) dv - \int_0^1 \left( \int_0^v x_i(z) dz \left[ F(v) - 1 \right] \right) \left[ \frac{v}{2} - \int_0^v (cF(v)-1) x(v) dv \right] \]

\[ = \int_0^1 x(v) (v f(v) + (F(v) - 1)) dv \]

\[ = \int_0^1 x(v) \left( v - \frac{1-F(v)}{F(v)} \right) f(v) dv \]
\[ E \left[ (v - \frac{1 - F(v)}{rC}) \times C(v) \right] \]