ADFOCS 2020: Fair Division
Problem Set 4

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1. Assume that a set M of m goods need to be fairly divided among a set N of n agents with additive valuations. Show that
   a. Envy-freeness up to one item (EF1) implies proportionality up to one item (Prop1)
   b. Prop1 does not imply EF1
2. Suppose there are n agents with additive valuations. Show that
   a. Show that MMS allocations exist when there are two agents.
   b. EF1 implies $1/n$-MMS.
   c. Show an example where EF1 implies $\Omega(1/n)$-MMS.
   d. Show an example where an MMS allocation is not EF1.
3. *a.* Show that 6/7-MMS allocation exists for three agents.[1]
   b. Show that 4/5-MMS allocation exists for four agents.
4. Consider the case of non-symmetric agents, where w_i is the weight of agent i. An allocation $A = (A_1, \ldots, A_n)$ is
   a. weighted Prop1 if there exists $g \in M$, such that $v_i(A_i \cup g) \geq \frac{w_i}{\sum_i w_i} v_i(M), \forall i$. Design a polynomial-time algorithm to output an allocation that is weighted Prop1 + PO.
   b. weighted EF1 if $\frac{v_i(A_i)}{w_i} \geq \frac{v_i(A_j \setminus g)}{w_j},\ g \in A_j,\ \forall i, j$. Show the existence of weighted EF1 + PO allocation. [Hint: Extend the EF1+PO approach in Lecture 3]
5. Assume that agents have additive valuations. Show that an allocation that maximizes the Nash welfare (MNW)
   a. is EF1 + PO.
   b. may not be EFX.
   c. is EFX when agents have identical valuations.
6. Recall the notations from the $O(n)$-algorithm for MNW under subadditive valuations done in the lecture, show that
   a. $v_i(A_i) \geq \frac{v_i(M \setminus Y)}{4n}$.
   b. $v_i(M \setminus Y) \geq v_i(M \setminus H_i) - n v_i(y_i^*)$. [Hint: $|Y| = n$.]

[1] The best known factor is 8/9 for three agents in *Approximate Maximin Share Allocations in Matroids* by Laurent Gourvès 1 Jérôme Monnot.