Fair Division of Indivisible Items

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UNIVERSITY OF ILLINOIS URBANA-CHAMPAIGN

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Recap

- Set N of n agents, Set M of m divisible items
- Agent i has a utility function $u_i\colon \mathbb{R}_+^m \to \mathbb{R}$ over bundle of items
- Goal: fair and efficient allocation $x = (x_1, \ldots, x_n)$

Fairness:

Envy-free (EF)

Proportionality (Prop)

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$\equiv$ Competitive Equilibrium with Equal Incomes (CEEI)

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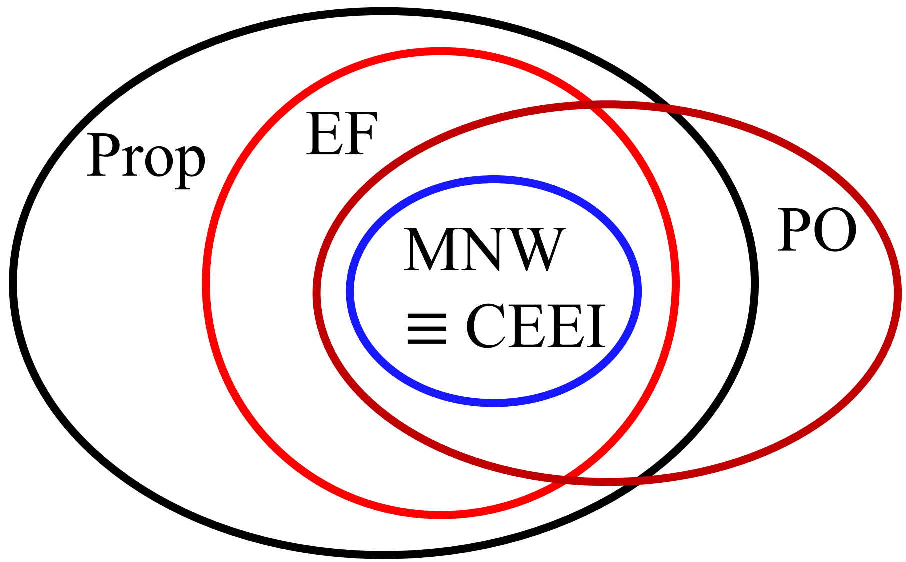

Maximum Nash Welfare (MNW)
≡ Competitive Equilibrium with Equal Incomes (CEEI)

Today: Indivisible Items

- n agents, m indivisible items (like cell phone, painting, etc.)
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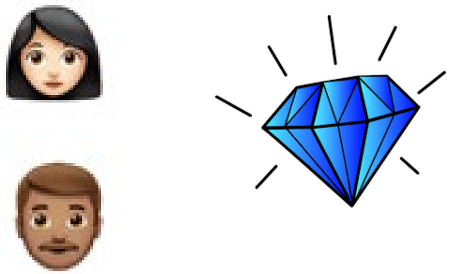

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Fairness Notions for Indivisible Items

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EF1 EFX	Lecture 3
MMS Prop1	Lecture 4
Guarantees	Lecture 5

Envy-Freeness up to One Item (EF1) [B11]

- An allocation $(A_1, \ldots, A_n)$ is EF1 if

$$v_i(A_i) \geq v_i\left(A_j \setminus g\right), \quad g \in A_j, \quad \forall i, j$$

That is, agent i may envy agent j, but the envy can be eliminated if we remove a single item from $j's$ bundle

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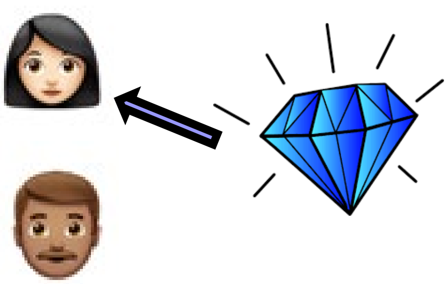
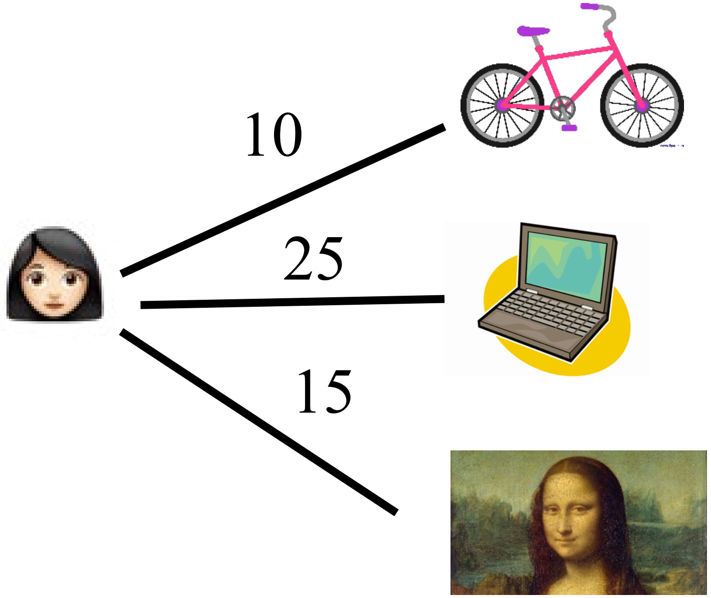
- Existence?



Additive Valuations: $v_i(S) = \sum_{j \in S} v_{ij}$



Round Robin Algorithm (Additive)

- Fix an ordering of agents arbitrarily
- While there is an item unallocated
  - i: next agent in the round robin order
  - Allocate i her most valuable item among the unallocated ones

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Claim: The final allocation is EF1

Observe that intermediate (partial) allocation is also EF1

Envy-Cycle Procedure (General) [LMMS04]

- General Monotonic Valuations: $v_i(S) \leq v_i(T), \ \forall S \subseteq T \subseteq M$

  (M: Set of all items)

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- Envy-graph of a partial allocation $(A_1, \dots, A_n)$ where $\cup_i A_i \subseteq M$
  - Vertices = Agents
  - Directed edge (i, j) if i **envies** j (i.e., $v_i(A_i) < v_i(A_j)$)

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  - Vertices = Agents
  - Directed edge (i, j) if i **envies** j (i.e., $v_i(A_i) < v_i(A_j)$)
- Suppose we have a partial EF1 allocation
- Then, we can assign one unallocated item j to a source i (in-degree 0 agent) and the resulting allocation is still EF1!
  - No agent envies i if we remove j

- If there is no source in envy-graph, then
  - there must be cycles
  - How to eliminate them?

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- EF1?
  - Valuation of each agent?
  - The bundles remain the same – We are only changing their owners!

Envy-Cycle Procedure [LMMS04]

$A \leftarrow (\emptyset, \dots, \emptyset)$

$R \leftarrow M$ // unallocated items

While $R \neq \emptyset$

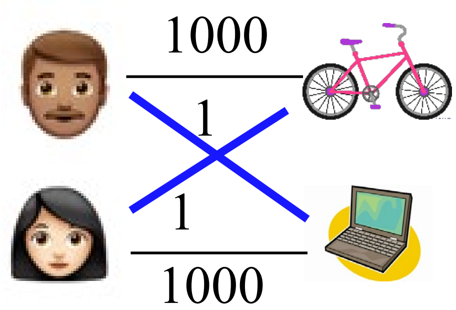
- If envy-graph has no source, then there must be cycles
- Keep removing cycles by exchanging bundles until there is a source
- Pick a source, say i, and allocate one item g from R to i

  $(A_i \leftarrow A_i \cup g;\ R \leftarrow R \setminus g)$

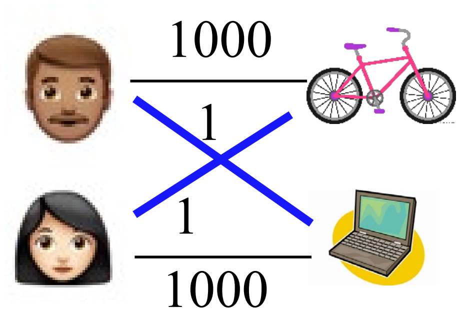
Output A

- Running Time?

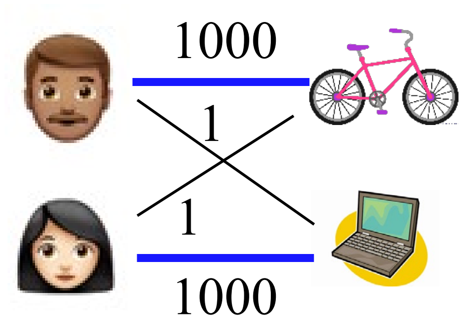
How Good is an EF1 Allocation?



How Good is an EF1 Allocation?



- Certainly not desirable!



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- Goal: EF1 + PO allocation
- Existence?
  - NO [CKMPS14] for general (subadditive) valuations
  - YES for additive valuations [CKMPS14]

submodular valuations

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submodular valuations

EF1+PO (Additive)

- Computation: pseudo-polynomial time algorithm [BKV18]

  

  Complexity of finding an EF1+PO allocation

- Difficulty: Deciding if an allocation is PO is co-NP-hard [KBKZ09]

EF1+PO (Additive)

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Complexity of finding an EF1+PO allocation

- Difficulty: Deciding if an allocation is PO is co-NP-hard [KBKZ09]

- Approach: Achieve EF1 while maintaining PO
  - PO certificate: competitive equilibrium!

Competitive Equilibrium (CE)

- m divisible items, n agents
- Utility of agent i : $\sum_j v_{ij} x_{ij}$
- Each agent has budget of B_i
- p_j: price of item j, f_{ij}: money flow from agent i to item j

Equilibrium (p, f):

1. Optimal bundle: $f_{ij} > 0 \Rightarrow \frac{v_{ij}}{p_j} = \max_{k \in M} \frac{v_{ik}}{p_k}$

   Maximum bang-per-buck (**MBB**) condition

2. Market clearing:

$$\sum_{j \in M} f_{ij} = B_i, \forall i \in N \quad \text{and} \quad \sum_{i \in N} f_{ij} = p_j, \ \forall j \in M$$

EF1+PO (additive) [BKV18]

- Approach: Achieve EF1 while maintaining PO
- Starting allocation $A = (A_1, \dots, A_n)$:
  - Each item j is assigned to an agent with the highest valuation
  - Set price of item j as $p_j = \max_i v_{ij}$
- $p(A_i)$: total price of all items in $A_i \equiv$ total valuation of i

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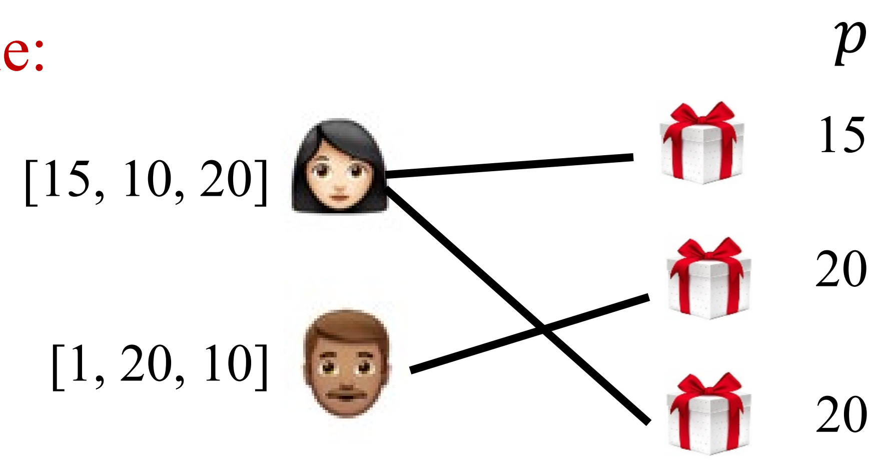
Example:



- Consider the integral allocation $A = (A_1, \dots, A_n)$
  - Each item j is assigned to an agent with the highest valuation
  - Set price of item j as $p_j = \max_i v_{ij}$
- $p(A_i)$: total price of all items in $A_i \equiv$ total valuation of i

Claim: (A, p) is (integral) CE when agent i has $p(A_i)$ budget and linear utility function $\sum_j v_{ij} x_{ij}$

Equilibrium (p, f):

1. Optimal bundle (**MBB**):

$$f_{ij} > 0 \Rightarrow \frac{v_{ij}}{p_j} = \max_{k \in G} \frac{v_{ik}}{p_k}$$

2. Market clearing:

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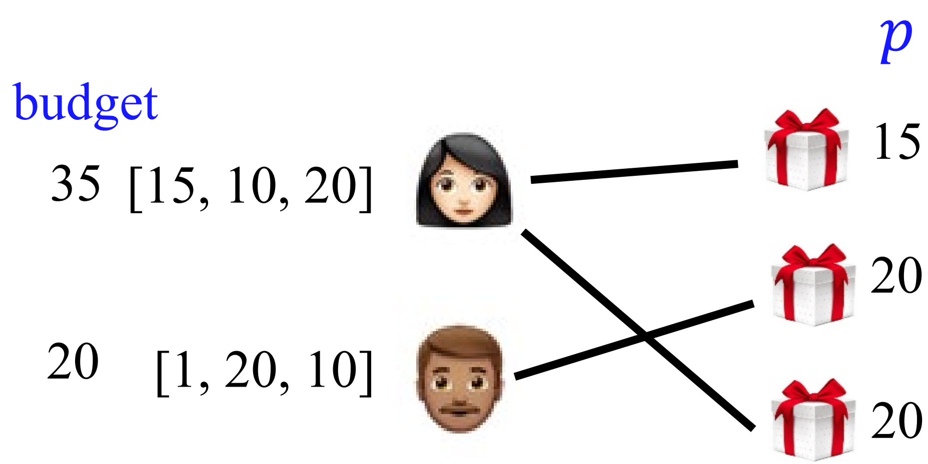

Scaling Valuations with Prices

- Recall that envy-freeness is scale-free
- (A, p): CE
- Let's scale $v_{ij} \leftarrow v_{ij} \cdot \min_k \frac{p_k}{v_{ik}}$

$\Longrightarrow v_{ij} \leq p_j$ and $v_{ij} = p_j$ if $j \in A_i$

Prices can be treated as valuations at CE!

Price-Envy-Free [BKV18]

- (A, p): CE
- A is Envy-Free (EF) if

$$v_i(A_i) \geq v_i(A_j), \qquad \forall i, j$$

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EF?

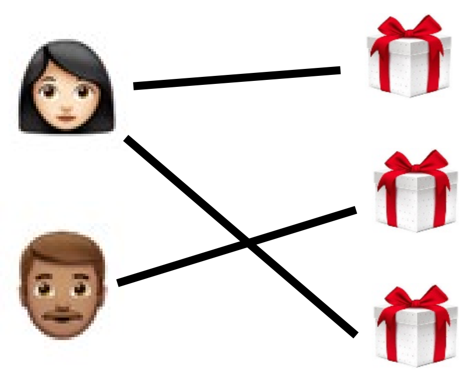
$$35 = v_1(A_1) \geq v_1(A_2) = 10$$

$$20 = v_2(A_2) \geq v_2(A_1) = 11$$

budget

35 [15, 10, 20]

20 [1, 20, 10]



p

15

20

20

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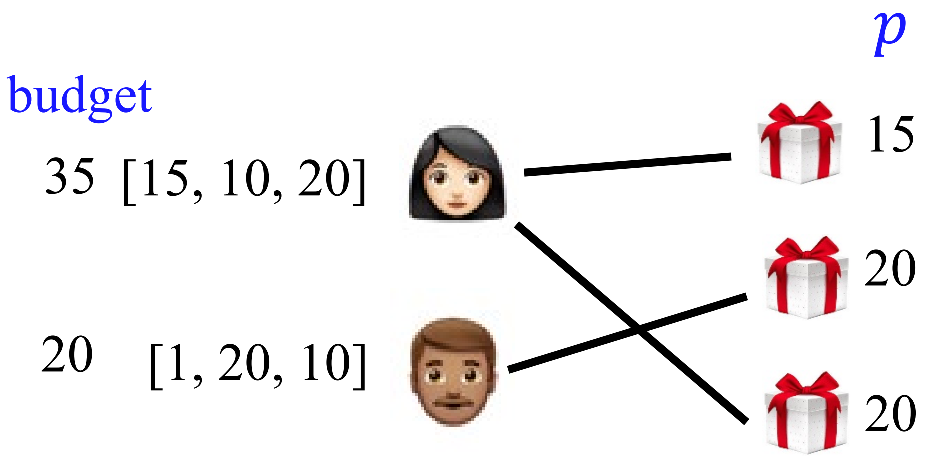

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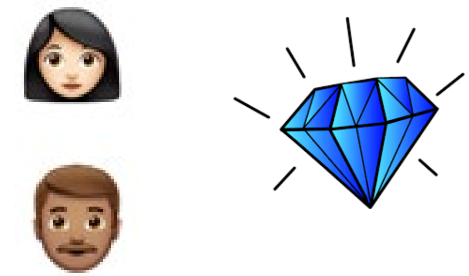

May not exist!

- A is Price-Envy-Free (pEF) if

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- pEF $\Rightarrow$ EF + PO

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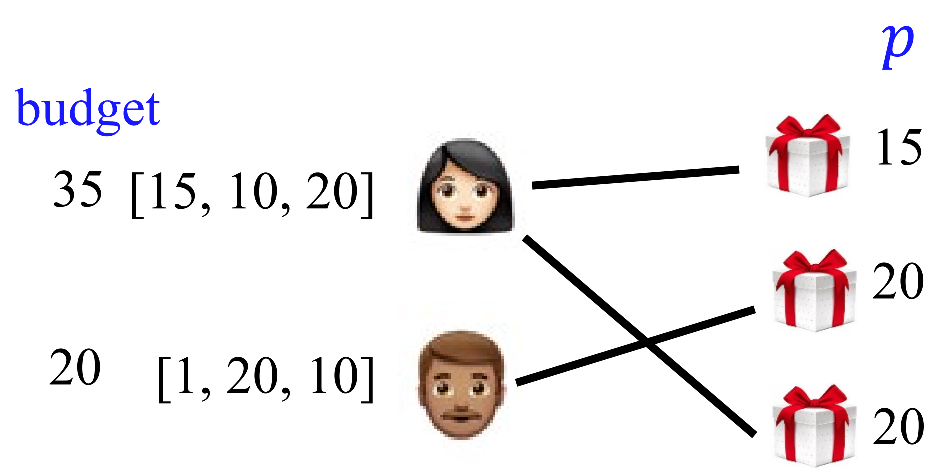

- (A, p): CE
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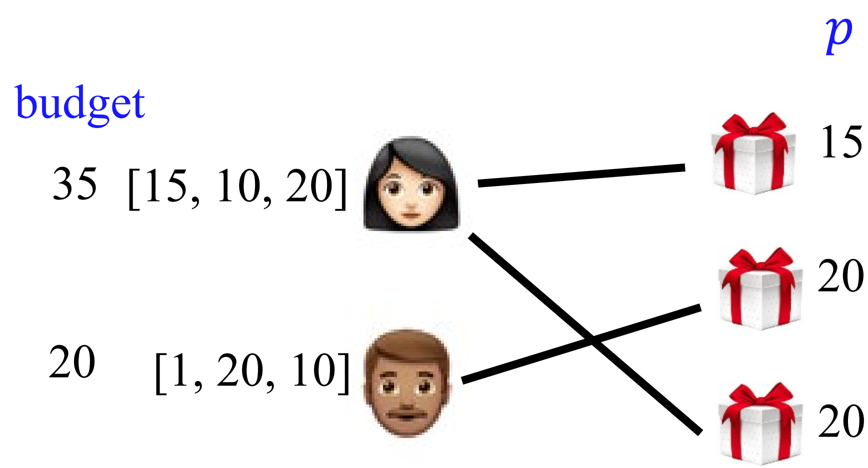
pEF1?

$$35 = p(A_1) > p(A_2 \setminus g_2) = 0$$

$$20 = p(A_2) > p(A_1 \setminus g_3) = 15$$



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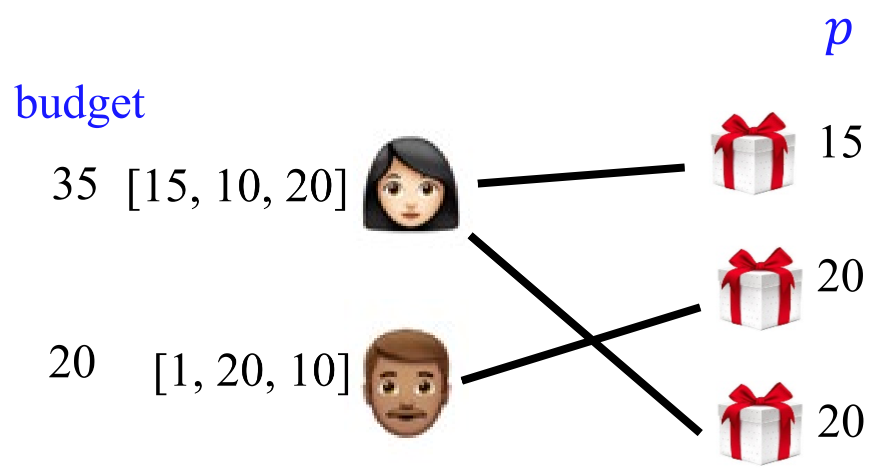


Theorem [BKV18]: There exists a pseudo-polynomial time procedure to find a pEF1 allocation

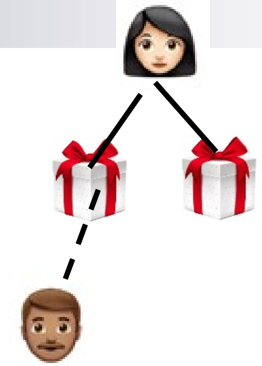
- (A, p): CE
- A is pEF1 if

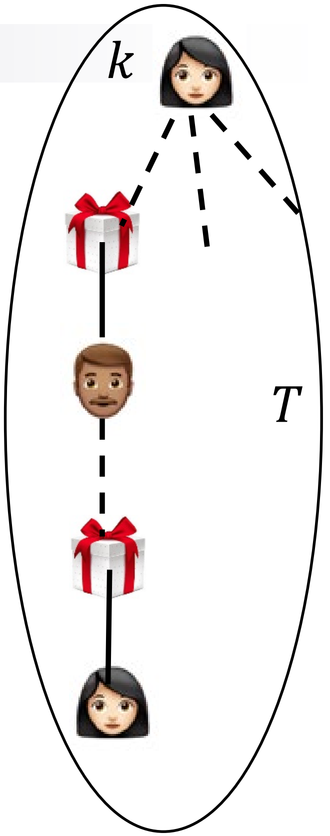
$$p(A_i) \geq p\left(A_j \setminus g\right), \qquad g \in A_j, \qquad \forall i, j$$

- If $\min_i p(A_i) \geq \max_j \min_{g \in A_j} p\left(A_j \setminus g\right)$ then ?

  (least spender) (big spender)

Procedure [BKV18]

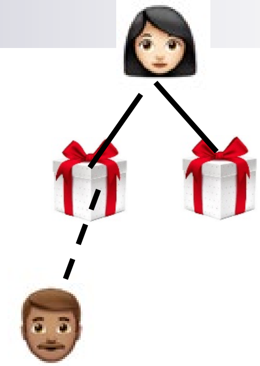




While A is not pEF1

$k \leftarrow \arg\min_i p(A_i)$ //least spender

$T \leftarrow$ Agents and items, k can reach in MBB residual network



While A is not pEF1

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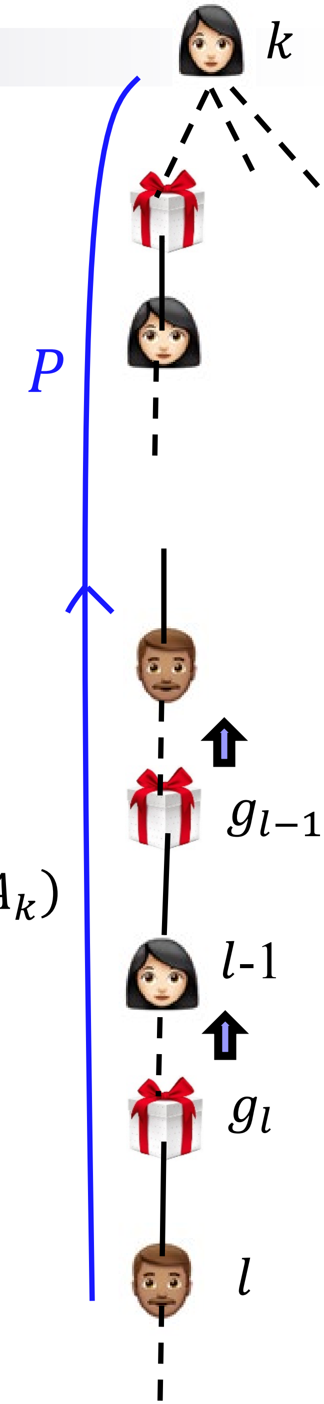
$T \leftarrow$ Agents and items, k can reach in MBB residual network

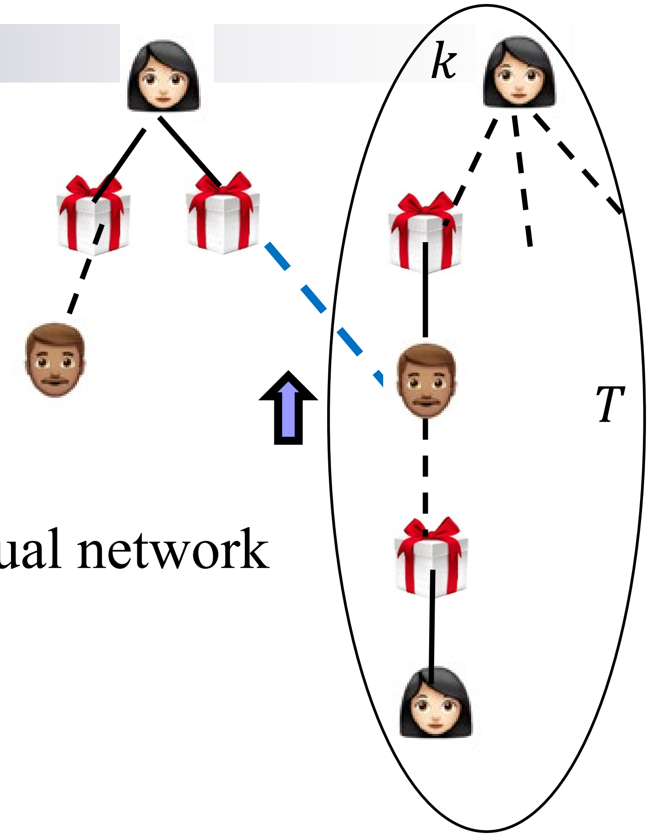
If k can reach l in T such that $p(A_l \setminus g_l) > p(A_k)$

Pick the nearest such l

$P \leftarrow$ Path from l to k

$A \leftarrow$ Reassign items along P until $p\big((A_j \cup g_{j+1}) \setminus g_j\big) \le p(A_k)$





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Pick the nearest such l

$P \leftarrow$ Path from l to k

$A \leftarrow$ Reassign items along P until $p\big((A_j \cup g_{j+1}) \setminus g_j\big) \leq p(A_k)$

else increase prices of items in T by a same factor until

Event 1: new MBB edge

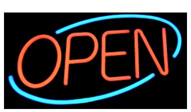
Event 2: k is not least spender anymore

Event 3: A becomes pEF1

$\min_i p(A_i)$ $\quad\quad$ $\max_j \min_{g \in A_j} p(A_j \setminus g)$

Lemma: The procedure converges to a pEF1 allocation in finite time!

Pseudo-polynomial time: Round $v'_{ij}s$ to the nearest integer powers of $(1+\epsilon)$ for a suitably small $\epsilon > 0$ and then run the procedure



Complexity of finding an EF1+PO allocation!

Analysis [BKV18]

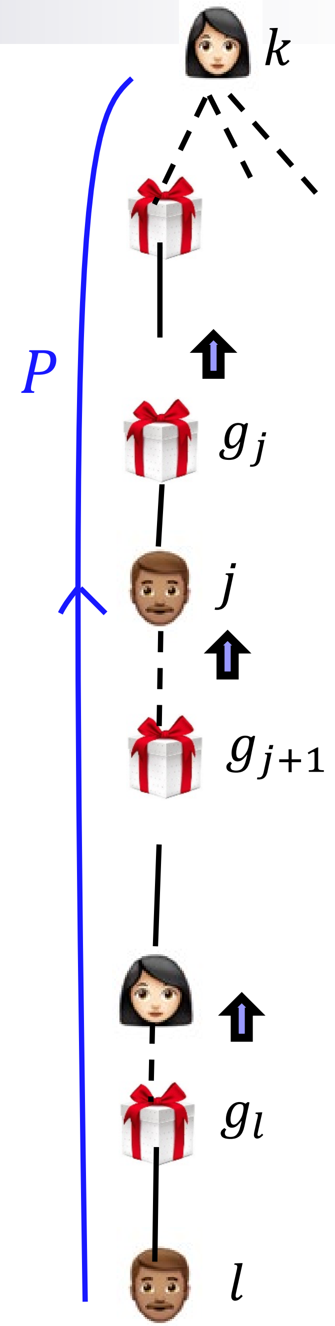
Lemma: $\min_i p(A_i)$ ⇧

Proof (sketch): prices ⇧

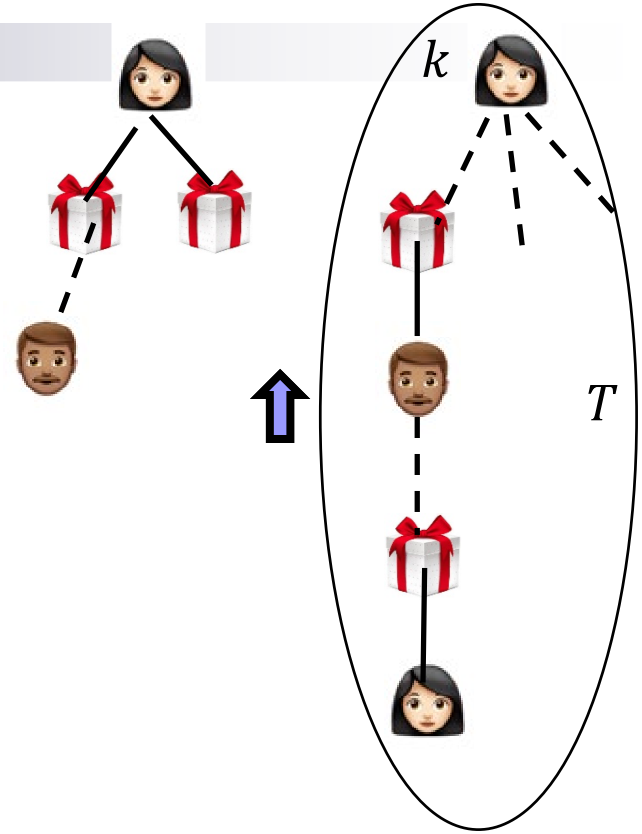
- $p(A_i)$ can only increase for agents not on P
- For agents on P

  l: $p(A_l \setminus g_l) > p(A_k)$

  j: $p\big((A_j \cup g_{j+1}) \setminus g_j\big) > p(A_k)$



Lemma: $\max_j \min_{g \in A_j} p(A_j \setminus g)$ ⇓ (big spender)



Proof (sketch)

- $\max_j \min_{g \in A_j} p(A_j \setminus g) > \min_i p(A_i)$
- Prices ⇑ ⇒ No big spender is in T

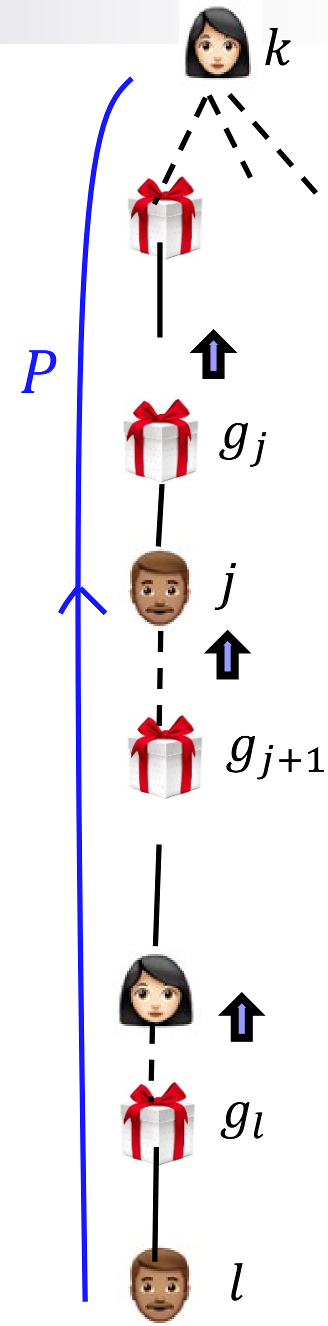
Lemma: $\max_j \min_{g \in A_j} p(A_j \setminus g)$ $\Downarrow$ (big spender)

Proof (sketch)

- $\max_j \min_{g \in A_j} p(A_j \setminus g) > \min_i p(A_i)$
- Prices $\Uparrow$ $\Rightarrow$ No big spender is in T
- On path P:
  - j: $\quad p(A_j \setminus g_j) < p(A_k)$

$$p((A_j \cup g_{j+1}) \setminus g_j) > p(A_k)$$

$$p((A_j \cup g_{j+1} \setminus g_j) \setminus g_{j+1}) = p(A_j \setminus g_j) < p(A_k)$$

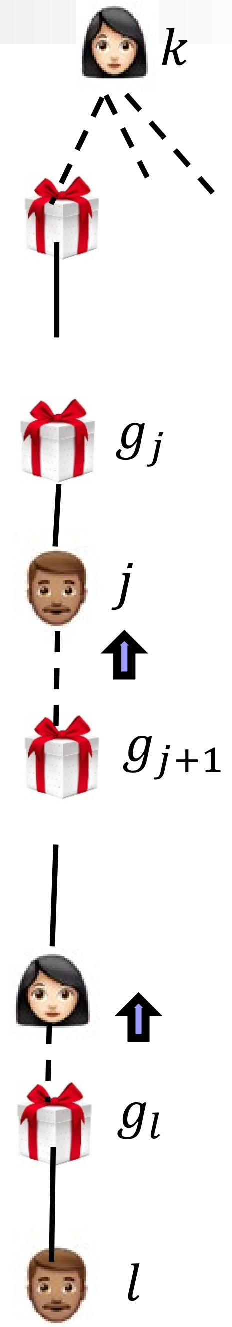


Lemma: $\max_j \min_{g \in A_j} p(A_j \setminus g)$ $\Downarrow$ (big spender)

Proof (sketch)

- $\max_j \min_{g \in A_j} p(A_j \setminus g) > \min_i p(A_i)$
- Prices $\Uparrow$ $\Rightarrow$ No big spender is in T
- On path P:
  - j: $p(A_j \setminus g_j) < p(A_k)$

    $p((A_j \cup g_{j+1}) \setminus g_j) \leq p(A_k)$



New Fairness Notions

- n agents, m indivisible items (like cell phone, painting, etc.)
- Each agent i has a valuation function over subset of items denoted by $v_i : 2^m \rightarrow \mathbb{R}$
- Goal: fair and efficient allocation

Fairness:
  Envy-free (EF)
  Proportionality (Prop)

Efficiency:
  Pareto optimal (PO)

Maximum Nash Welfare (MNW)

EF1 EFX	Lecture 3
MMS Prop1	Lecture 4
Guarantees	Lecture 5

Envy-Freeness up to One Item (EF1)

- An allocation $(A_1, \ldots, A_n)$ is EF1 if

$$v_i(A_i) \geq v_i\left(A_j \setminus g\right), \qquad g \in A_j, \qquad \forall i, j$$

That is, agent i may envy agent j, but the envy can be eliminated if we remove a single item from $j's$ bundle

Envy-Freeness up to Any Item (EFX) [CKMPS14]

- An allocation $(A_1, \ldots, A_n)$ is EFX if

$$v_i(A_i) \geq v_i(A_j \setminus g), \quad \forall g \in A_j, \quad \forall i, j$$

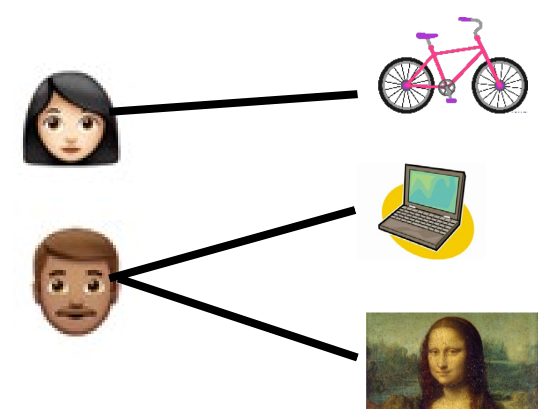
That is, agent i may envy agent j, but the envy can be eliminated if we remove any single item from $j's$ bundle

EF1 ?

EFX ?

[15, 10, 20]

[1, 20, 10]



EFX: Existence

- General Valuations [PR18]
  - Identical Valuations
  - $n = 2$

- Additive Valuations
  - $n = 3$ [CG.M20]

Additive ($n > 3$), General ($n > 2$)

"Fair division's biggest problem" [P20]

Summary

Covered

- EF1 (existence/polynomial-time algorithm)
- EF1 + PO (existence/pseudo-polynomial time algorithm)
- EFX

Not Covered

- EFX for 3 (additive) agents
- Partial EFX allocations
  - Little Charity [CKMS20]
  - High Nash welfare [CGH19]
- Chores
  - EF1 (existence/ polynomial-time algorithm)

Major Open Questions (additive valuations)

- EF1+PO: Polynomial-time algorithm
- EF1+PO: Existence for chores
- EFX : Existence

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