Two-Sided Matching

Match agents to agents.

[[Both sides have preferences.]]

- Job markets: match workers to firms (e.g., NRMP, crowdwork)
- School choice: match students to schools (if schools have priorities over students)
- Marriage markets: match people to each other (with two distinct groups that match to each other)

Goals:

- Pareto efficiency: no one can improve without harming others
- No justified envy: if $a$ envies $b$’s match, then $b$’s match prefers $b$ to $a$
- Algorithmic: polytime alg to find matching
- Strategyproof (SP): reporting true pref’s maximizes rank of match

Model

Def: A two-sided one-to-one matching market has

- set $M$ of $m$ men
- set $W$ of $n$ women

Def: Preferences $\succeq_x$ of agent $x$ are strict total orders over

- $W \cup \{m\}$ for man $m$
- $M \cup \{w\}$ for woman $w$

where $a \succ_x b \rightarrow$ agent $x$ prefers $a$ to $b$ and $x \succ_x a \rightarrow$ agent $x$ prefers being unmatched to $a$.

Def: A matching $\mu : M \rightarrow W$ is a one-to-one mapping. Overloading notation, if $\mu(m) = w$, we say $(m, w) \in \mu$ and $\mu(w) = m$.

Def: A matching $\mu$ is Pareto efficient (PE) if there is no matching $\nu$ s.t.

- $\nu(x) \succeq_x \mu(x)$ for all agents $x$
- and $\nu(x) \succ_x \mu(x)$ for some agent $x$.

Def: A matching $\mu$ is stable (aka has no justified envy) if it is

- individually rational (IR): $x$ prefers $\mu(x)$ to being single,
- and there is no blocking pair $(m, w)$ s.t. $m \succ_w \mu(m)$ and $w \succ_m \mu(m)$.

Goal: Find a PE and stable $\mu$ if it exists.

Example: Can you find a stable matching here?
Matching \( \mu = \{(m_1, w_1), (m_2, w_2), (m_3, w_3)\} \) is PE but not stable because \((m_3, w_2)\) are a blocking pair.

**Note:** When preferences are strict, a stable matching is always PE.

**Deferred Acceptance**

**Algorithm:** Tattonnement (Sketch)

1. Buyers increasing prices
2. Sellers tentatively accept highest offer, rejecting others

Buyers’ options get worse, sellers’ get better.

**Algorithm:** Men-proposing Deferred Acceptance (Sketch)

1. Men successively offer to marry favorite woman who hasn’t rejected them
2. Women tentatively accept best man, rejecting others

Men’s options get worse, women’s get better.

**Note:** For convenience, assume complete lists, i.e., \( a \succ x x \) for all agents \( a \).

**Algorithm:** Men-Proposing Deferred Acceptance (m-DA)

1. Let \( \mu(m) = m \) for all \( m \in M \).
2. Let \( S \) be the set of unmatched men, i.e., \( S = \{ m : \mu(m) = m \} \).
3. While there’s an unmatched man \( m \in S \),

(a) Man \( m \) applies to favorite woman \( w \) who has not yet rejected him.

(b) Let \( m' = \mu(w) \) be \( w \)'s current match. If \( m \succ_w m' \), \( w \) rejects \( m' \) \( (\mu(m') = m') \) and tentatively accepts \( m \) \( (\mu(m) = w) \).

4. Return matching \( \mu \).

**Example:** For pref’s in previous example,

1. \( m_1 \rightarrow w_1, \mu = \{(m_1, w_1)\} \)
2. \( m_2 \rightarrow w_2, \mu = \{(m_1, w_1), (m_2, w_2)\} \)
3. \( m_3 \rightarrow w_2, \mu = \{(m_1, w_1), (m_3, w_2)\} \)
4. \( m_2 \rightarrow w_3, \mu = \{(m_1, w_1), (m_2, w_3), (m_3, w_2)\} \)

**Theorem 1** DA computes a stable matching.

**Proof:**

- Terminates: each man proposes to each woman at most once.

- Stable: IR since pref’s complete. No blocking pairs since,

  - if \( w \succ m \mu(m) \), \( m \) proposed to \( w \)
  - if \( w \) rejected \( m \), it was for \( m' \) where \( m' \succ_w m \)
  - \( w \)'s options only improve, so \( \mu(w) \succ_w m' \succ_w m \)

so \((m, w)\) don’t block.
Properties

Example: w-DA for running example

1. \( w_3 \rightarrow m_1, \mu = \{(m_1, w_3)\} \)
2. \( w_2 \rightarrow m_1, \mu = \{(m_1, w_2)\} \)
3. \( w_3 \rightarrow m_3, \mu = \{(m_1, w_2), (m_3, w_3)\} \)
4. \( w_1 \rightarrow m_3, \mu = \{(m_1, w_2), (m_3, w_3)\} \)
5. \( w_1 \rightarrow m_2, \mu = \{(m_1, w_2), (m_2, w_1), (m_3, w_3)\} \)

Note: Every person prefers outcome when they propose!

Claim: There’s a unique man-optimal stable matching and m-DA finds it. In fact, stable matchings form a lattice.

Claim: m-DA is group SP for men.

Example: women’s incentives for m-DA, suppose \( w_1 \) claims \( m_1 \) unacceptable.

1. \( \ldots, \mu = \{(m_2, w_3), (m_3, w_2)\} \)
2. \( m_1 \rightarrow w_1, \mu = \{(m_2, w_3), (m_3, w_2)\} \)
3. \( m_1 \rightarrow w_2, \mu = \{(m_1, w_3), (m_2, w_3)\} \)
4. \( m_3 \rightarrow w_3, \mu = \{(m_1, w_2), (m_3, w_3)\} \)
5. \( m_2 \rightarrow w_1, \mu = \{(m_1, w_2), (m_2, w_1), (m_3, w_3)\} \)

This is a rejection chain.

Note: No mech that always outputs stable \( \mu \) is SP: if \( x \) doesn’t always get favorite stable partner \( y \), can report \( y \) as only acceptable unique.

Claim: Best partner agent can receive is favorite stable partner.

Unique stable partners

Can you identify a restriction on prefs in which stable partners are unique?

Theorem 2 (Rural hospital or lone wolf theorem.) Set of unmatched agents same at every stable matching.

Proof:

- for matching \( \mu \), let \( \mu(M) = \) matched women, \( \mu(W) = \) matched men.
- consider \( \mu^M \), man-opt SM, and \( \mu \), another SM: \( \mu^M \) is
  - worst for women:
    \[ \mu(M) \supseteq \mu^M(M) \]
    so
    \[ |\mu(M)| \geq |\mu^M(M)| \]
  - best for men:
    \[ \mu(W) \subseteq \mu^M(W) \]
    so
    \[ |\mu(W)| \leq |\mu^M(W)| \]

but then, as the number of matched men and women are equal for all matchings, \( |\mu(M)| \geq |\mu^M(M)| = |\mu^M(W)| \geq |\mu(W)| \). Hence all cardinalities equal and so set containment relation implies sets are equal as well.

Note: Can count stable partners of \( x \) by

- have \( x \) start rejection chain (i.e., truncate list just above current stable partner)
• stop if single agent receives proposal
• stop if married man runs through his list

**Example:** 2 men, 3 women, \( w_1 \succ_m w_2 \succ_m w_3, m_1 \succ_w m_2 \)

To show \( \{(m_1, w_1), (m_2, w_2)\} \) unique, have \( w_2 \) truncate at \( m_2 \). Then \( m_2 \) applies to \( w_1 \) and is rejected, then applies to \( w_3 \) violating rural hospital theorem.

**Claim:** If men’s lists are random lists of length \( k << n \), almost all agents have unique stable partner.


**Reasoning:** Balls and bins

First compute \( m - DA \):

• women are bins \( \rightarrow n \) bins

• men’s proposals are balls \( \rightarrow k \) balls per man

  − \( m \) throws ball into random bin

  − if \( m \) rejected, try again up to \( k \) times

To see if \( w \) has > 1 stable partner, have \( w \) reject partner, continue alg., halt if single woman gets proposal \( \rightarrow \) probability \( 1/(\#singles + 1) \approx e^k/n. \)

**Claim:** If lists are random and \( |M| = |W| + 1 \), then ave. rank of men in w-DA is \( \approx \) ave. rank of men in m-DA!


**Discussion**

*What do you think about applying this to our examples from the beginning of class?*

• Job markets like NRMP

• School choice with coarse priorities and random tie-breaking

• Marriage markets like OKCupid