Lecture 6: Revenue Maximization

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How to sell used car?
- market research to figure out worth
- advertise to recruit buyers
- offer fixed price/negotiate
- sell at an auction

Applications: listing items on eBay, search ads 0 rooms on AirBnB

Given:
- 1 item
- n buyers, \( v_i \in \mathbb{F} \)
Sell item to maximize revenue

Mechanism:

\[
\begin{array}{ccc}
\text{Buyers \( v \)} & \text{\( v \in \mathbb{F} \)} & \text{\( v \)} \\
\end{array}
\]

Bayesian assumption: Buyers' values drawn from a known distribution.

Example: 1 buyer, \( v \sim U[0,1] \) Will show this is best among all mech.
- Optimal posted price 
  \[
  p^* = 0.5, \quad \text{rev}(p^*) = 0.25
  \]
  \[
  p^* = \arg\max p (1-p) = 0.5, \quad 2p^* p - p^* = 1 - 2p^* = 0
  \]

Bayes-Nash equilibrium (BNE)
- Strategies \( s_i : \{\text{values}\} \rightarrow \{\text{bids}\} \)
- Common prior \( v_i \sim F_i \)
- Outcomes \( x_i(s(v)) \equiv x_i(v), \quad p^*_i(s(v)) = p_i(v) \)
- Interim outcomes \( x_i(v_i) \equiv E[v_i | v_i], \quad p_i(v_i) = E_x[p_i(v_i) | v_i] \)
- Interim utility \( U_i(v_i) = v_i - p_i(v_i) \)

\[\text{def BNE} \iff v_i x_i(v_i) - p_i(v_i) \geq v_i x_i(v_i) - p_i(v_i) \quad \forall i, v_i, v_i, \quad \text{assume } s_i \text{ is onto} \]

Example A: 2nd price auction \( \omega / \) 2 bidders \( v_i \sim U[0,1] \)

Mechanism: solicit bids \( b_1, b_2 \).
If \( b > b_2 \), 1 wins and pays \( b_2 \), else 2 wins and pays \( b_2 \)

Equilibrium: what is 1's best response to \( b_2 \)?

\[
\begin{array}{c}
\text{\( V_i \)} \\
\text{\( b_2 \)} \\
\text{\( b_1 \)}
\end{array}
\]

\[
\text{\( V_i - b_2 \)} \\
\text{\( \text{bid of 1} \)} \\
\text{\( b_2 \)}
\]

\[
\text{\( V_i \)} \\
\text{\( b_2 \)} \\
\text{\( b_1 = V_2 \)}
\]

\[\Rightarrow b_2 = V_2 \quad \text{a best response} \]

\[\text{case 1: } V_i \leq b_2, \text{ bid } h \leq b_2 \]

\[\text{case 2: } V_i > b_2, \text{ bid } h < b_2 \]

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case 1: \( v_i \geq \frac{b_2}{2} \) : bid \( \geq \frac{b_2}{2} \)

Case 2: \( v_i < \frac{b_2}{2} \) : bid < \( \frac{b_2}{2} \)

A truthful \( (b_i = v_i) \) dominant strategy (best response independent of \( b_2 \)) equilibrium.

Example B: 1st price auction w/ 2 bidders \( v_i \sim U[0,1] \)

Mechanism: solicit bids \( b_1, b_2 \).

If \( b_1 \geq b_2 \), I win and pays \( b_2 \), else 2 wins and pays \( b_2 \).

Equilibrium: Guess and check, symmetric equil: \( s(v) = \frac{v}{2} \)

- Suppose opponent bids according to \( s(v) \)
- if \( I \) bid \( b \), \( p_i[ \text{I win} ] = \frac{b > s(v)}{s(v)} = \frac{b > v/2}{v/2} = \frac{v}{2} \)
- best response: given value \( v_i \), pick \( b \) s.t.
  \[
  b^* = \arg\max_{b} \left[ p_i[\text{win}] \times (v - b) \right]
  \]
  \[
  = \arg\max_{b} \left[ 2b \left( v - b \right) \right] = \frac{v}{2}
  \]

Same as opponent strategy, so equilibrium.

Question: \( E_{v_i, v_i}[\text{Rev}(A)] = ? \) Answer: \( \frac{1}{3} \), \( E_{v_i, v_i}[\text{Rev}(B)] = ? \) Answer: \( \frac{1}{8} \) !!

Better revenue? Example C.

2nd price auction w/reserve \( r \): if higher bid \( > r \), win and pay \( \max(r, \text{2nd highest bid}) \)

Equilibrium: \( s(v) = v \)

Revenue: case analysis, label bidders so \( v_i \geq v \)

- \( v_i \geq v_2 \)
- \( v_i \geq v_2 \)

Total expected revenue: \( \left( \frac{2}{3} r + \frac{1}{3} \right) \left( 1-r \right)^2 + 2(1-r)r^2 \), optimized \( r = \frac{1}{2} \) same as price for 1 buyer!

Defn. A mechanism is direct if actions of buyers are to report values.

Revelation Principle. Any outcome implemented by some mechanism can be implemented by an incentive-compatible direct mechanism.

Proof: Construct alternate mech. that inputs value and simulates actions of buyer w/that value.

*Only need to consider mechanisms w/truthful BNE.

Characterization Theorem. BNE if:

1. Monotonicity: \( x_i(v_i) \) monotone non-decreasing
2. Payment identity: \( p_i(v) = v_i x_i(v_i) - \sum_{j \neq i} x_j(2^j b_i \cdot 2^{i-1} + \lambda) \)

Proof. \( v \) and (ii) \( \rightarrow \) BNE
Can bidder improve by impersonating \( v' < v \)?

\[ \text{BNE} \Rightarrow (i), \text{BNE}+(i) \Rightarrow (ii): \text{follows from incentive constraint} \]
\[ v \times (v) - p(v) \geq v \times (v') - p(v') \quad \forall v, v' \]

Consequence (Revenue Equivalence):
Auctions with same allocation in BNE have same revenue.

Example D: 1st price auction: guess \( s(v) \) monotone in \( v \)

\[ p(v) = E[\text{2nd price payment} \mid v] = E[\text{payment if } v \text{ wins}] = \frac{v}{2} \]

check guess: \( s(v) \) is increasing in \( v \).

Optimizing BNE:

Lemma: [Myerson '81] In BNE, \( E[p_i(v_i)] = E[\phi_i(v_i) x_i(v_i)] \)
where \( \phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{F_i(v_i)} \) is the virtual value.

Approach for deriving max revenue auction:
- calculate virtual values \( \phi_i \)
- choose \( x \) to maximize \( E[\sum \phi_i(v_i) x_i(v_i)] \)
  - subject to constraints on \( x \) (eg only allocate 1 item)
  - check \( x_i \) is monotone (it will be if \( \phi_i = v \))
  - use payment identity to calculate payments.

Example A': 1 buyer, \( v \sim U[0,1] \)

\[ \phi(v) = v - \frac{1 - F(v)}{F(v)} = v - \frac{v - 1}{v} = 2v - 1 \]

\[ p(v) = \begin{cases} \frac{v}{2}, & v > \frac{1}{2} \\ 0, & \text{otherwise} \end{cases} \]

\( \Rightarrow \) posted price of \( 1/2 \)!

Example E: \( n \) buyers, \( v_i \sim U[0,1] \)

\[ \phi_i(v_i) = 2v_i - 1 \]

\[ \max_x \sum \phi_i(v_i) x_i(v_i) \Rightarrow \text{allocate to highest bidder} \]

\[ \ldots \ldots \text{allocated} \]
\[ \phi_i(v) = 2v - 1 \]

\[
\max_{x_i} \sum_{i} \Phi_i(v) x_i(v) \quad \Rightarrow \text{allocate to highest bidder if } \text{value } > 0 \]

2\textsuperscript{nd} price auction w/reserve of \( 1/2 \)

Ref. (cf Myerson's lemma)

\[
E_{\nu} \left[ \phi_i(v) \right] = E_{\nu} \left[ v x_i(v) - \sum_{x_i(\omega) = 1} \int_0^v d\omega \right]
\]

\[
= \sum_{x_i(\omega) = 1} \int_0^v d\omega f(v) dv = \sum_{x_i(\omega) = 1} \int_0^v \frac{f(v) \nu v x_i(v) f(v) dv}{\int_0^v d\omega}
\]

Recall integration by parts: \[
\int_0^v h dg = h g |_0^v - \int_0^v g dh
\]

\[
= \sum_{x_i(\omega) = 1} \int_0^v \frac{f(v) \nu x_i(v) f(v) \nu v x_i(v) f(v) dv}{\int_0^v d\omega} - \left. (F(v) - 1) \right|_0^v - \sum_{x_i(\omega) = 1} (F(v) - 1) x_i(v) dv
\]

\[
= \sum_{x_i(\omega) = 1} \int_0^v \frac{f(v) \nu x_i(v) f(v) dv}{\int_0^v d\omega} - (1 - F(v)) x_i(v) dv
\]

\[
= \sum_{x_i(\omega) = 1} \int_0^v \left( v - \frac{F(v)}{f(v)} \right) x_i(v) f(v) dv
\]

Discussion: Apply optimal auctions to sell search ads
- What is \( F \)?
- Separate auction for each keyword? But then auctions interact.
- Multiple slots per keyword
- What about the competition?