Computing (with) Curves on Surfaces II

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ADFOCS
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The problems

- Given two paths on a surface, are they homotopic?
- Given a cycle on a surface, is it contractible?
- Given a path or cycle π on a surface, compute the shortest path or cycle homotopic to π.

What's a "surface"?

A space obtained from a set of triangles by gluing pairs of equal-length edges together.

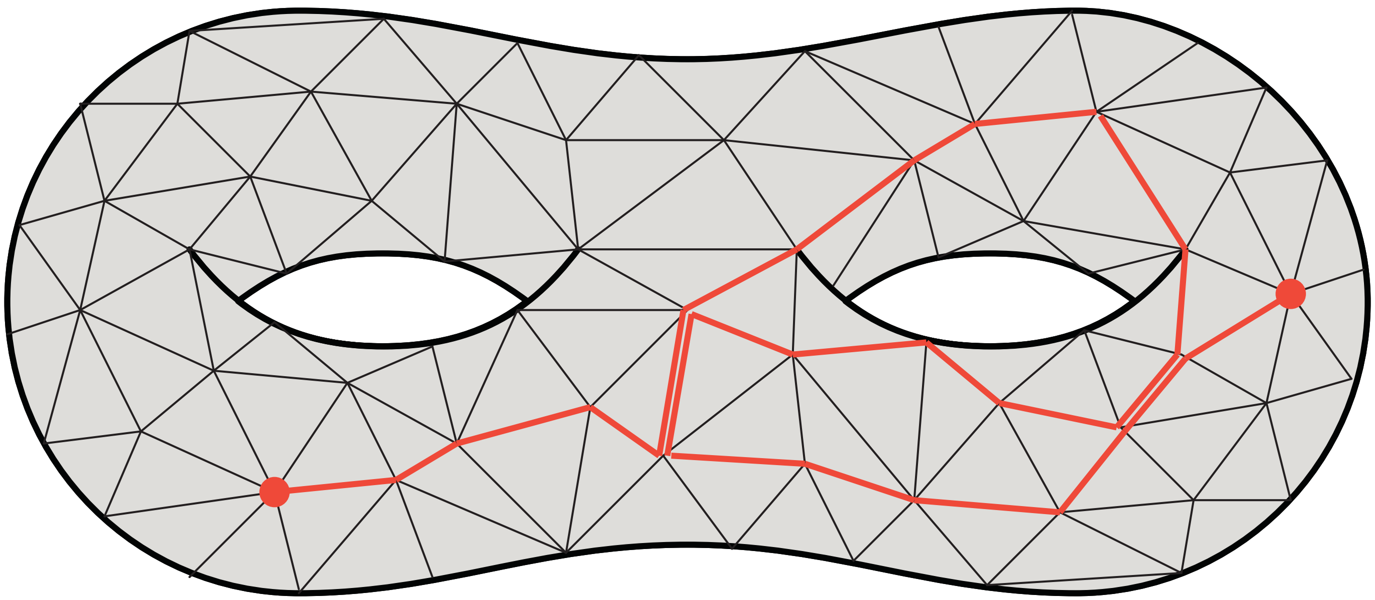
- **Boundary:** union of unglued edges
- **Interior:** everything else
- We consider only **orientable** surfaces

Combinatorial surface

- Abstract 2-manifold **Σ** with a weighted graph **G** embedded so that every face is a disk.
- Curves are restricted to walks in G.
- The length of a curve is the total weight of its edges, counted with multiplicity

[..., Mohar Thomassen 01, E Har-Peled 04, Colin de Verdière Lazarus 04]

Example



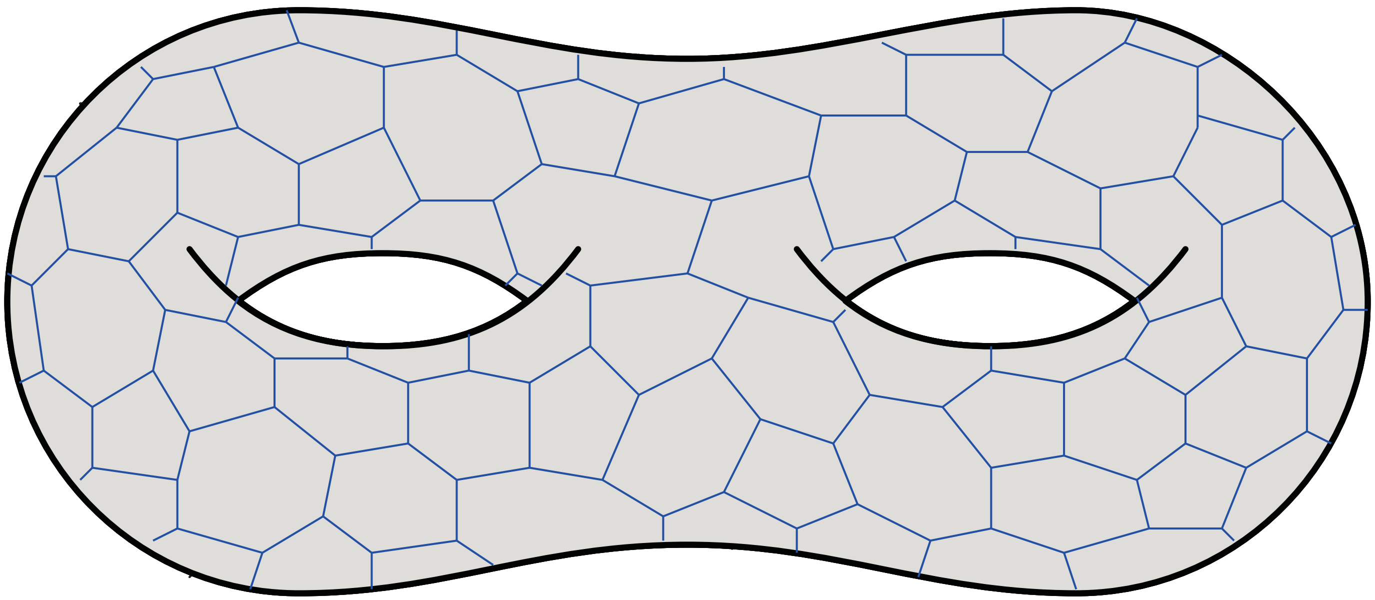
Shortest paths

Computed directly in the surface graph G

- O(n log n) [Dijkstra 56]
- O(n) if $g=O(n^{1-\epsilon})$ [Henzinger Klein Rao Subramanian 97]
- O(n) if all edges have unit length [Adam 4004BC]

Dual graph

Every edge e in G has a dual edge e* in G*

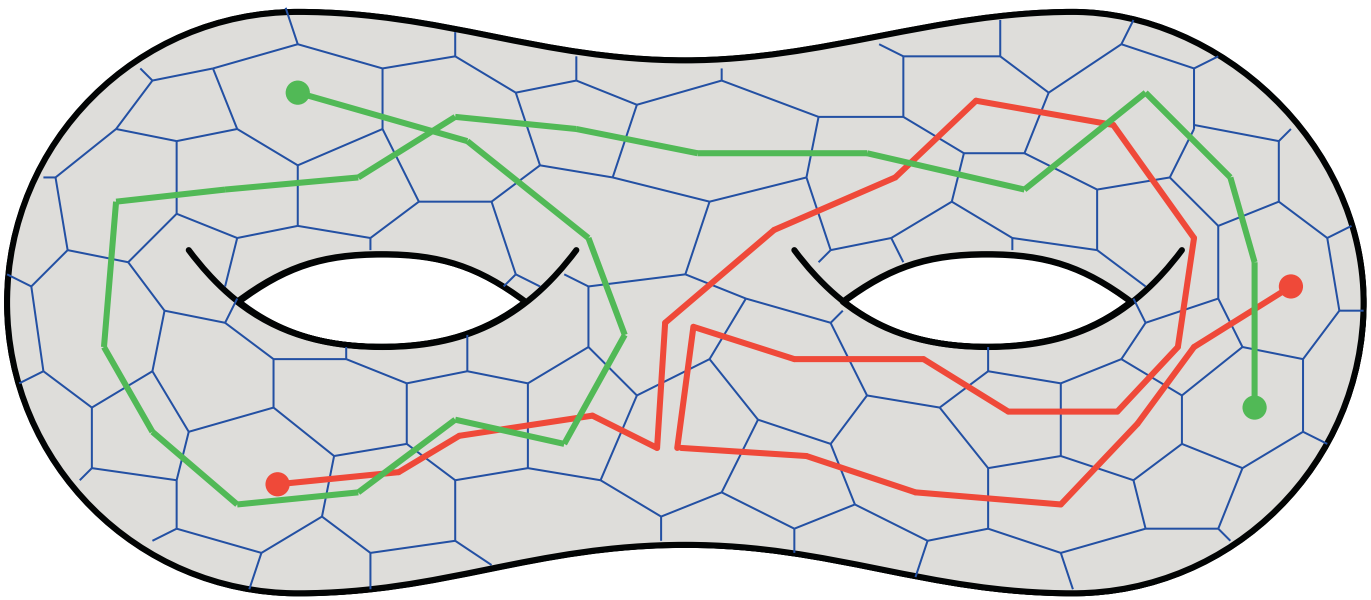


Dual formulation

- Abstract 2-manifold **Σ** with a weighted graph **G*** embedded so that every face is a disk.
- Curves must avoid vertices of G* and cross edges transversely.
- The length of a curve is the total weight of the edges it crosses, counted with multiplicity

[Colin de Verdière E 05]

Dual example



Notation

- **n** = complexity of Σ
  = #vertices + #edges + #faces of G
- **b** = number of boundary cycles
- **g = genus** = maximum # disjoint simple cycles that can be removed without separating Σ
- **k** = # edges in input path or cycle

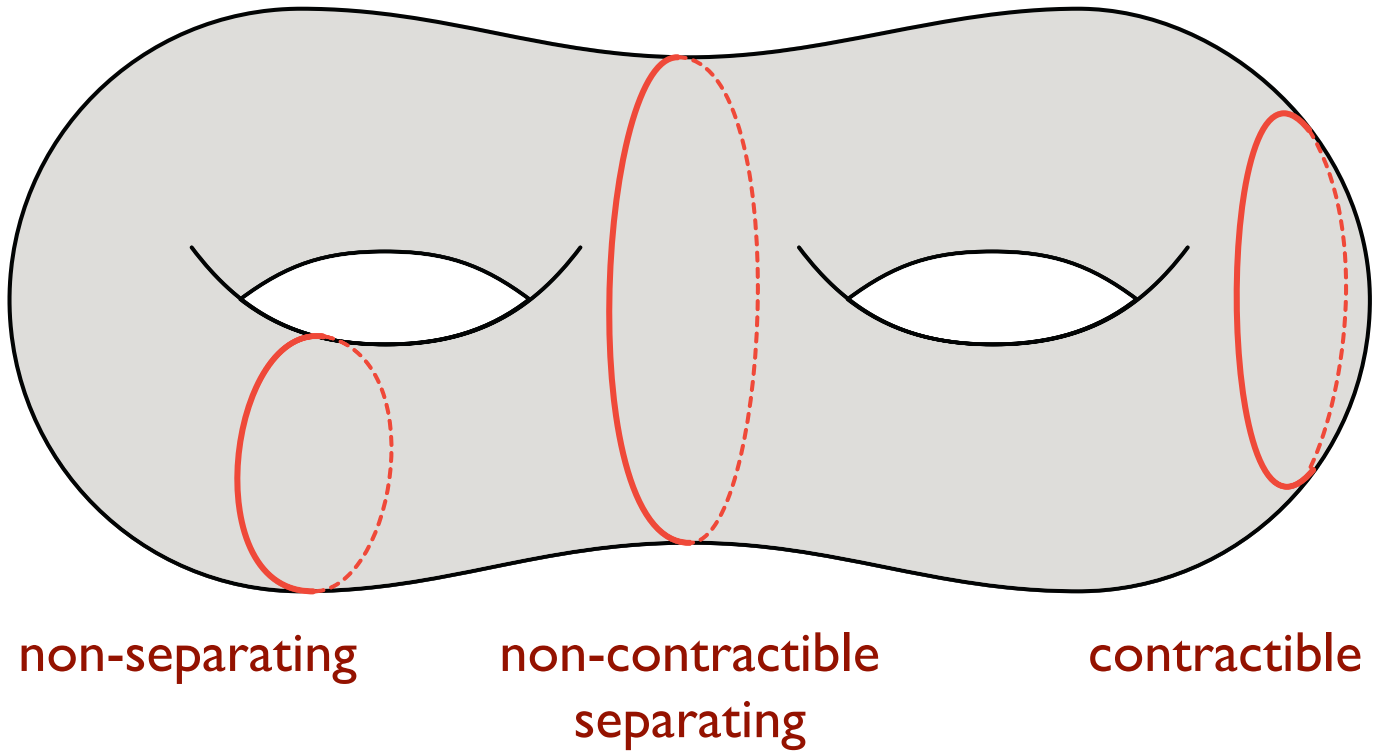
Free Homotopy

- A (free) homotopy between cycles γ and γ' is a continuous function $h:[0,1]\times S^1\rightarrow\Sigma$ such that $h(0,t) = \gamma(t)$ and $h(1,t) = \gamma'(t)$ for all t.
- A contractible cycle is (freely) homotopic to a constant cycle ($\gamma(t)=\gamma(0)$ for all t).
- A **tight** cycle is as short as possible in its (free) homotopy class.

Simple cycles

- A simple contractible cycle bounds a disk.
- We can decide whether a simple cycle is contractible in $O(n)$ time by depth-first search

Simple cycles



Testing contractibility

How??

1. Cut the surface into one or more disks
2. Compute the crossing sequence of the cycle
3. Reduce the crossing sequence
4. The cycle is contractible if and only if the reduced crossing sequence is empty.

Path homotopy

- A path homotopy between paths π and π' is a continuous function $h:[0,1]\times[0,1]\rightarrow\Sigma$ such that
    1. $h(0,t) = \pi(t)$ and $h(1,t) = \pi'(t)$ for all t
    2. $h(s,0) = \pi(0) = \pi'(0)$ and $h(s,1) = \pi(1) = \pi'(1)$ for all s
- Two paths are homotopic if there is a path homotopy between them.
- A **tight** path is as short as possible in its (path) homotopy class.

Tightening a path

How??

1. Cut the surface into one or more disks
2. Compute the crossing sequence of the path
3. Reduce the crossing sequence
4. Build a "sleeve" by gluing disks together along the reduced path
5. Compute the shortest path within the sleeve

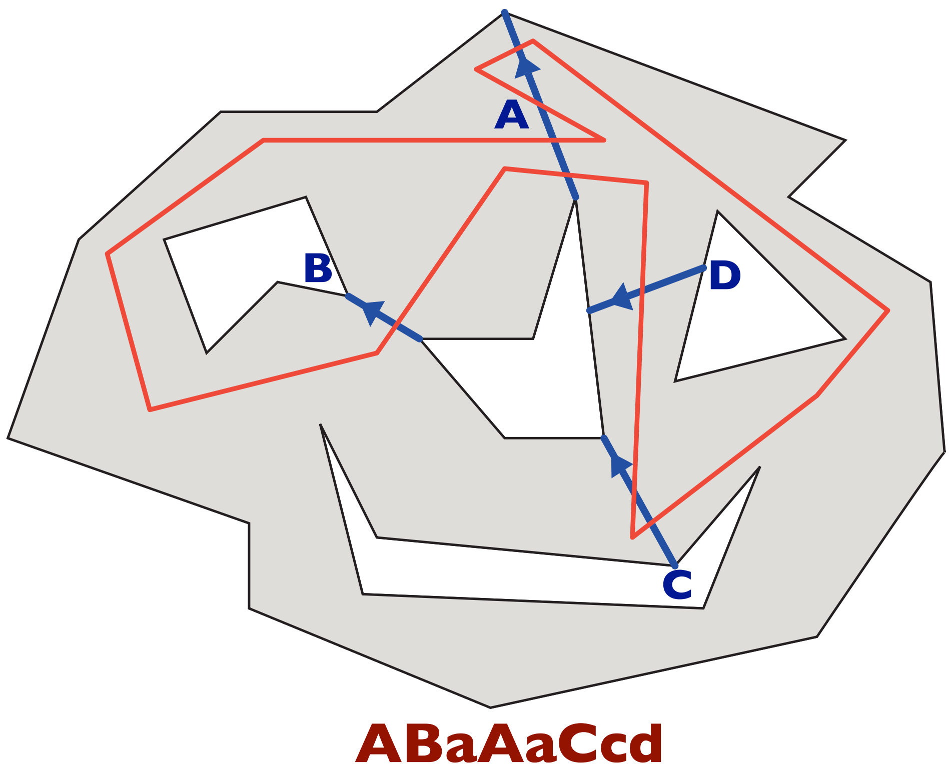
Genus-0 surfaces with boundary

System of arcs

- **arc** = path with both endpoints on the boundary of Σ.
- **system of arcs** = set of b-1 arcs whose removal cuts Σ into a disk
  - ◇ Easy to build in O(n) time [BFS + DFS]
  - ◇ System of **tight** arcs in O(n log n) time [Dijkstra's shortest path + Kruskal's MST]

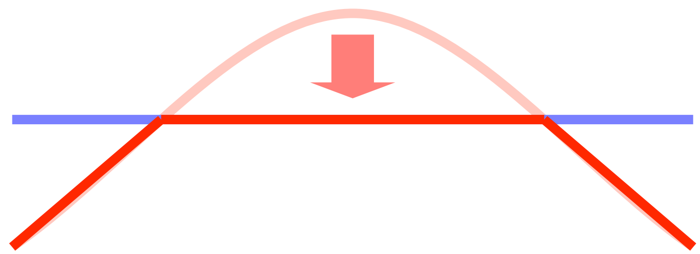
Crossing sequence

- Sequence of oriented crossings of arcs by π
- **x** = # crossings ≤ k(b-1)



Reduction

A reduced crossing sequence has no bigons (Aa or aA)



We can reduce any crossing sequence in O(x) time by removing all bigons in any order

Reduction

Theorem: *A cycle is contractible if and only if its reduced crossing sequence is empty.*

Testing contractibility

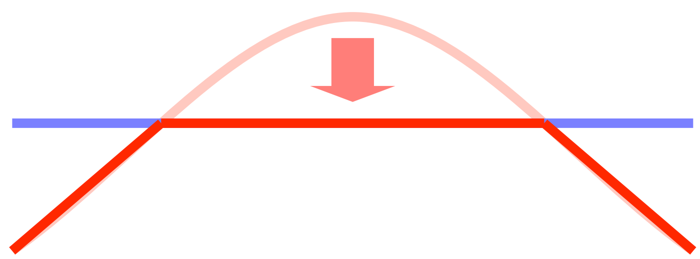
To test whether a cycle γ is contractible:

1. Cut Σ into a disk using b-1 arcs — $O(n)$
2. Compute the crossing sequence of γ — $O(k + x)$
3. Reduce the crossing sequence — $O(x)$

Total time = $O(n + k + x)$ = $O(n + kb)$

No bigons

Lemma: In any surface, two tight paths never bound a disk.



(but they might cross more than once)

Tightening a path

To compute the shortest path homotopic to π:

1. Cut Σ into a disk D using b-1 tight arcs — $O(n \log n)$
2. Compute the crossing sequence of π — $O(k+x)$
3. Reduce the crossing sequence — $O(x)$
4. Glue copies of D along reduced crossing sequence — $O(xn)$
5. Compute shortest path in resulting sleeve — $O(xn)$

Total time = $O(n \log n + xn) = O(n \log n + nkb)$

Positive genus surfaces with boundary

System of arcs

Forest/cotree decomposition:

- **F** = spanning forest where every tree touches the boundary once
- For any edge $e \notin F$, **α(e)** = unique αrc in F+e
- **T*** = spanning tree of (G\F)*
- **X** = extra edges { e | $e \notin F$ and $e^* \notin T^*$ }
- **S** = system of arcs { α(e) | $e \notin F$ and $e^* \notin T^*$ }

Adapted from [Eppstein 04]

System of arcs

Lemma: $\Sigma \setminus \bigcup S$ *is a topological disk.*

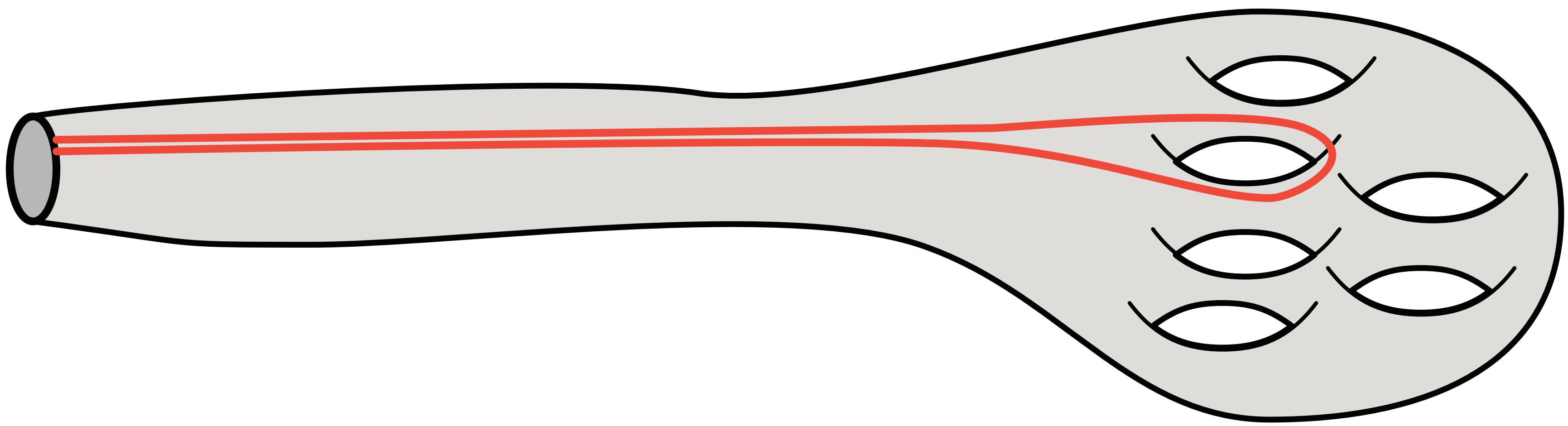
Lemma: $|X| = 2g+b-1$

Construction time

- forest F: O(n) [BFS]
- co-tree T*: O(n) [BFS]
- extras X: O(n)
- system S: O(s)
  where s=O(ng) is the total complexity of the arcs in S
- **Total:** O(s) = O(ng)

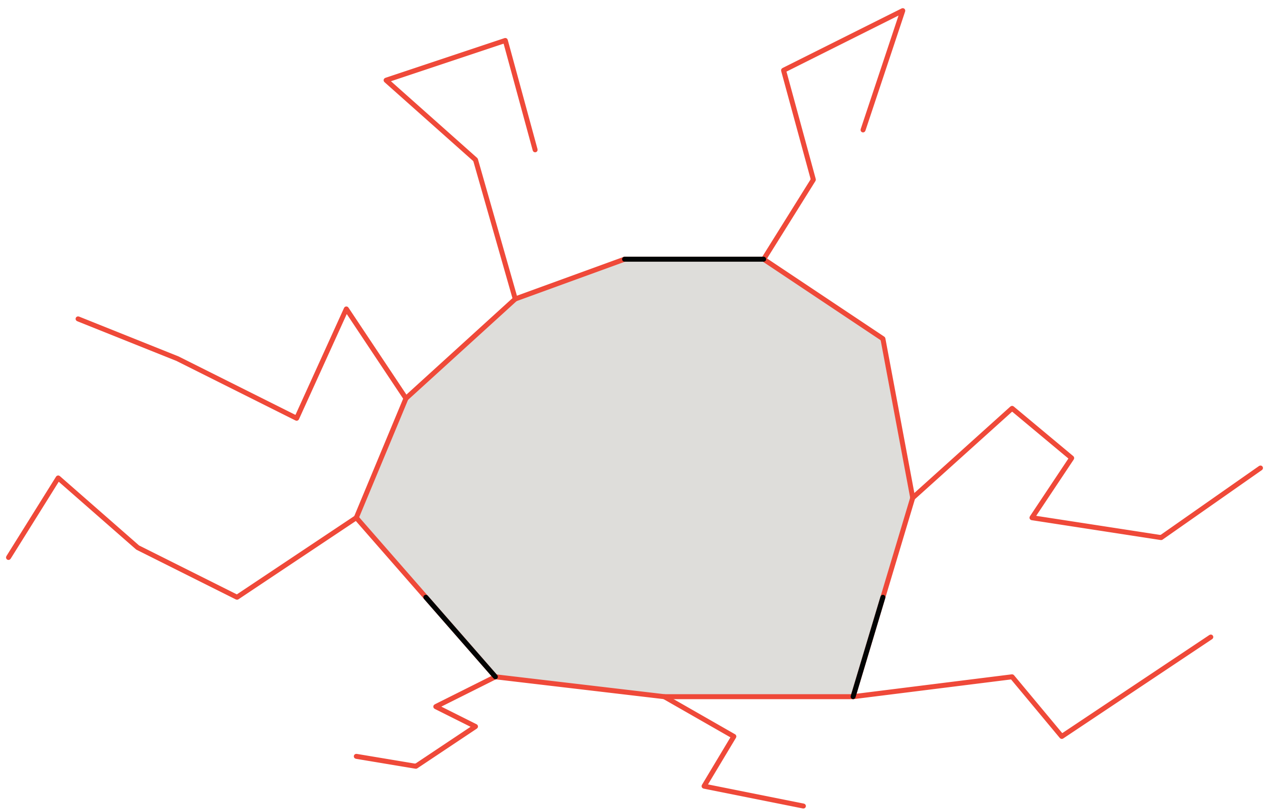
Size

In the worst case, $s=\Theta(ng)$



Size

The disk D=Σ\⋃S has complexity O(n+s)



main body: n ≤4g noodly appendages: s

Multiplicity

- Each arc $\alpha(e)$ traverses any edge in G at most twice (at most once if the endpoints of $\alpha(e)$ are on different boundaries)
- Thus, $x \leq (4g+b-1)k$.

Testing contractibility

To test whether a cycle γ is contractible:

1. Build a system of arcs — $O(s)$
2. Compute the crossing sequence of γ — $O(k + x)$
3. Reduce the crossing sequence — $O(x)$

Total time = $O(s + k + x) = O(gn + (b+g)k)$

System of tight arcs

- F = forest of shortest paths from the boundary
- T* = maximum spanning tree of $(G \setminus F)^*$,
  where $\text{weight}(e^*) = \text{length}(\alpha(e))$
- $S = \{ \alpha(e) \mid e \notin F \text{ and } e^* \notin T^* \}$

S can be constructed in time $O(n \log n + s)$.

Adapted from [E Whittlesey 05]

System of tight arcs

Lemma: *Every arc in S is tight.*

Proof (sketch): Let α_i be the shortest arc in Σ such that $\Sigma \backslash (\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_i)$ is connected. Each α_i is tight, and $S = \{\alpha_i\}$.

Tightening a path

To compute the shortest path homotopic to π:

1. Construct a system of tight arcs — $O(n \log n + s)$
2. Compute the crossing sequence of π — $O(k+x)$
3. Reduce the crossing sequence — $O(x)$
4. Glue copies of D along reduced crossing sequence — $O(xs)$
5. Compute shortest path in resulting sleeve — $O(xs)$

Total time = $O(n \log n + xs)$ = $O(n \log n + (b+g)gnk)$

Improvements

- Don't compute S explicitly, just the subgraph $\bigcup S$
  Removes s=O(ng) term from construction time
- Compress crossing words [Dey Guha 97]
  Improves crossing word length to O(k)
- Remove "noodly appendages" from the disk D
  Improves complexity of D to O(n)
- Output sensitivity [Colin de Verdière E 05]
  Analyze sleeve in terms of *reduced* crossing sequence

Improved time bounds

Testing contractibility: $O(n + k)$

Tightening paths: $O(n \log n + (b+g)k + (b+g)n\underline{k})$

- k = input path complexity
- k' = output path complexity
- $\underline{k} = \min\{k, k'\}$

[Dey Guha 97, Colin de Verdière E 05]

— Break —

Surfaces without boundary

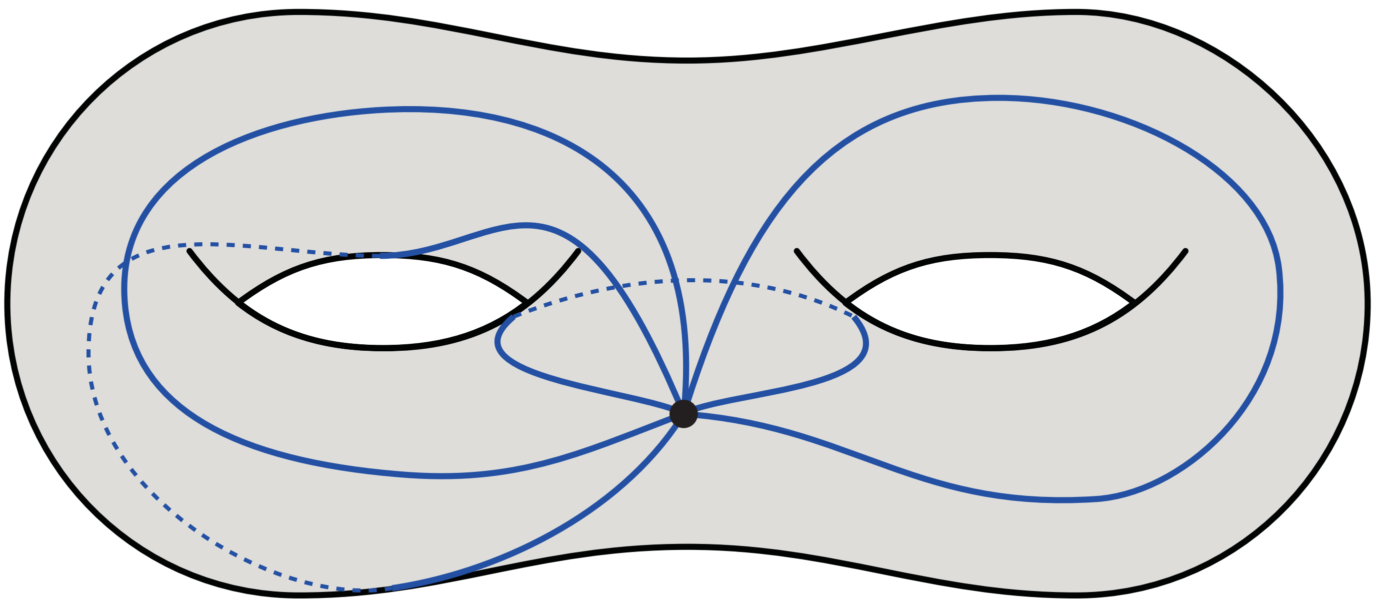
Boundaries helped us

No boundaries $\Rightarrow$ no arcs $\Rightarrow$ no tight arcs

We need yet another way to cut up the surface!

System of loops

- **loop** = path with both ends at fixed **basepoint**
- **system of loops** = 2g loops whose removal leaves a disk



System of loops

Tree/cotree decomposition:

- **T** = spanning tree
- For any edge e∉T, **λ(e)** = unique loop in T+e
- T* = spanning tree of (G\T)*
- X = extra edges { e | e∉T and e*∉T* }
- S = system of loops { λ(e) | e∉T and e*∉T* }

[Eppstein 04]

System of loops

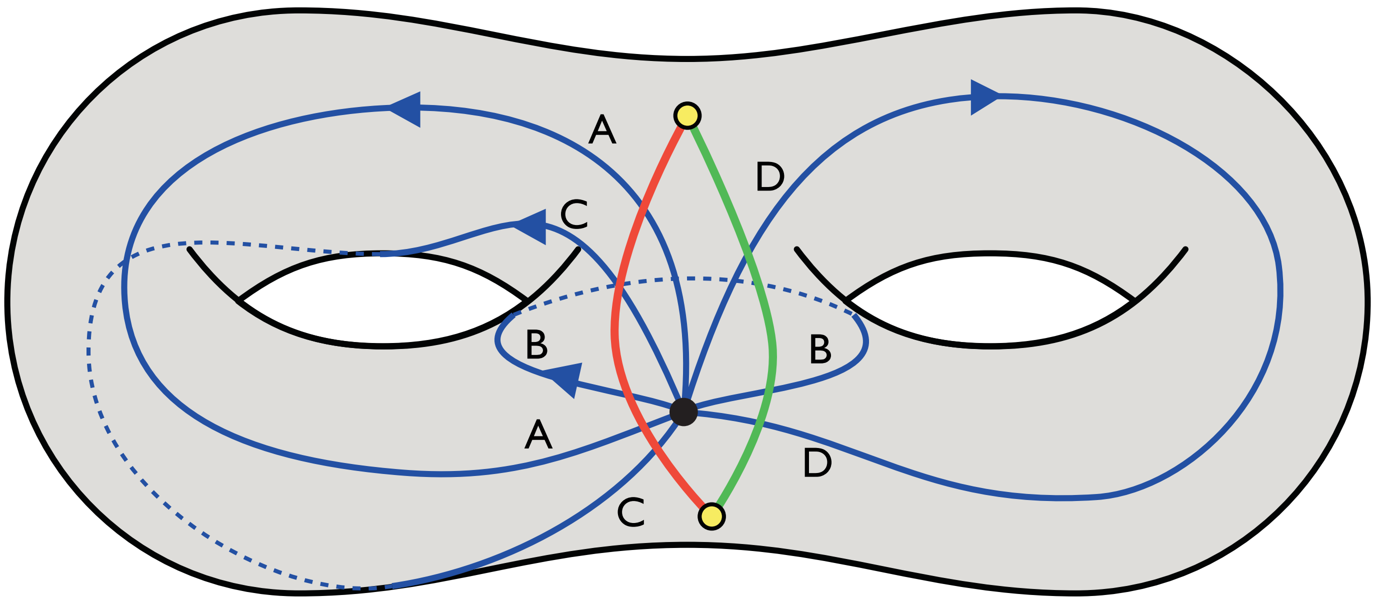
Lemma: $|X| = 2g$

Lemma: $\Sigma \backslash \cup S$ *is a topological disk.*

Lemma: S *uses each edge of* $G \leq 4g$ *times.*

Reduction

Equivalence of crossing words is more complex



ACBac = dBD

Relators: cyclic shifts of ACBacdbD or CABcadBD

Dehn's algorithm

Lemma: The crossing sequence of a contractible cycle either is empty, contains a bigon (aA or Aa) or contains most of a relator.

Thus, any crossing sequence can be reduced in $O(x)$ elementary reductions

Each elementary reduction takes $O(g)$ time

[Dehn 17]

Testing contractibility

To test whether a cycle γ is contractible:

1. Build a system of loops — $O(n + s)$
2. Compute the crossing sequence of γ — $O(k + x)$
3. Reduce the crossing sequence — $O(xg)$

Total time = $O(n + s + k + x) = O((n+k)g^2)$

Can be improved to $O(n + k)$ [Dey Guha 97]

Tightening paths

To exploit the "no bigons" rule, we need a system of tight **cycles** (not just tight loops).

But how is such a system defined?

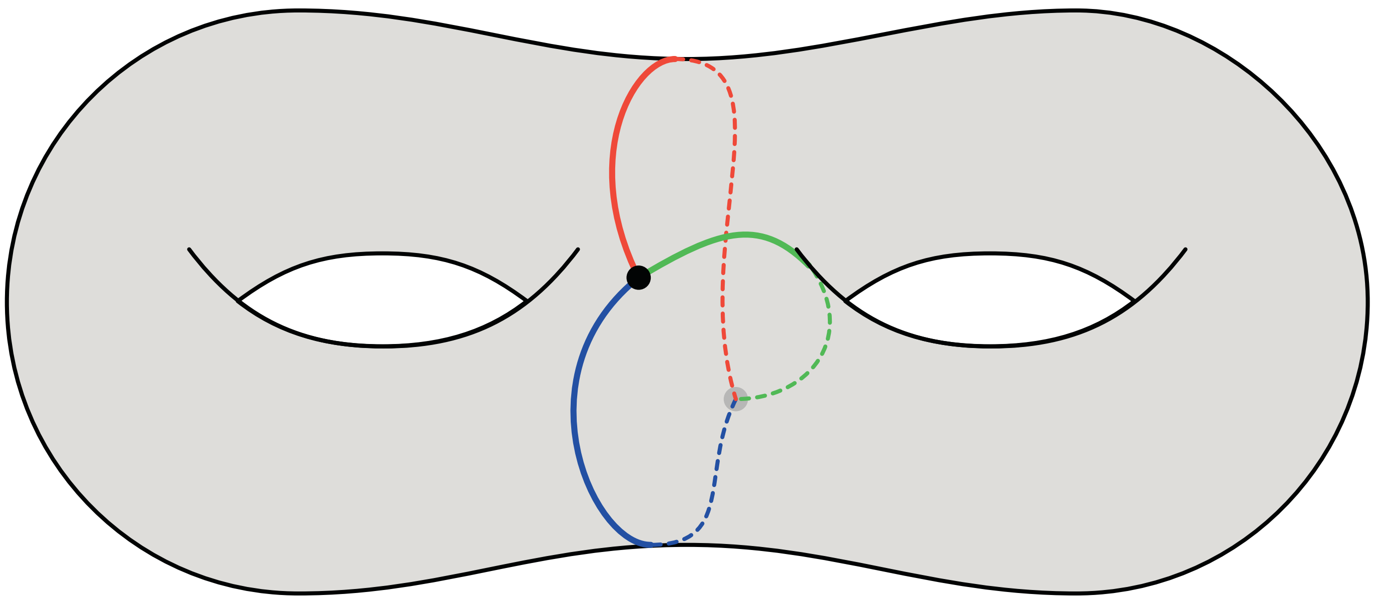
How do we build *any* tight cycles?

Two tight cycles

- The shortest non-contractible cycle is tight.
- The shortest non-separating cycle is tight.

3-path property

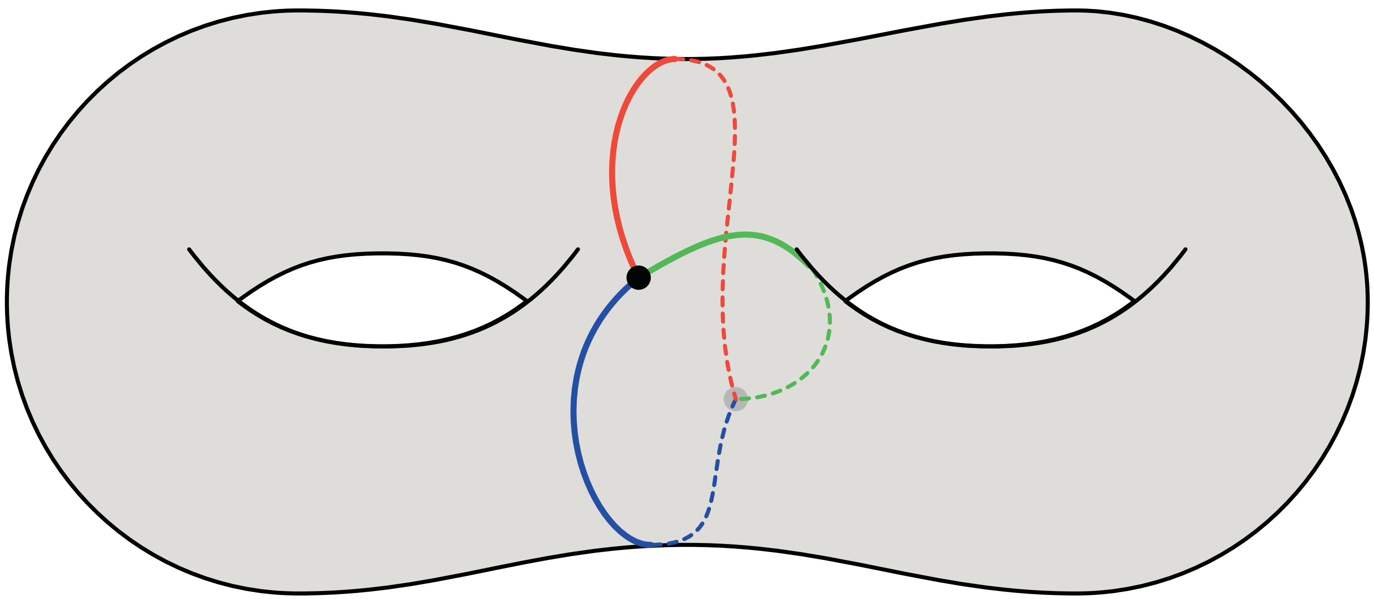
For any three paths with the same endpoints, if two of the cycles they define are contractible, then the third cycle is also contractible.



[Mohar Thomassen 01]

3-path property

Lemma: *The shortest non-contractible cycle is the union of two equal-length shortest paths.*



[Mohar Thomassen 01]

Shortest non-contractible loop

- Run Dijkstra's algorithm starting at basepoint v
- When wavefront meets itself, test resulting simple cycle for contractibility in O(n) time
- If contractible, charge search time to bounded disk and discard it.
- Total time is O(n log n).

[E Har-Peled 04]

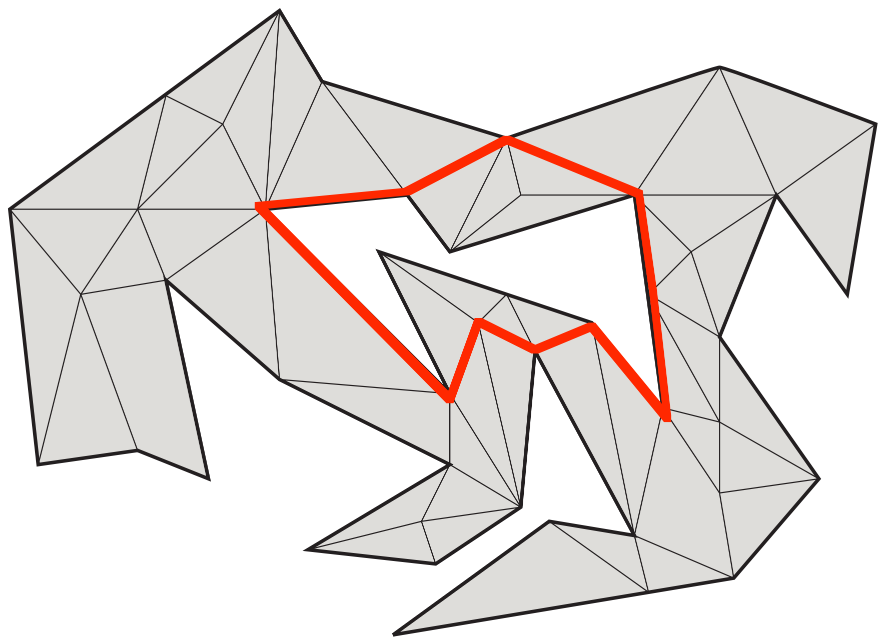
Shortest non-contractible cycle

- Find the shortest non-contractible loop based at each vertex v
- Return the shortest of these loops
- $O(n^2 \log n)$ time

[E Har-Peled 04]

Shortest cycle in annulus

Crosses the minimum cut in G* that separates the two boundaries

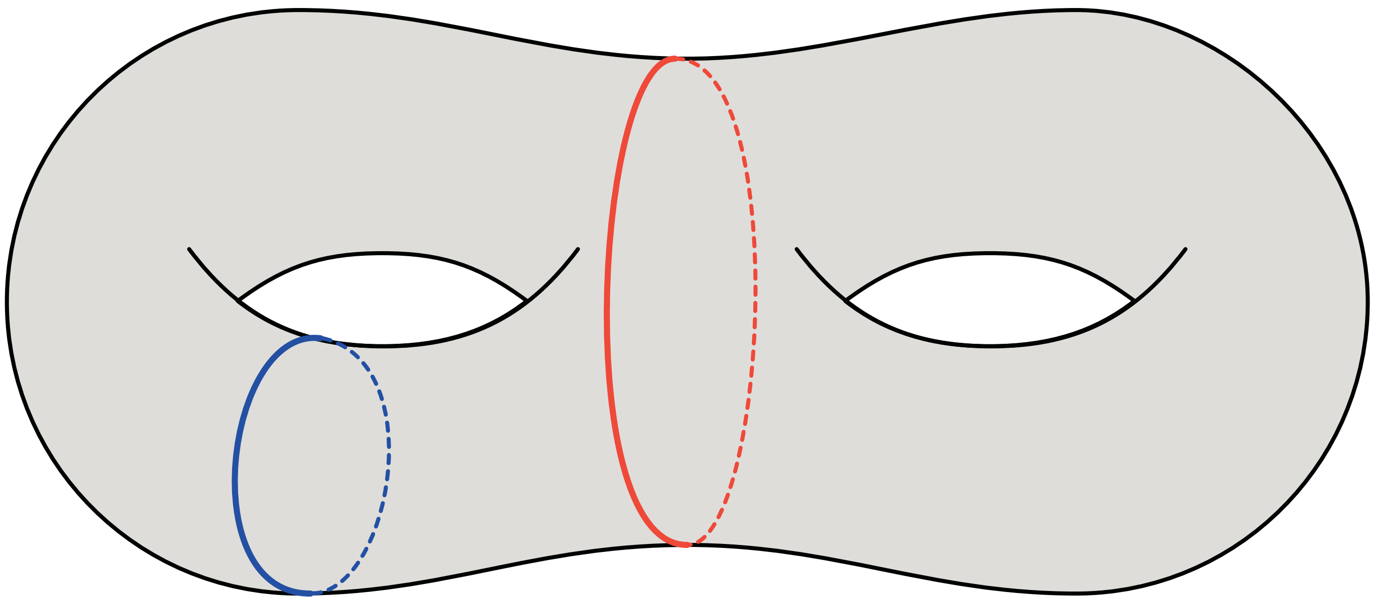


Can be computed in $O(n \log n)$ time.

[Frederickson 87]

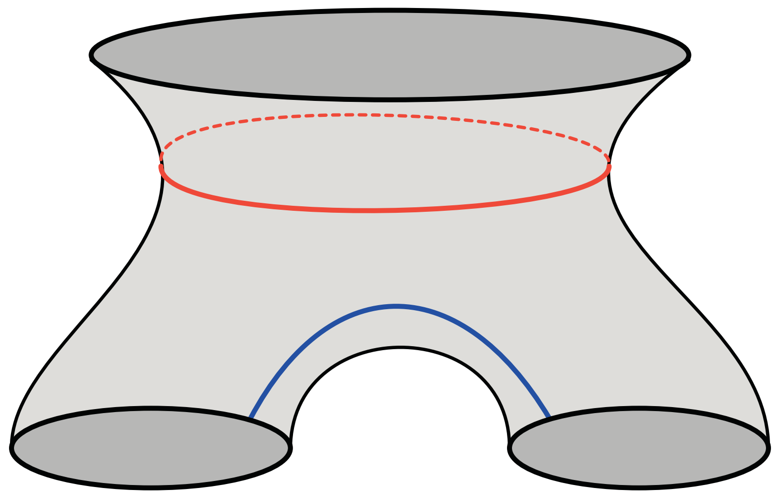
Bootstrapping

Lemma: *Let α be a tight simple arc or cycle in Σ, and let β be a simple path or cycle in $\Sigma\backslash\alpha$. If β is tight in $\Sigma\backslash\alpha$, then β is also tight in Σ.*



Pair of pants

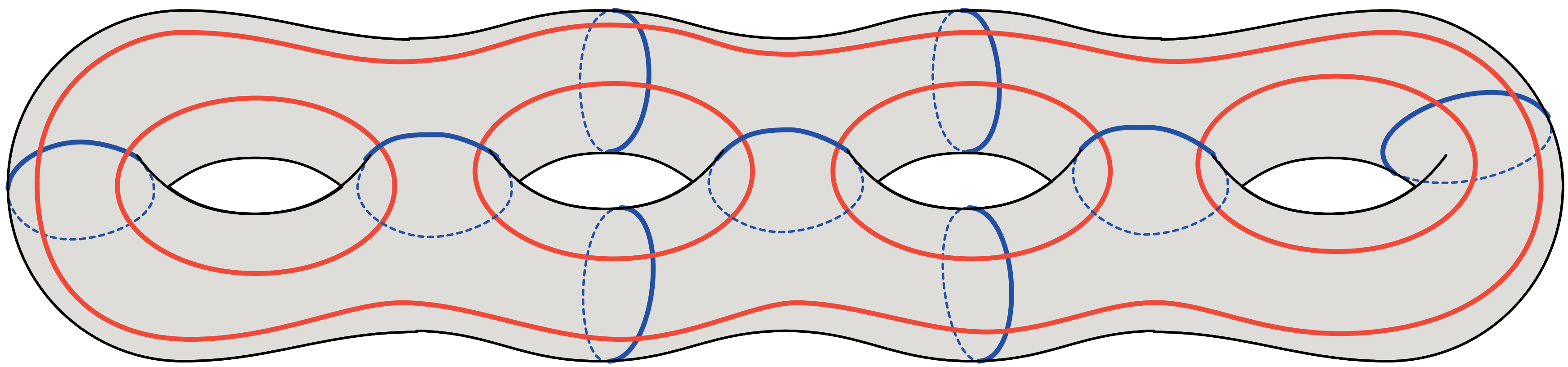
Genus-zero surface with 3 boundaries $\delta_1, \delta_2, \delta_3$



To compute the shortest cycle homotopic to δ_1:

1. Compute the shortest arc α between δ_2 and δ_3.
2. Compute the shortest nontrivial cycle δ^* in $\Sigma \backslash \alpha$.

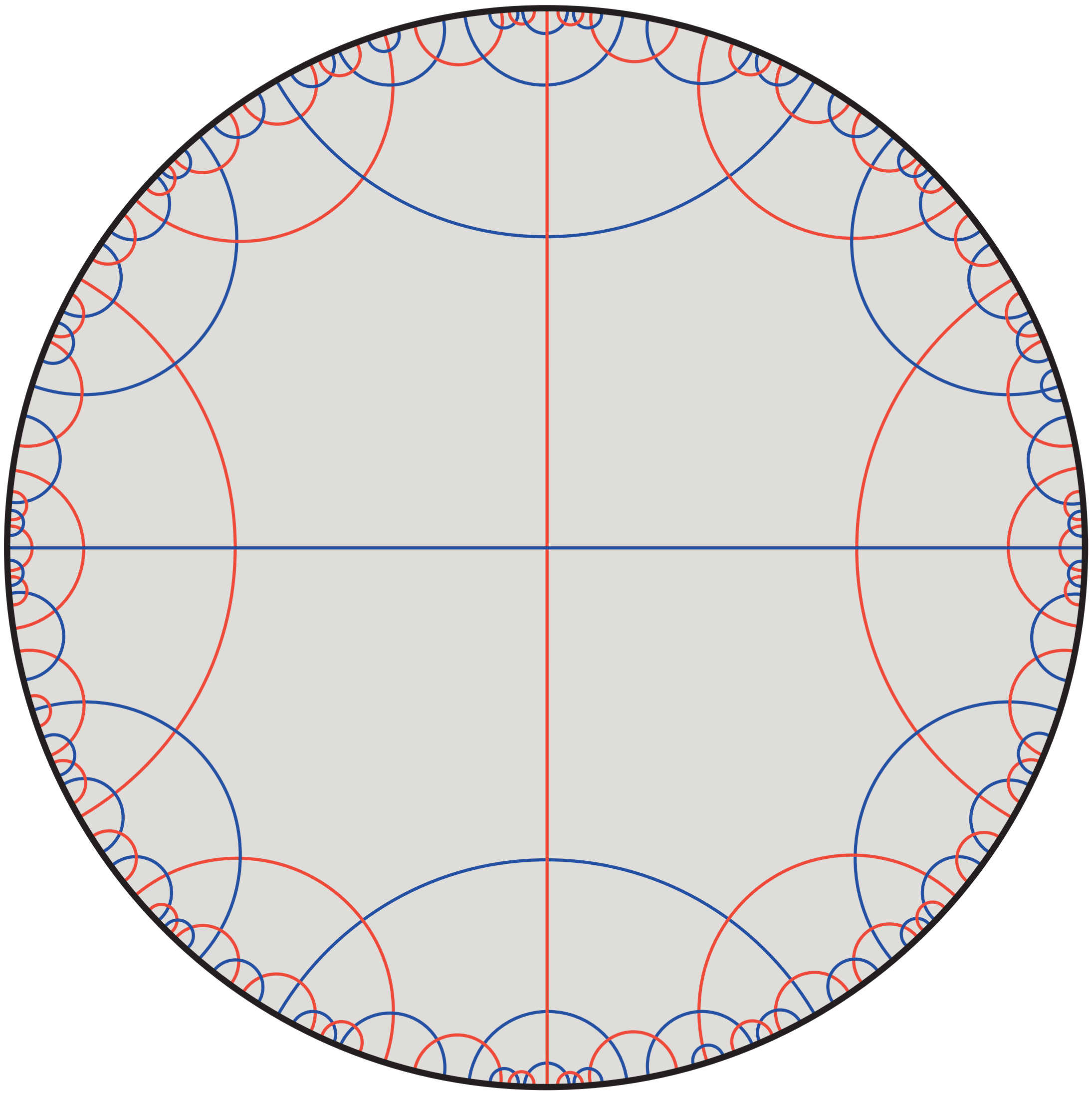
Hexagonal decomposition



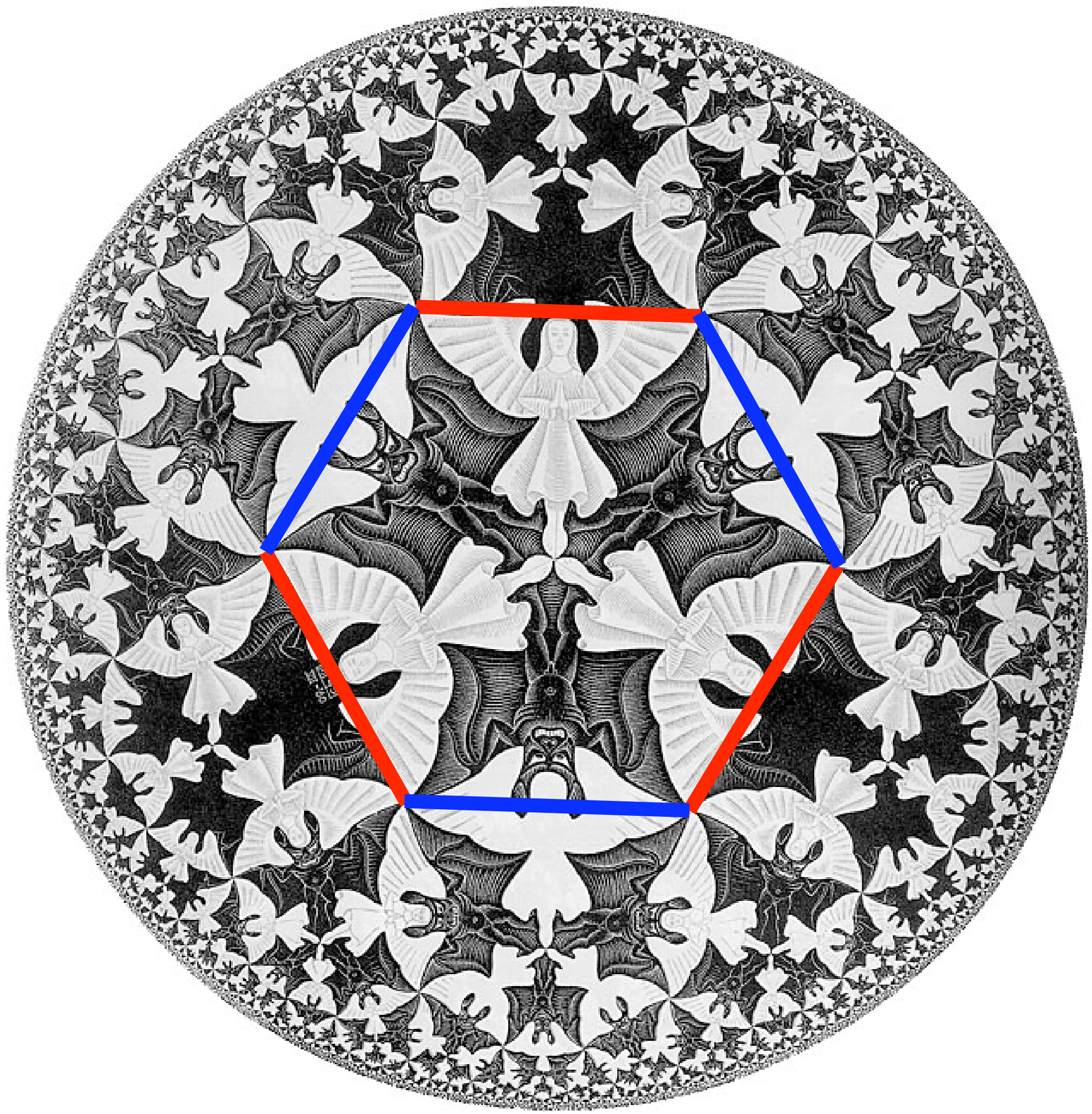
4g-2 tight cycles, 4g-4 hexagons
Built in O(gn log n) time

[Colin de Verdière E 05]

Universal cover

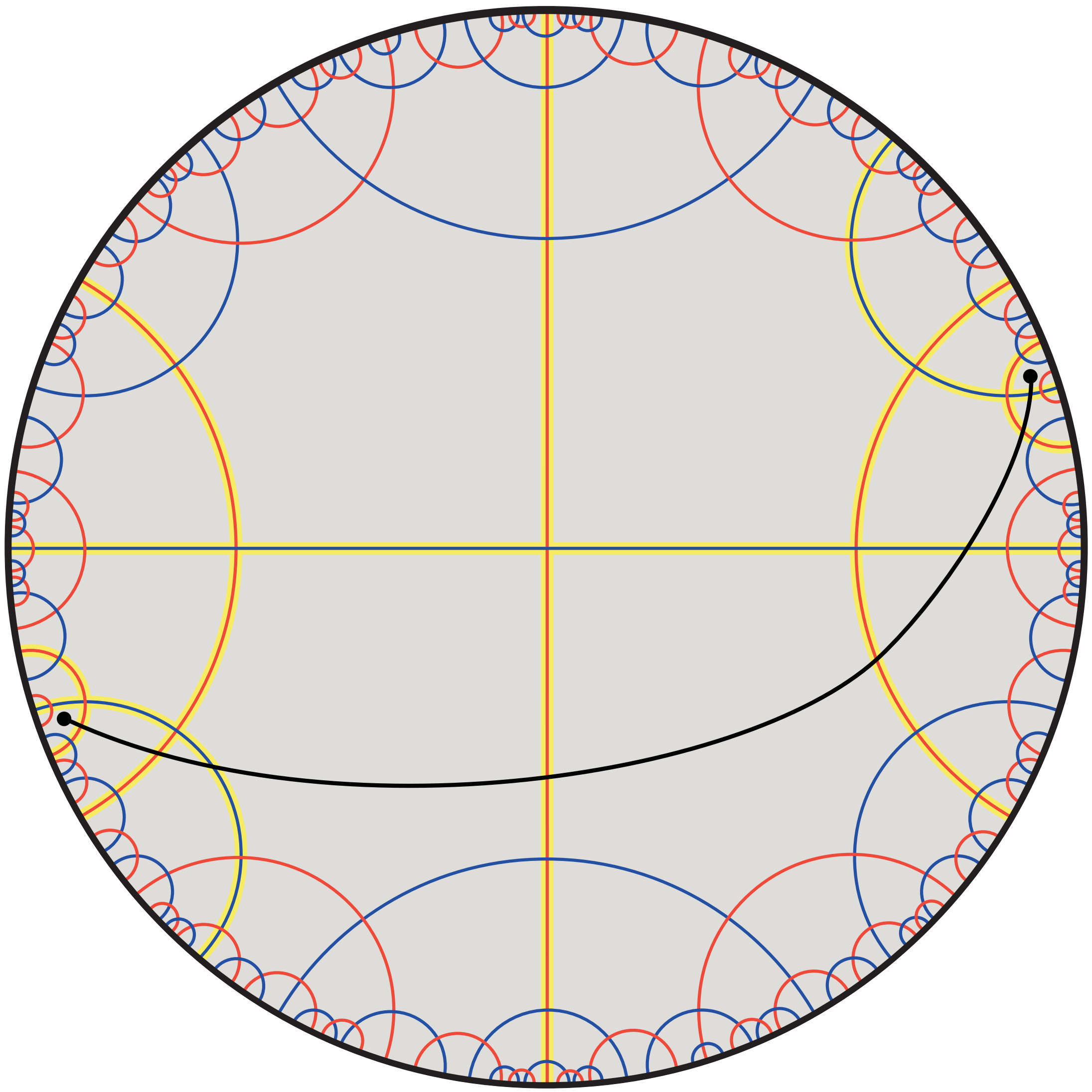


Universal cover



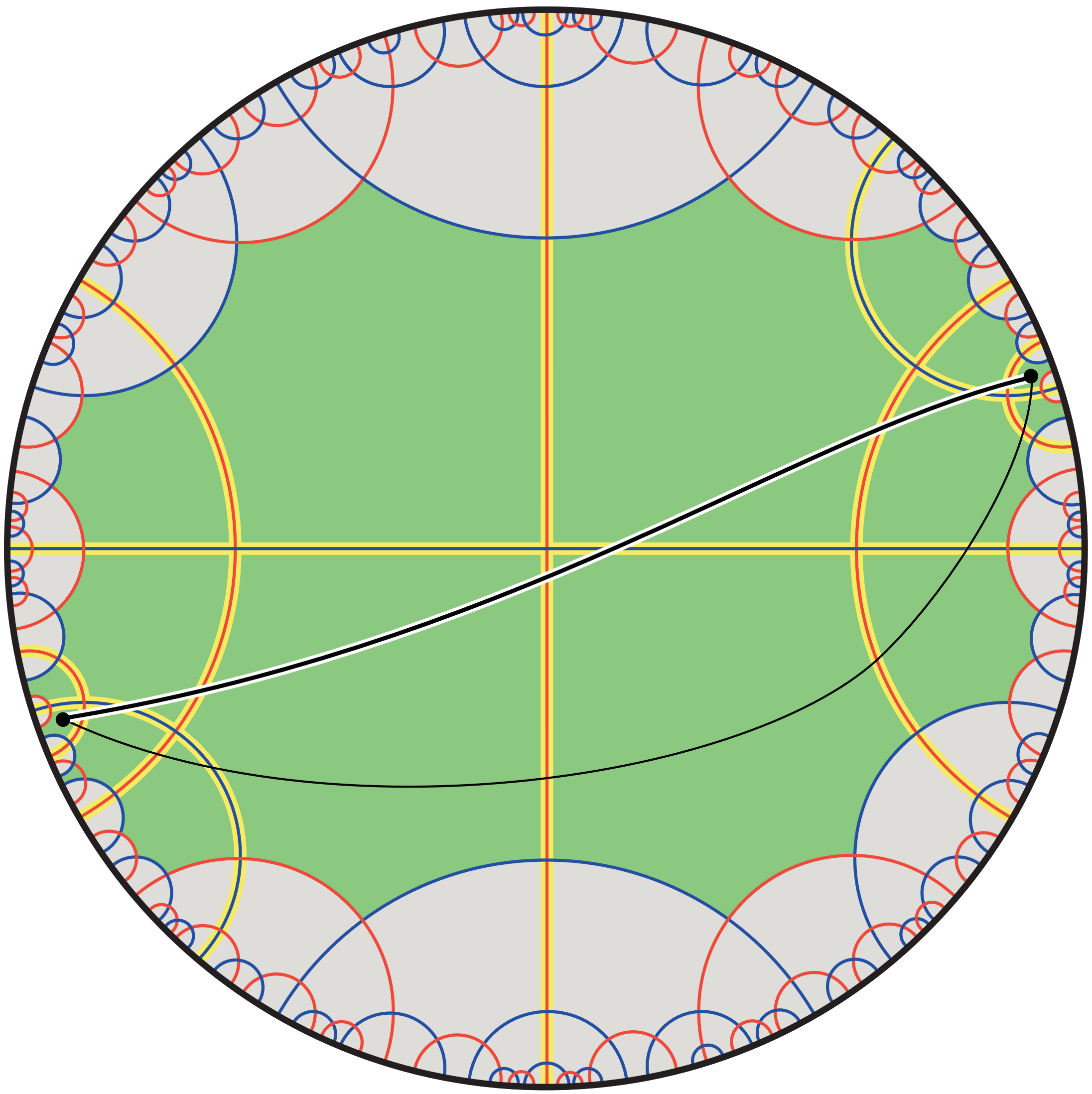
M. C. Escher, Circle Limit IV: Heaven and Hell (1960)

Relevant lines



[Colin de Verdière E 05]

Relevant region



[Colin de Verdière E 05]

Tightening a path

To compute the shortest path homotopic to π:

1. Build a hexagonal decomposition — $O(gn \log n)$
2. Compute the crossing sequence of π — $O(k + x)$
3. Reduce the crossing sequence — $O(x)$
4. Construct the relevant region — $O(nx)$
5. Compute the shortest path in R — $O(nx)$

Total = $O(gn \log n + k + nx) = O(gn(\log n + k))$

[Colin de Verdière E 05]

That's enough.