Pseudotriangulations

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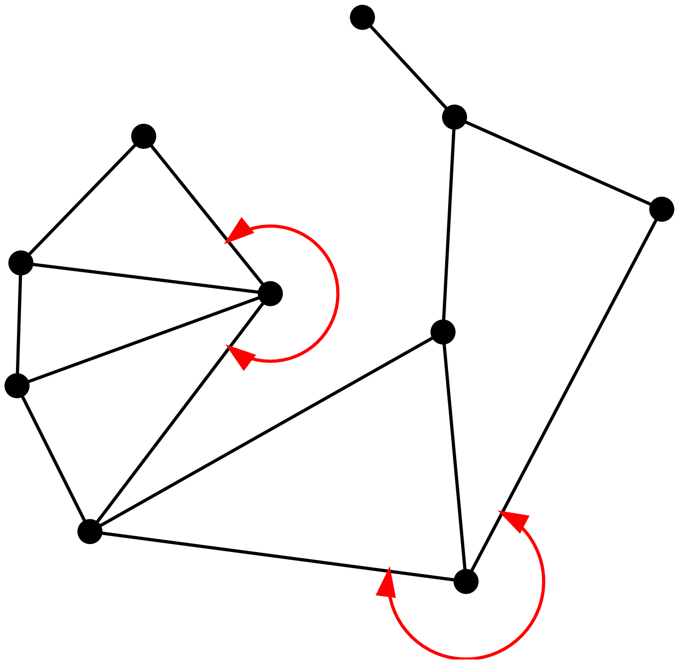
Part I: 0. Introduction, definitions, basic properties
1. Application: Ray shooting in a simple polygon
2. Rigid and flexible frameworks
3. Planar Laman graphs
4. Combinatorial Pseudotriangulations
5. Tutte embeddings

Part II: 6. Stresses and reciprocals
7. Unfolding of frameworks
8. Liftings and surfaces

Part III: 9. kinetic data structures, PPT-polytope, counting and enumeration, visibility graphs, flips, combinatorial questions

0. BASIC PROPERTIES. Pointed Vertices

A *pointed* vertex is incident to an angle $> 180^\circ$ (a *reflex* angle or *big* angle).



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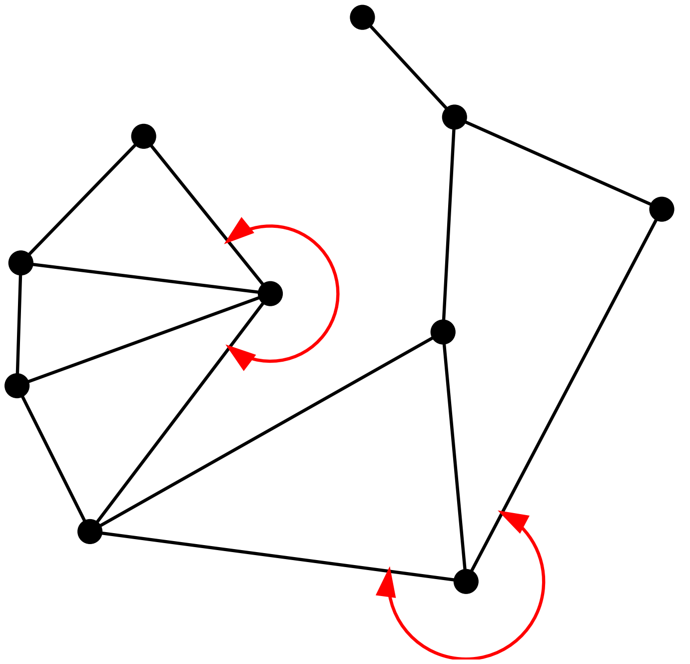

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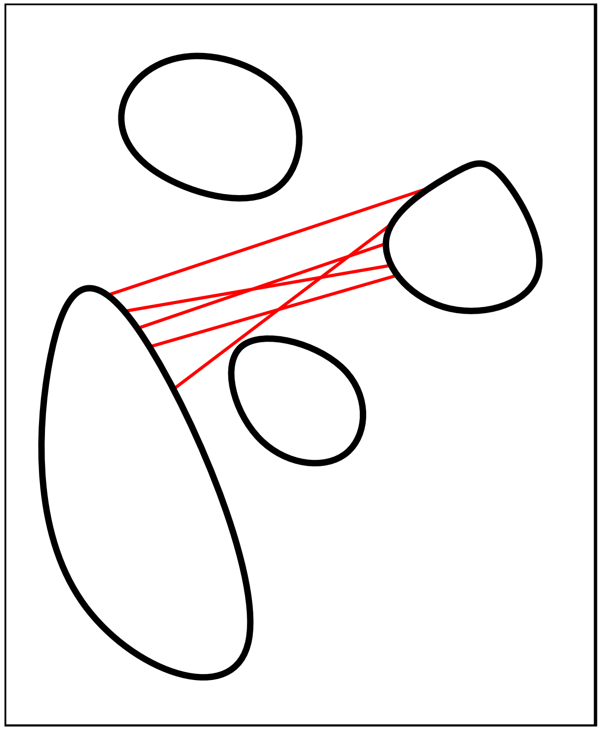
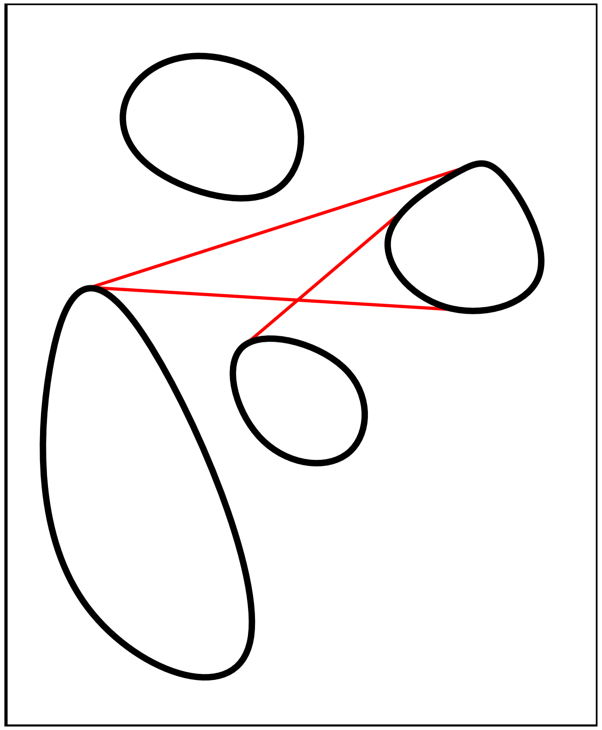
Where do pointed vertices arise?

Visibility among convex obstacles

Equivalence classes of *visibility segments*. Extreme segments are *bitangents* of convex obstacles.

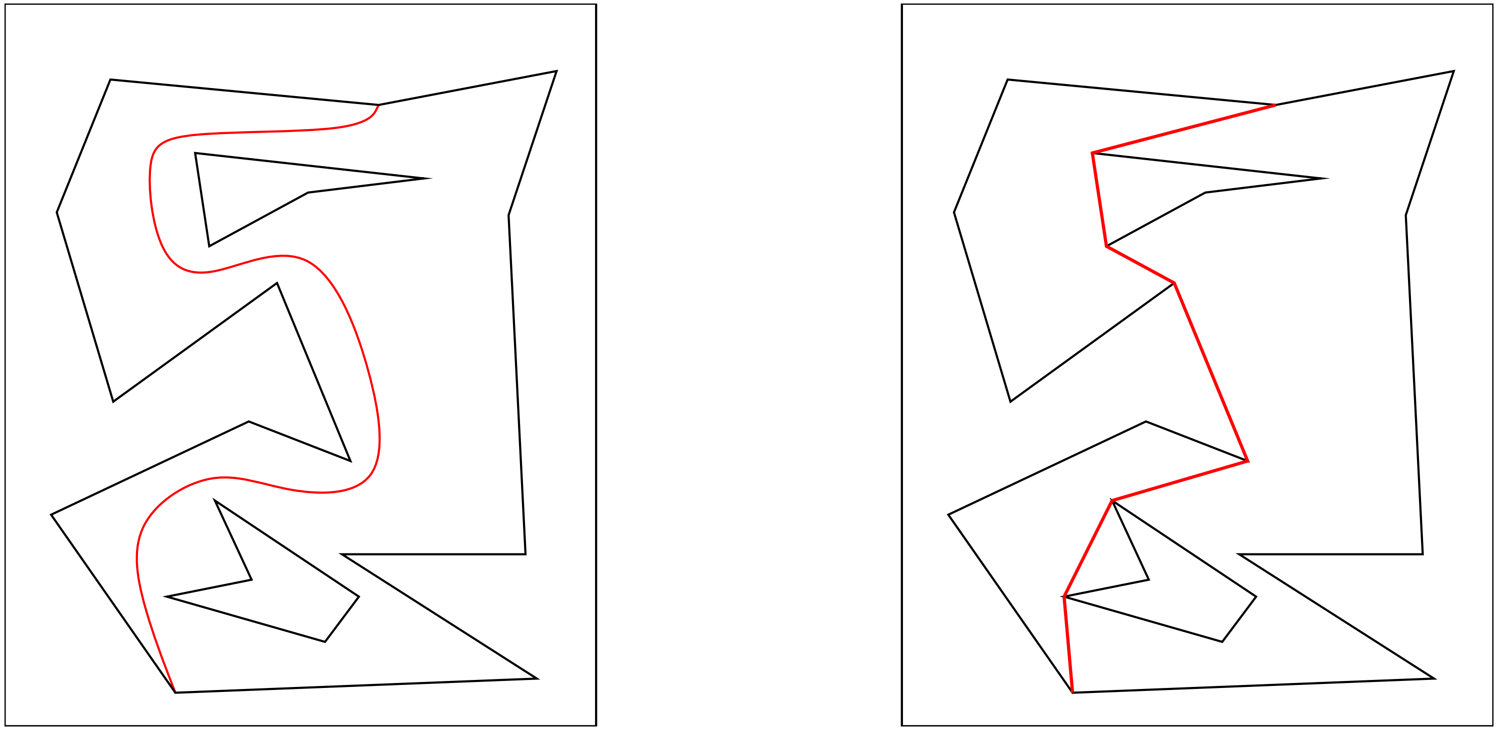




[Pocchiola and Vegter 1996]

Geodesic shortest paths

Shortest path (with given homotopy) turns only at pointed vertices. Addition of shortest path edges leaves intermediate vertices pointed.



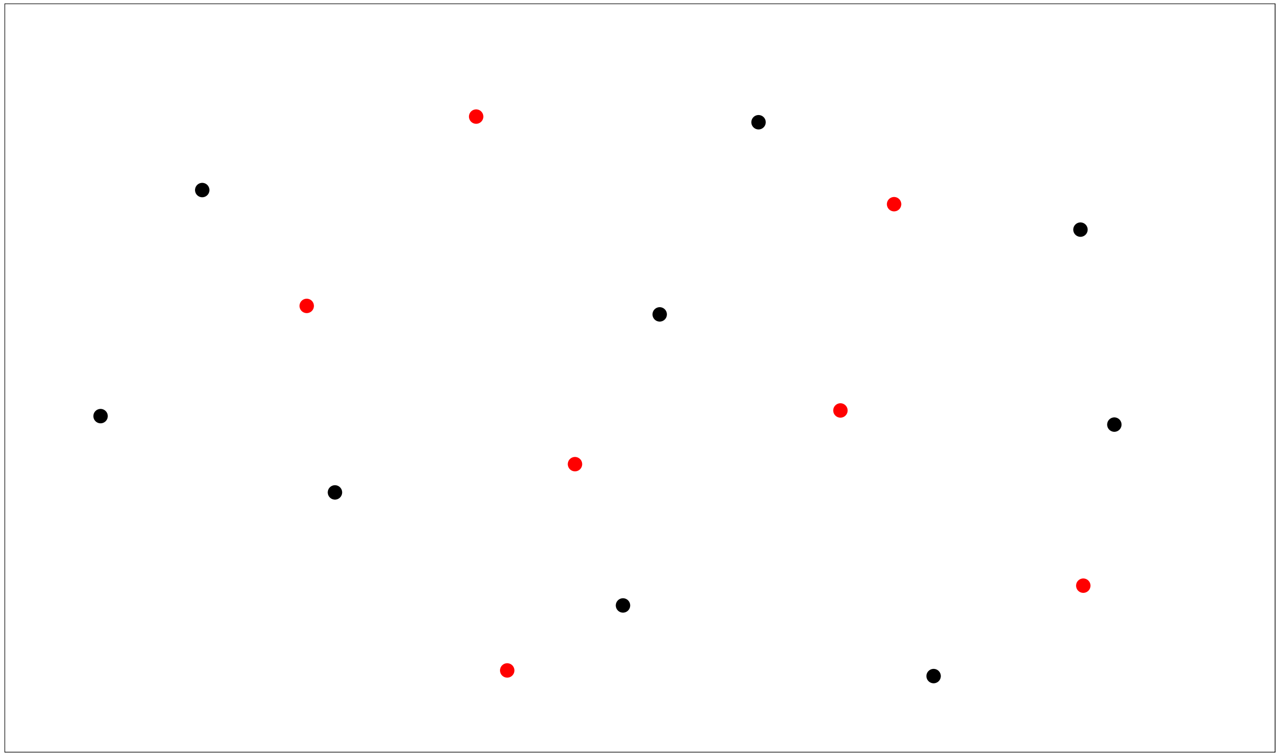
→ *geodesic* triangulations of a simple polygon

[Chazelle, Edelsbrunner, Grigni, Guibas, Hershberger, Sharir, Snoeyink '94]

Pseudotriangulations

Given: A set V of vertices, a subset $V_p \subseteq V$ of *pointed vertices*.

A *pseudotriangulation* is a maximal (with respect to $\subseteq$) set of non-crossing edges with all vertices in V_p pointed.



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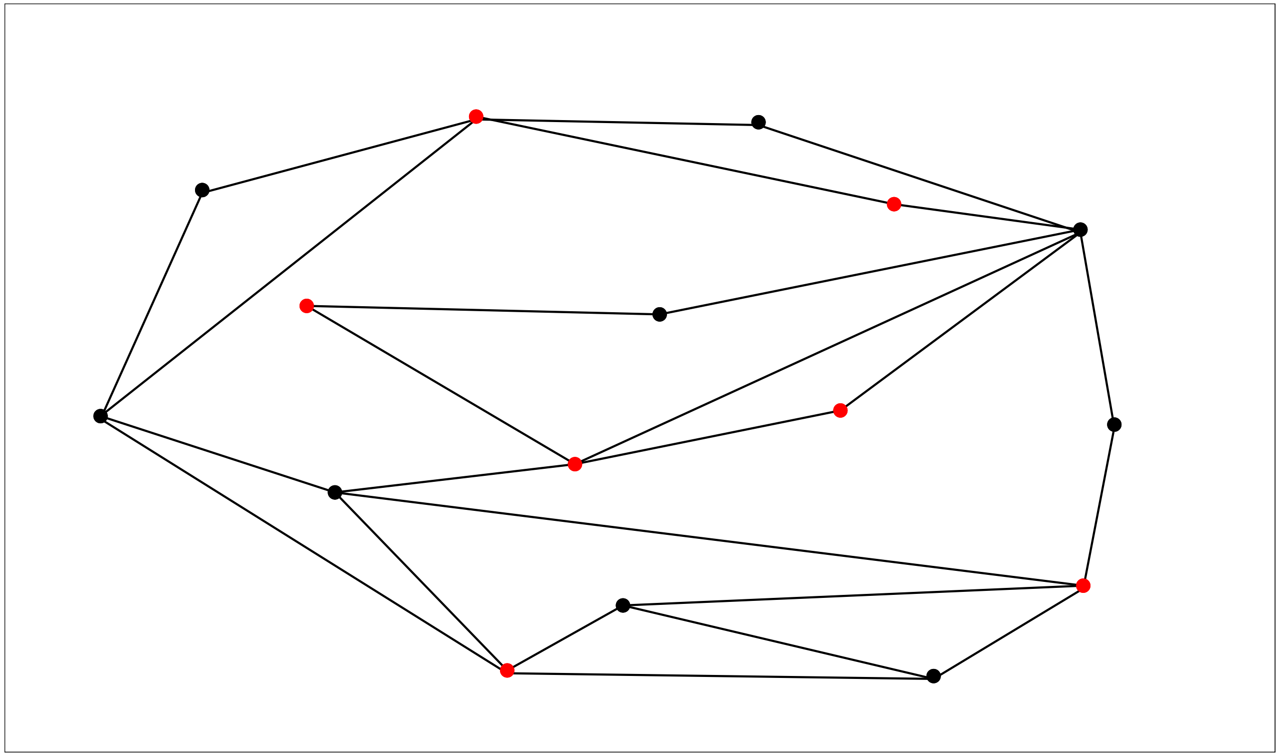

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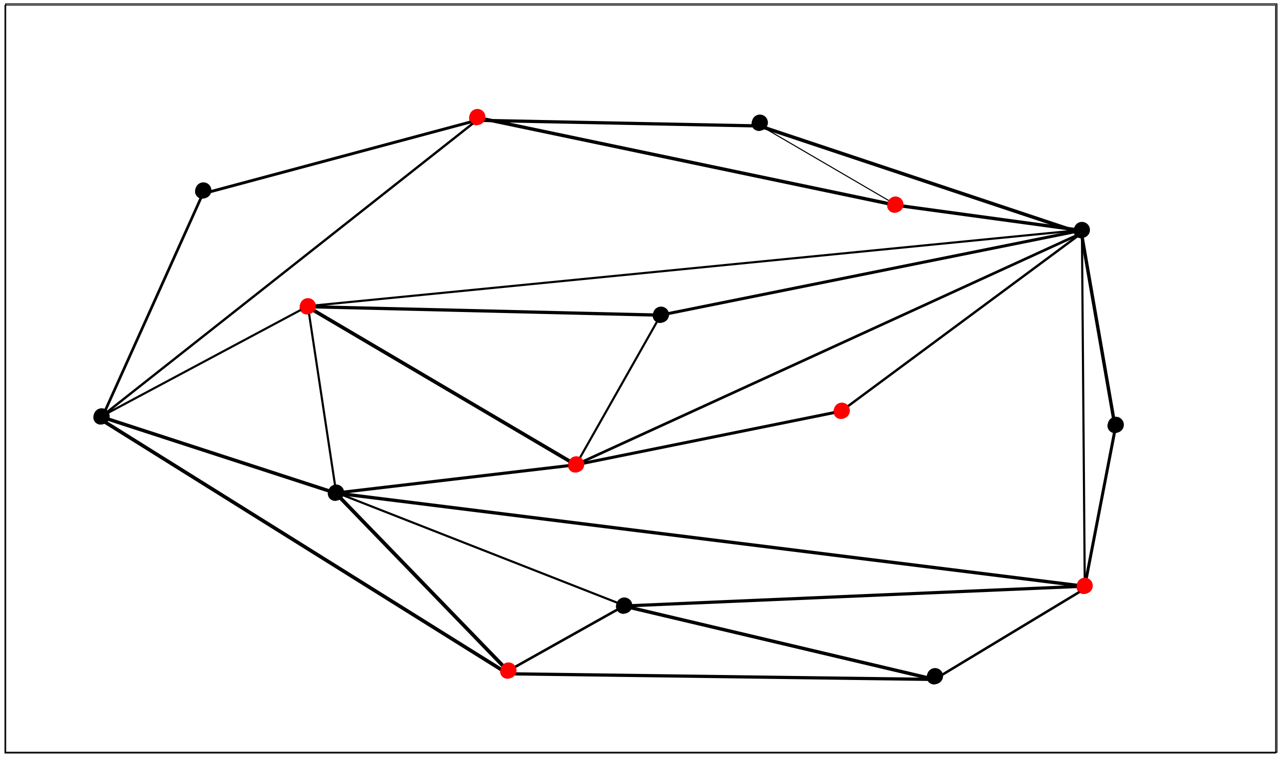

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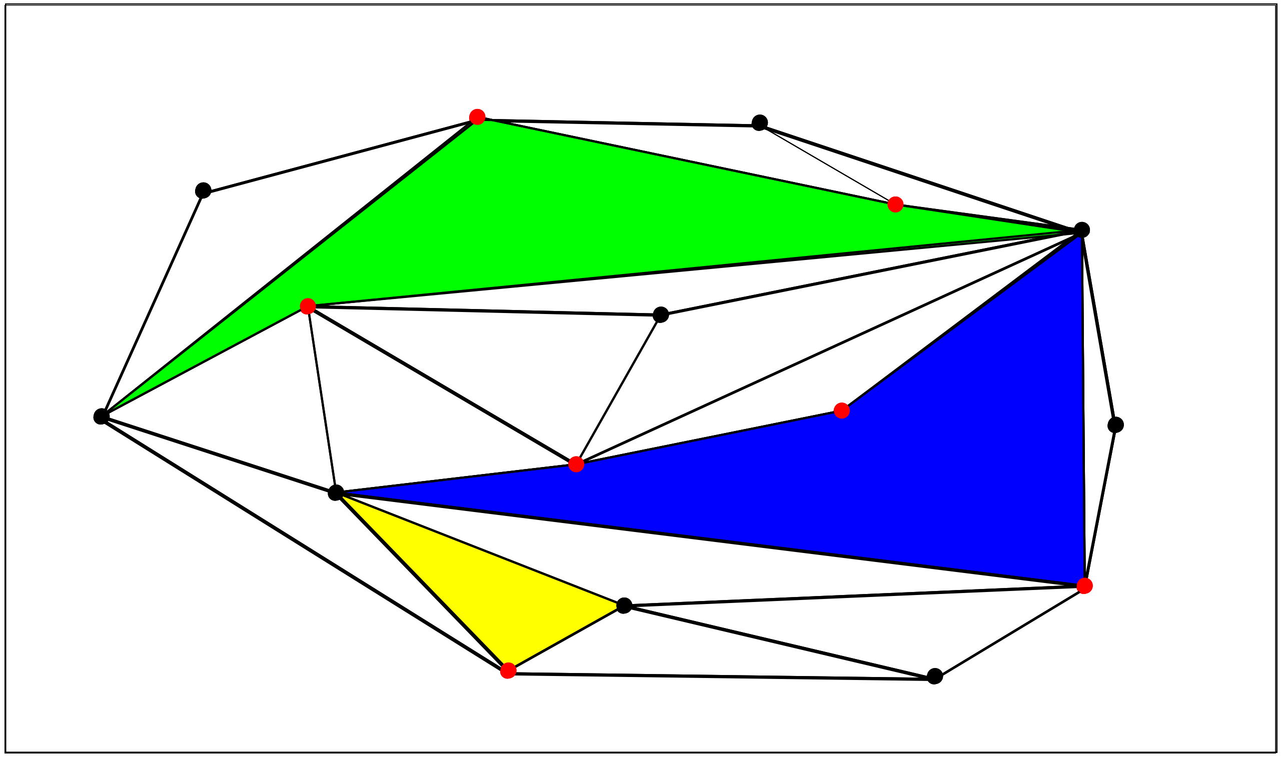
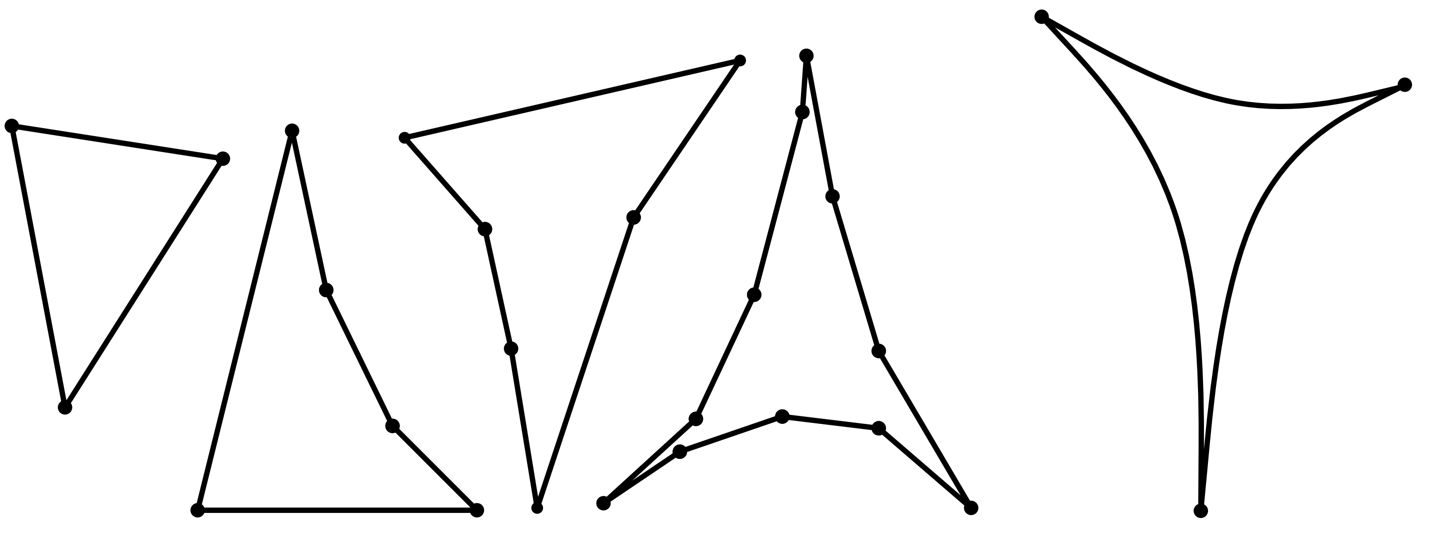


Pseudotriangles

A pseudotriangle has three convex *corners* and an arbitrary number of reflex vertices ($> 180^\circ$).



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Proof. $(2) \implies (1)$ No edge can be added inside a pseudotriangle without creating a nonpointed vertex.
Proof. $(1) \implies (2)$ All convex hull edges are in E.
$\rightarrow$ decomposition of the polygon into faces.
Need to show: If a face is not a pseudotriangle, then one can add an edge without creating a nonpointed vertex.

Characterization of pseudotriangulations

Lemma. *If a face is not a pseudotriangle, then one can add an edge without creating a nonpointed vertex.*

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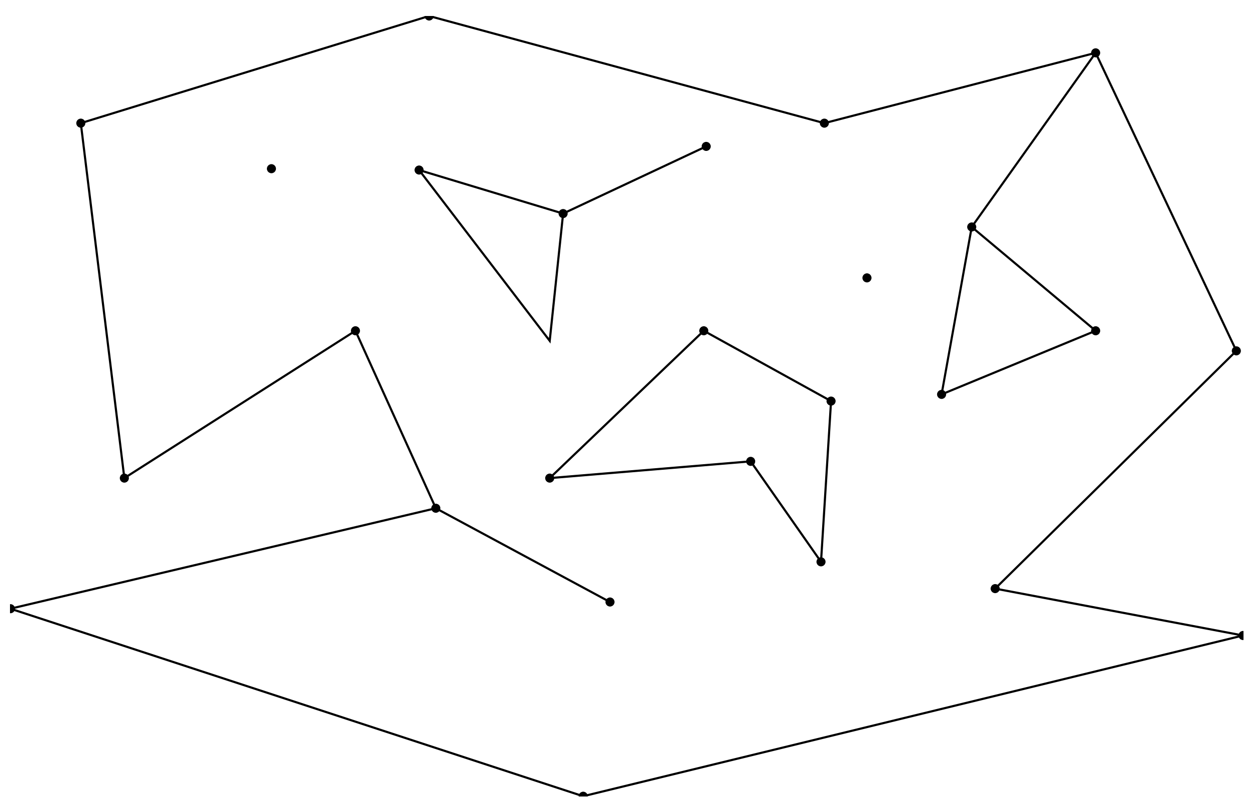
Go from a convex vertex along the boundary to the third convex vertex. Take the shortest path.



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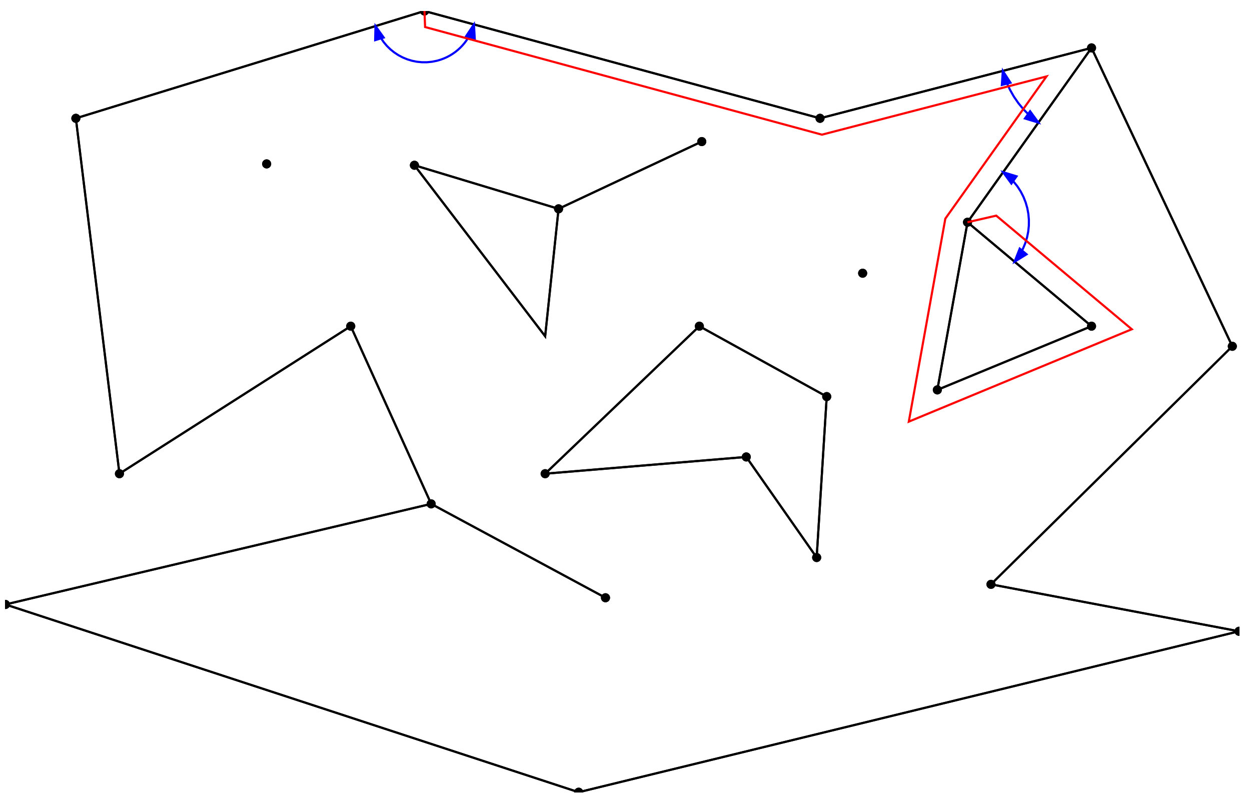

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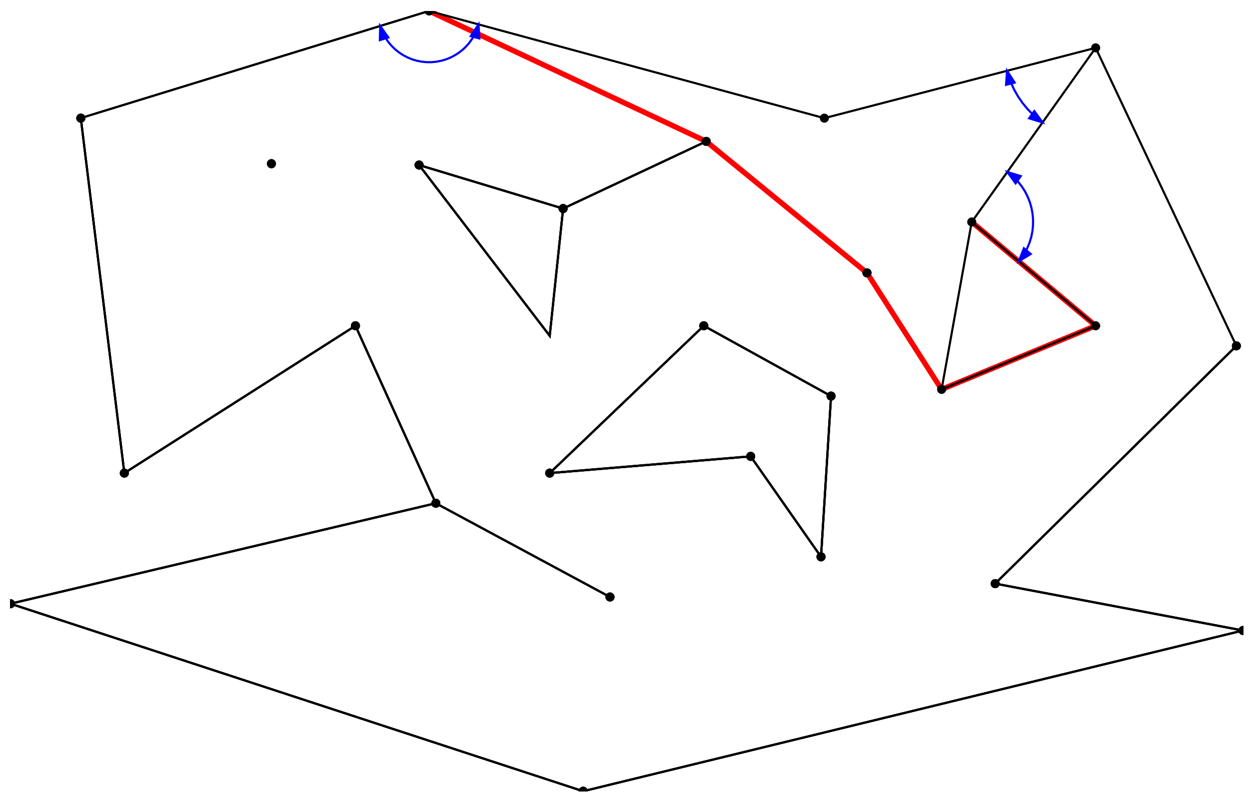
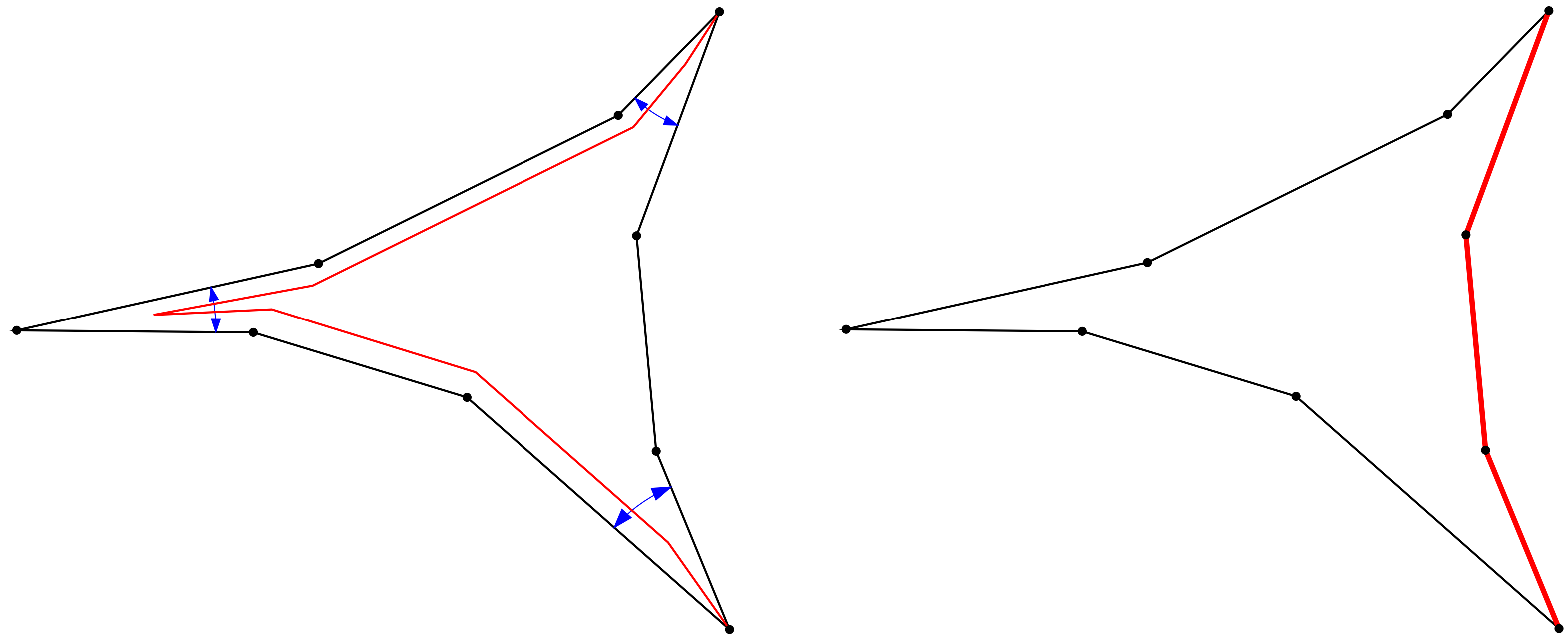


Characterization of pseudotriangulations, continued

A new edge is always added, unless the face is already a pseudotriangle (without inner obstacles).



[Rote, C. A. Wang, L. Wang, Xu 2003]

Vertex and face counts

Lemma. *A pseudotriangulation with* x *nonpointed and* y *pointed vertices has* $e = 3x + 2y - 3$ *edges and* $2x + y - 2$ *pseudotriangles. (Exercise 1)*

Corollary. *A pointed pseudotriangulation with* n *vertices has* $e = 2n - 3$ *edges and* $n - 2$ *pseudotriangles.*

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BETTER THAN TRIANGULATIONS!

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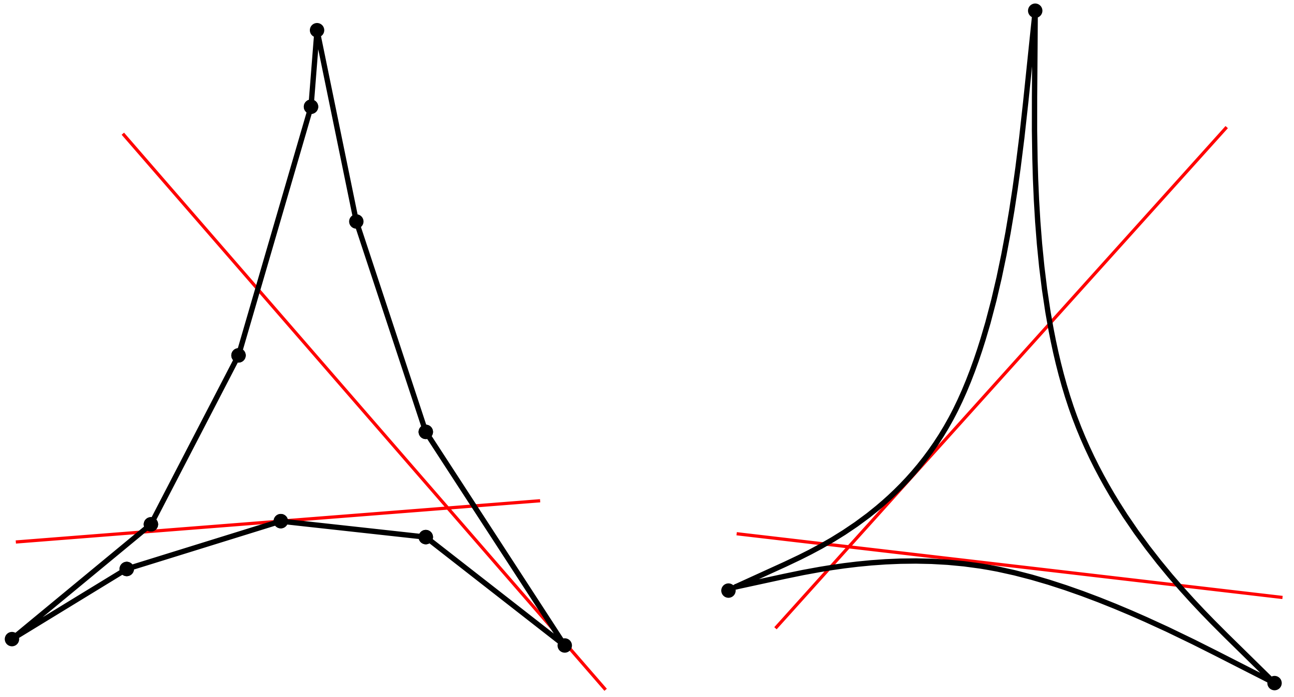
BETTER THAN TRIANGULATIONS!

Corollary. *A pointed graph with* $n \geq 2$ *vertices has at most* $2n - 3$ *edges.*

Tangents of pseudotriangles

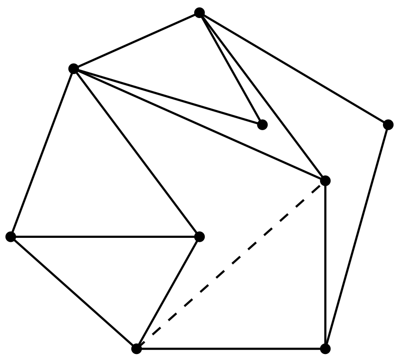
"Proof. $(2) \implies (1)$ No edge can be added inside a pseudo-triangle without creating a nonpointed vertex."

For every direction, there is a unique line which is "tangent" at a reflex vertex or "cuts through" a corner.

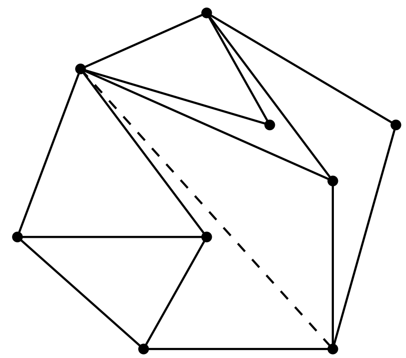


Flipping of Edges

Any interior edge can be flipped against another edge. That edge is unique.



before



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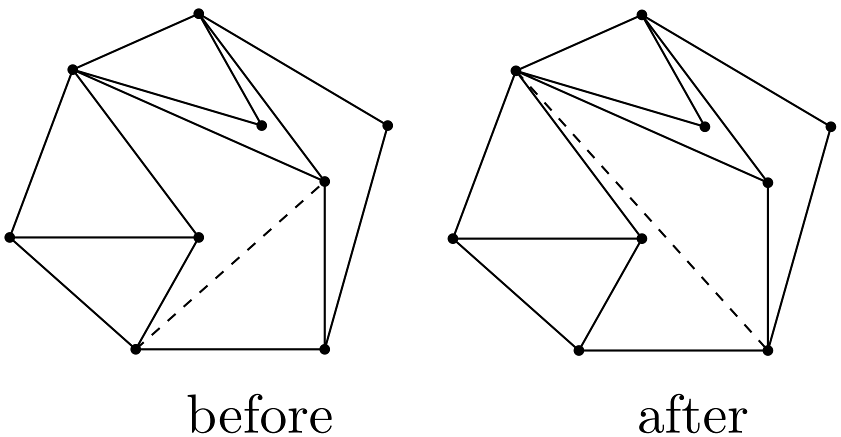

The flip graph is connected.
Its diameter is $O(n \log n)$. [Bespamyatnikh 2003]

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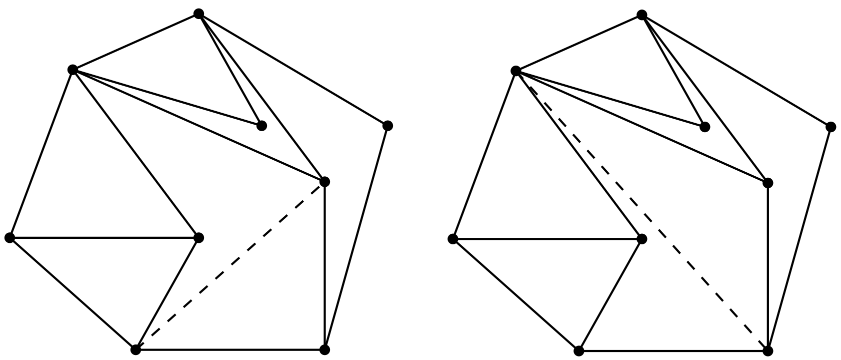

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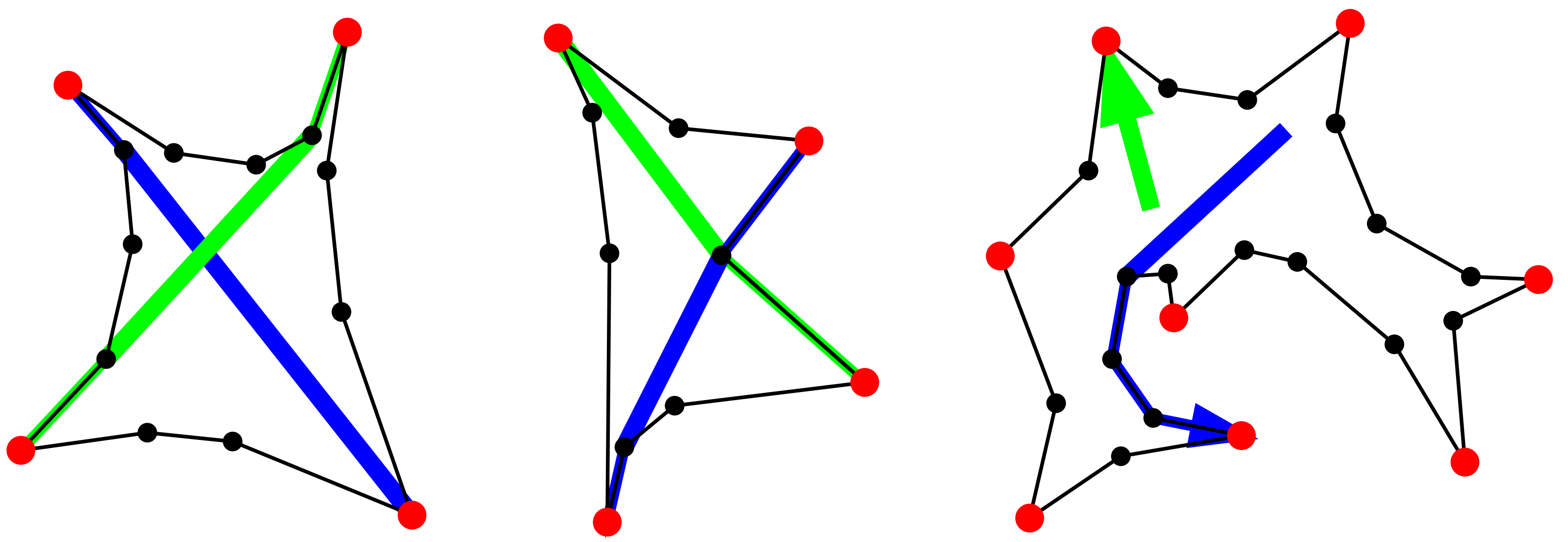
BETTER THAN TRIANGULATIONS!

Flipping

Every *tangent ray* can be continued to a geodesic path running along the boundary to a corner, in a unique way.

Every pseudoquadrangle has precisely two diagonals, which cut it into two pseudotriangles. (see (Exercise 6)

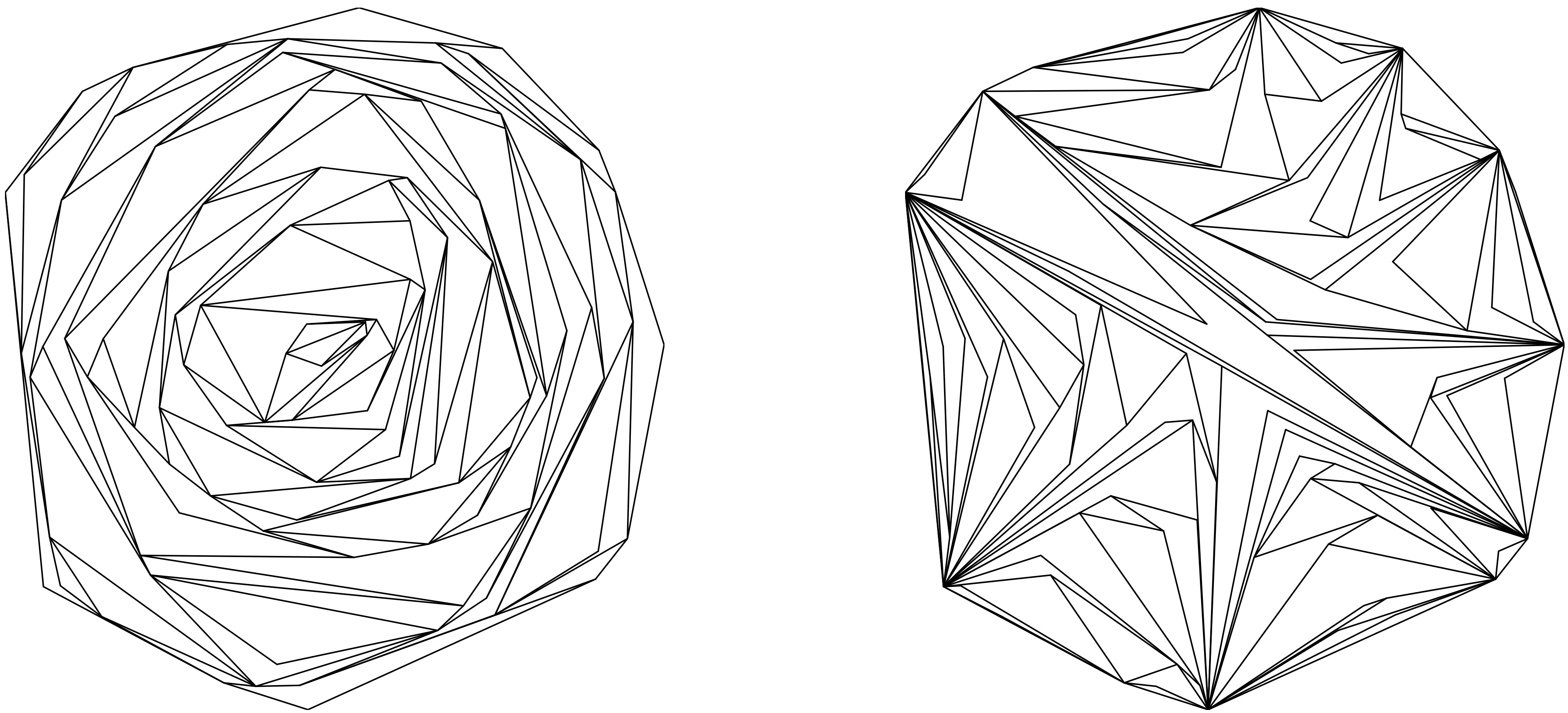


Pseudotriangulations/ Geodesic Triangulations

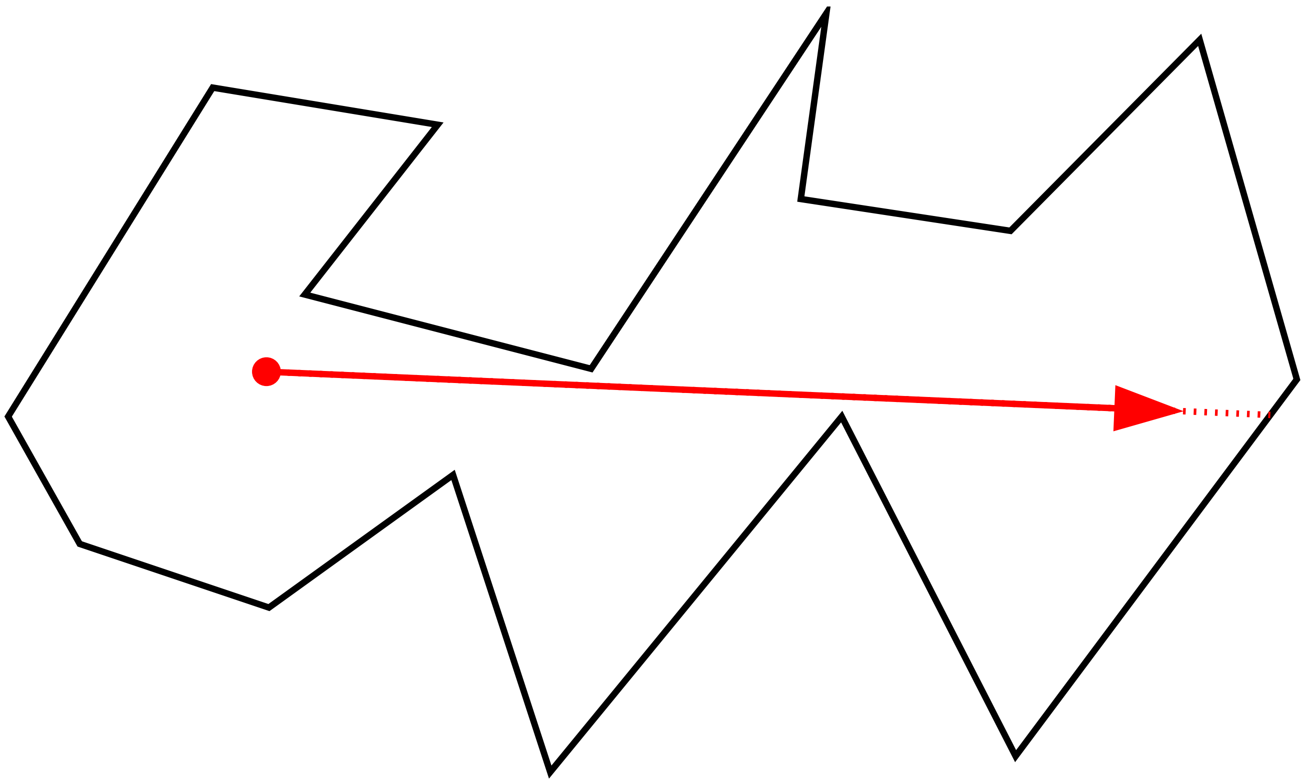
Applications:

- motion planning, unfolding of polygonal chains [Streinu 2001]
- data structures for ray shooting [Chazelle, Edelsbrunner, Grigni, Guibas, Hershberger, Sharir, and Snoeyink 1994] and visibility [Pocchiola and Vegter 1996]
- kinetic collision detection [Agarwal, Basch, Erickson, Guibas, Hershberger, Zhang 1999–2001] [Kirkpatrick, Snoeyink, and Speckmann 2000] [Kirkpatrick & Speckmann 2002] (see (Exercises 3 and 4)
- art gallery problems [Pocchiola and Vegter 1996b], [Speckmann and Tóth 2001]

Two pseudotriangulations for 100 random points

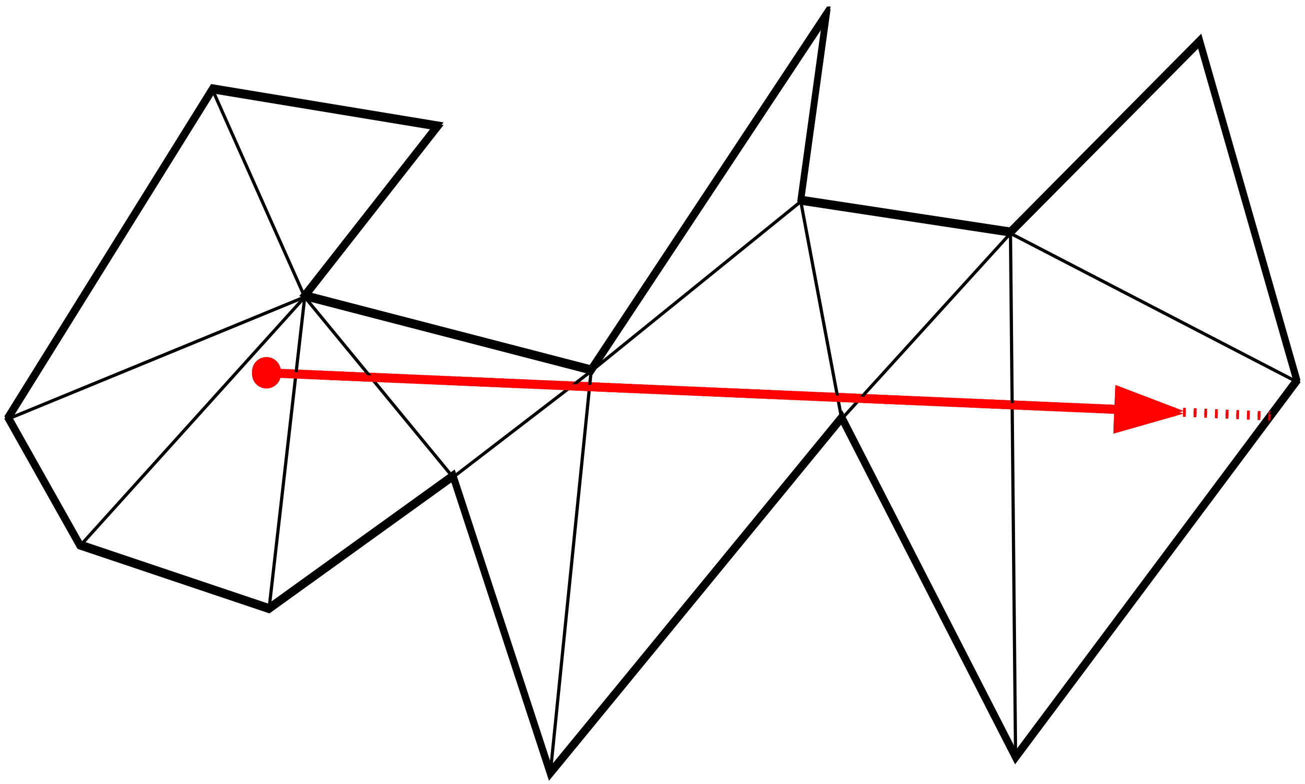


1. Application: Ray Shooting in a Simple Polygon



1. Application: Ray Shooting in a Simple Polygon

Or: Computing the crossing sequence of a path π



Walking in a triangulation:

Walk to starting point. Then walk along the ray.

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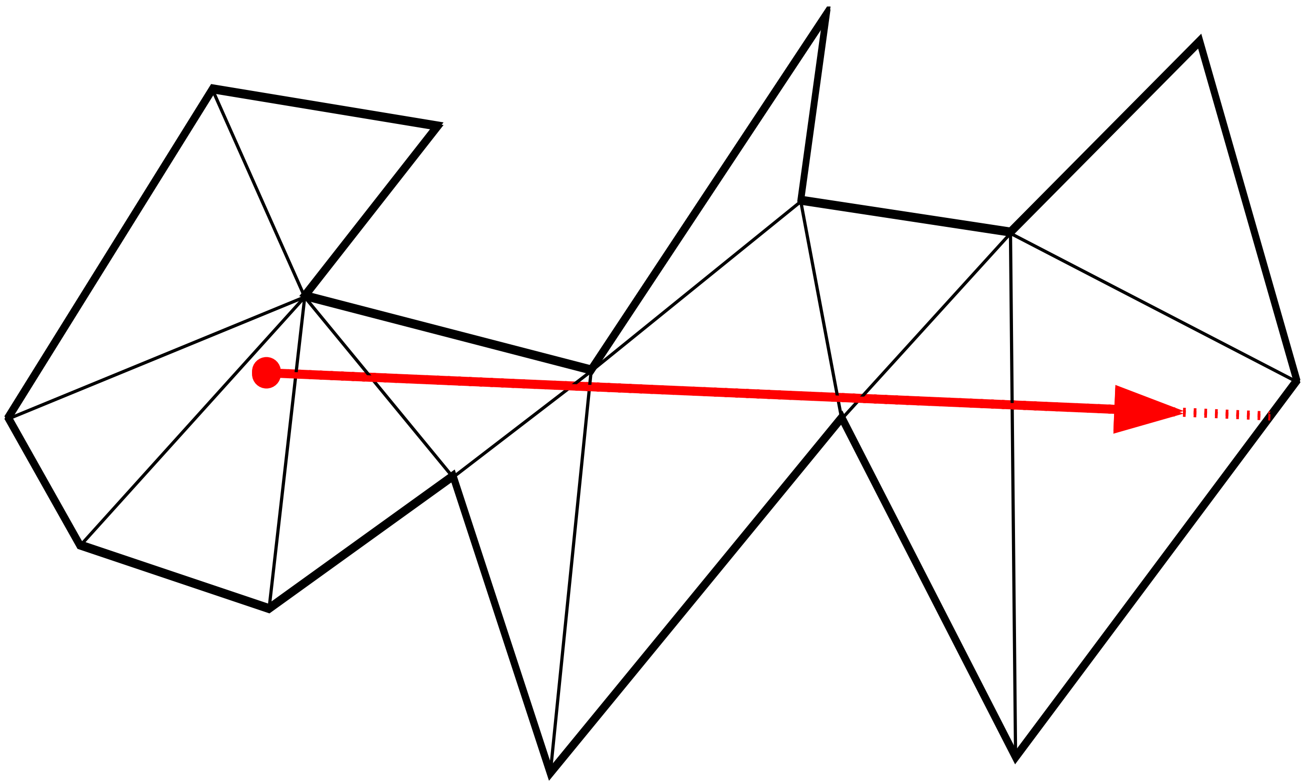
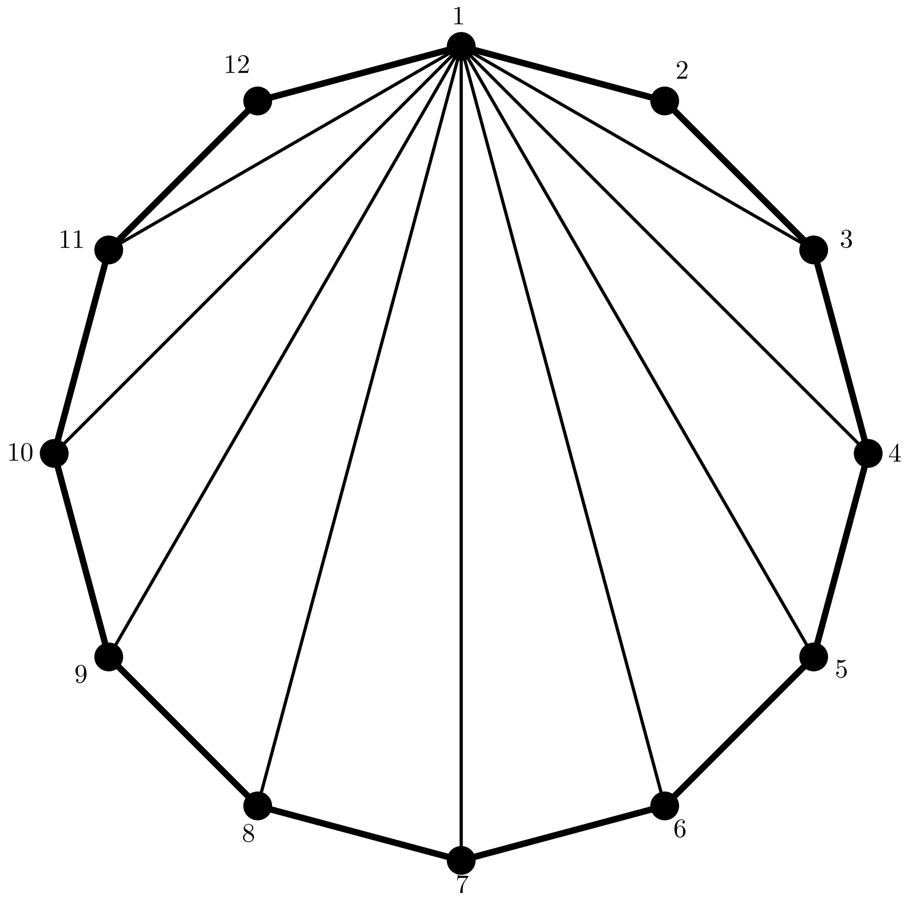


Walking in a triangulation:

Walk to starting point. Then walk along the ray.

$O(n)$ steps in the worst case.

Triangulations of a *convex* polygon



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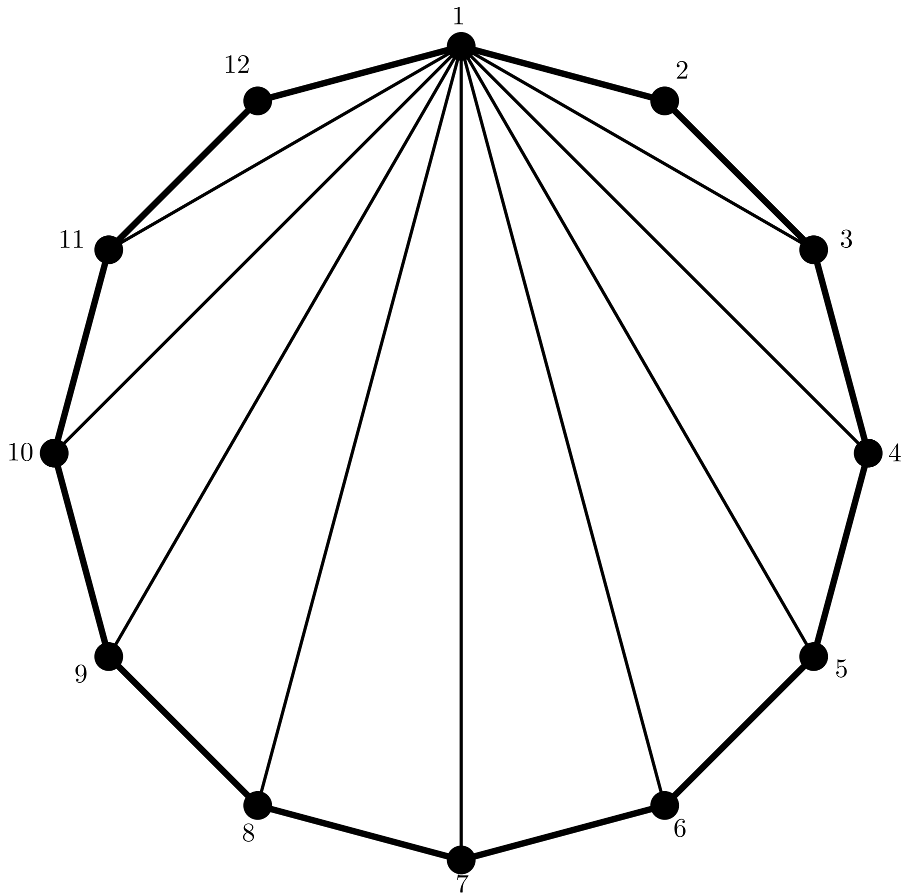
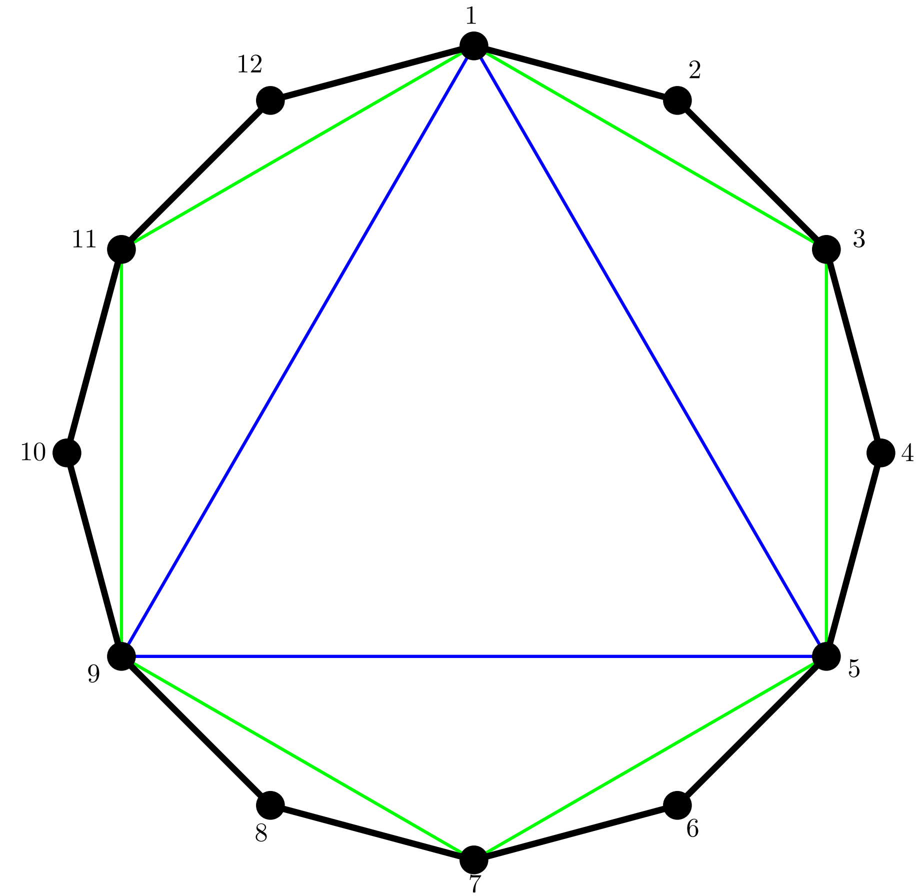
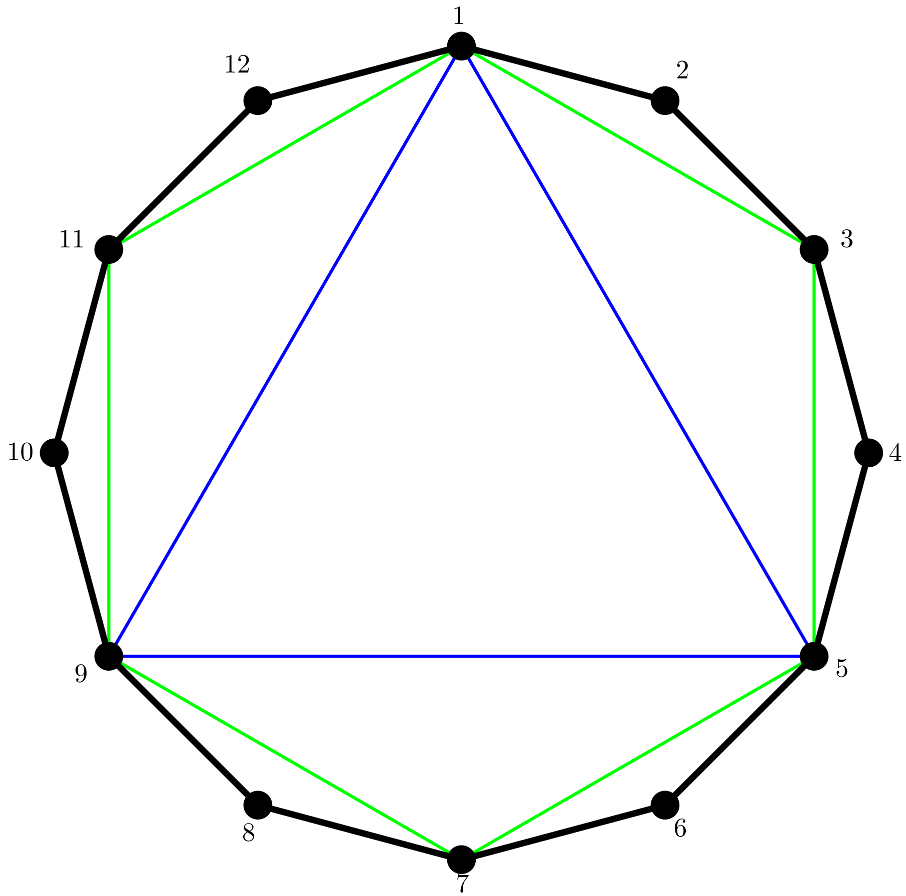
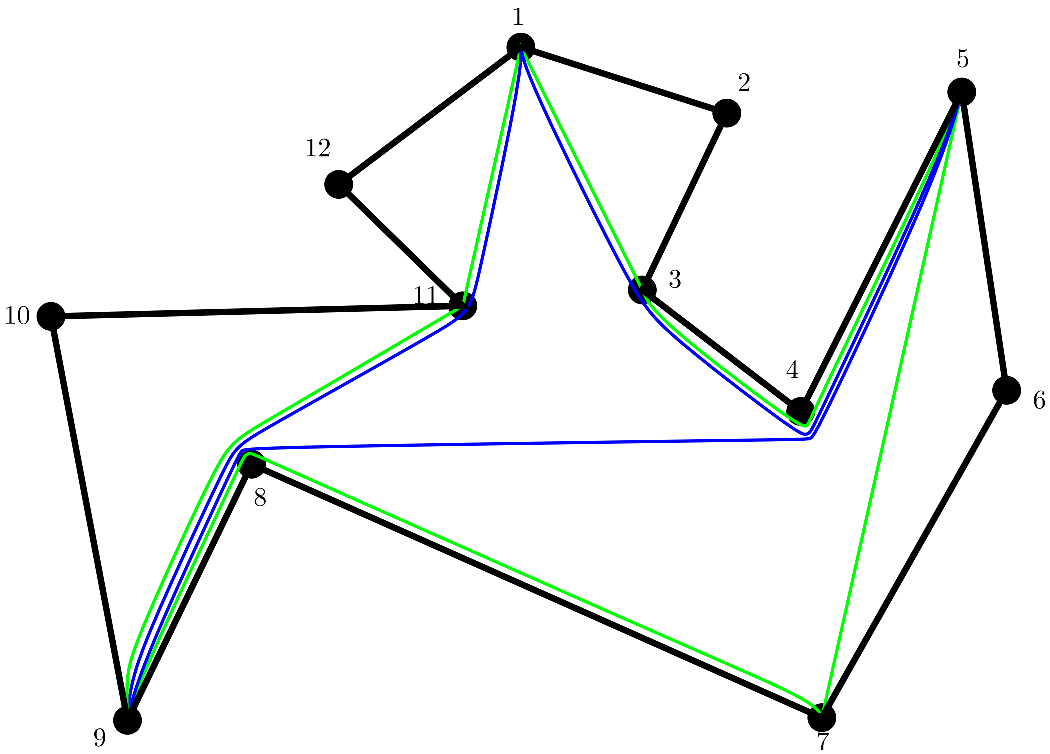




balanced triangulation
A path crosses $O(\log n)$
triangles.

Triangulations of a *simple* polygon





balanced triangulation:
An edge crosses $O(\log n)$
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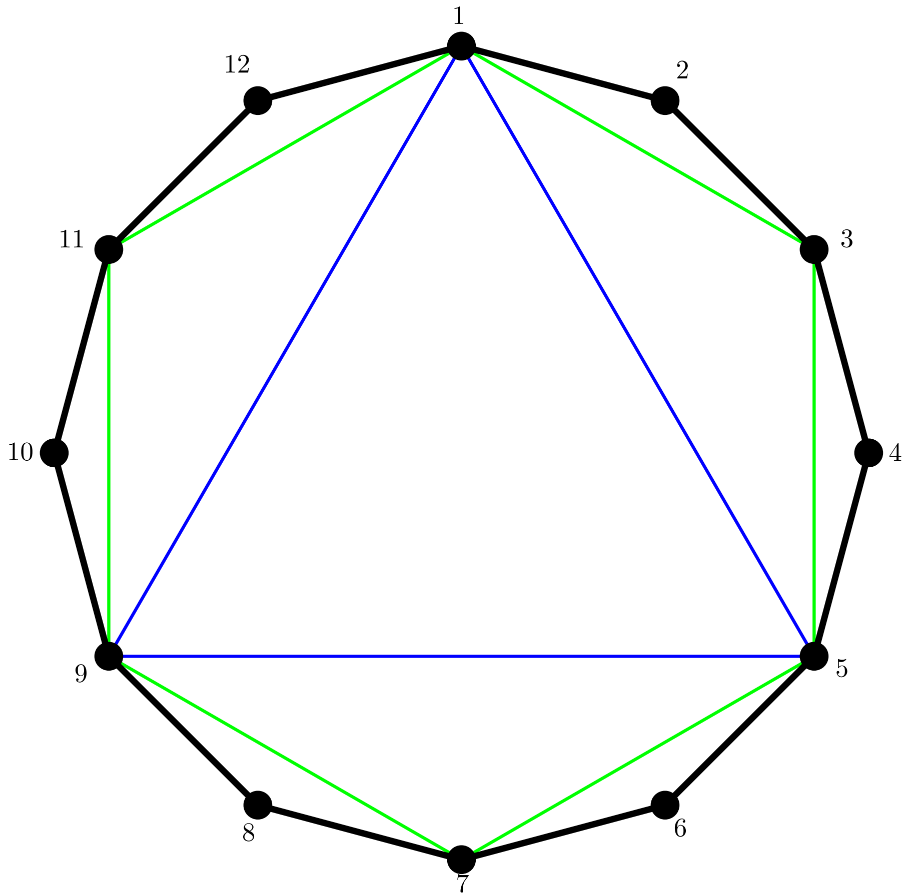
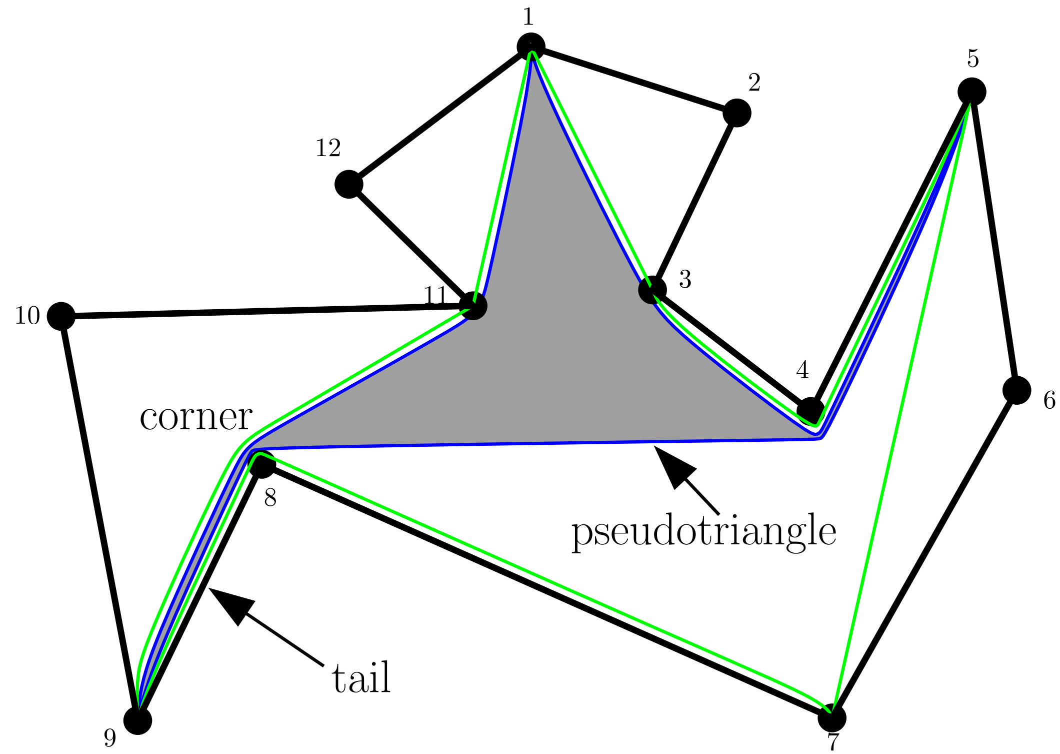
balanced *geodesic* triangulation:
An edge crosses $O(\log n)$
pseudotriangles.

[Chazelle, Edelsbrunner, Grigni, Guibas, Hershberger, Sharir, Snoeyink 1994]

Triangulations of a *simple* polygon





balanced triangulation:
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balanced *geodesic* triangulation:
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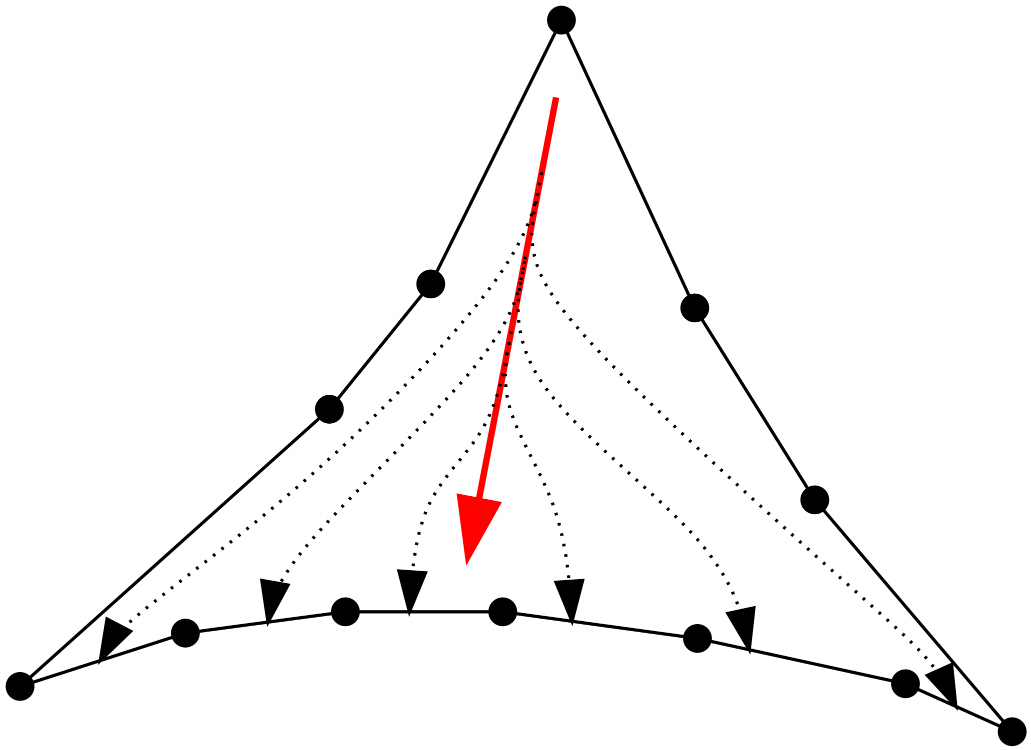
Going through a single pseudotriangle



balanced binary tree for each pseudo-edge:
$\rightarrow$ $O(\log n)$ time per pseudotriangle
$\rightarrow$ $O(\log^2 n)$ time total

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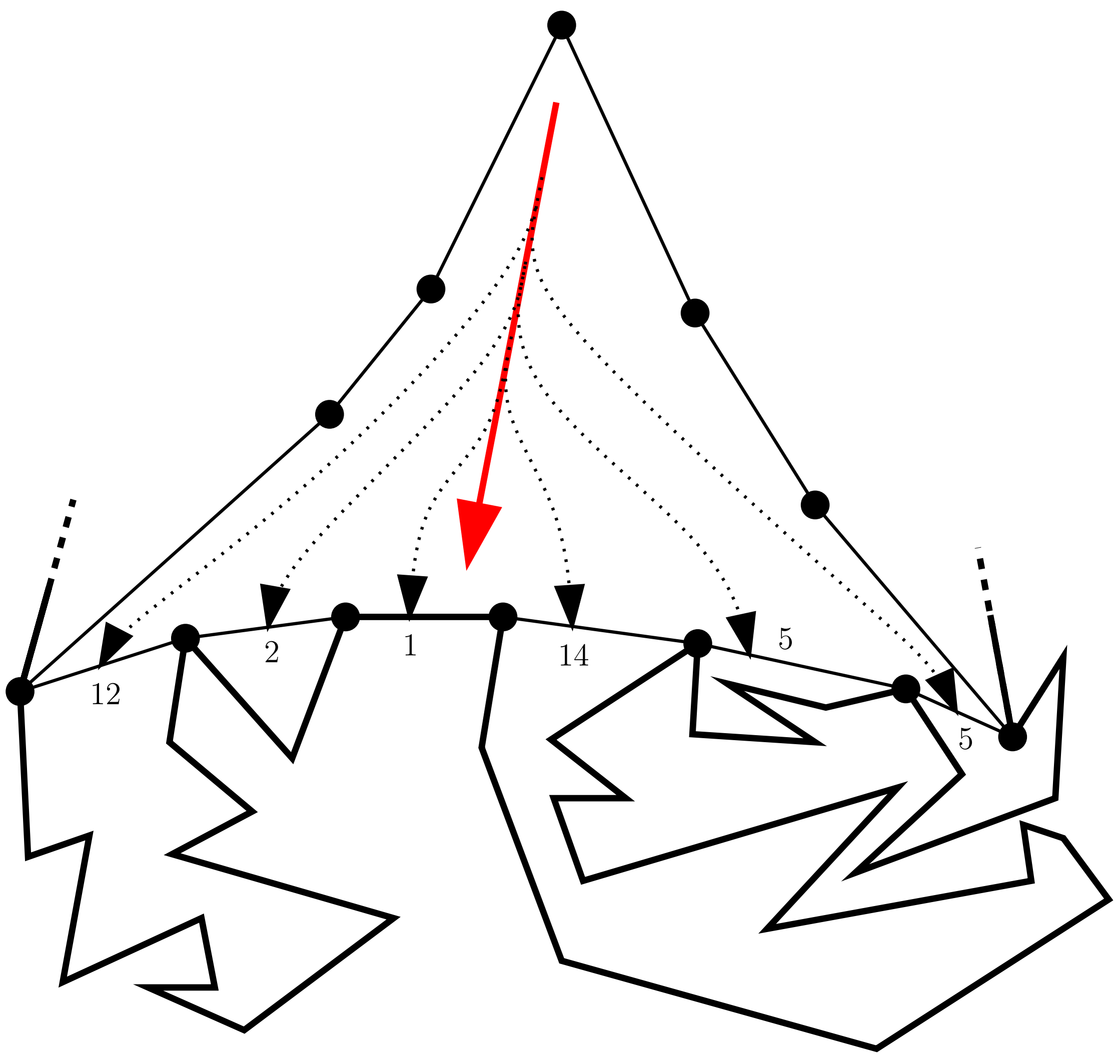

balanced binary tree for each pseudo-edge:
$\rightarrow O(\log n)$ time per pseudotriangle
$\rightarrow O(\log^2 n)$ time total

weighted binary tree:
$\rightarrow O(\log n)$ time total

2. Rigidity and Motions
Unfolding of polygons

Theorem. *Every polygonal arc in the plane can be brought into straight position, without self-overlap.*

Every polygon in the plane can be unfolded into convex position.

[Connelly, Demaine, Rote 2001], [Streinu 2001]

Infinitesimal motions — rigid frameworks

n vertices $p_1, \ldots, p_n$.

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$$v_i = \frac{d}{dt} p_i(t) = \dot{p}_i(0)$$

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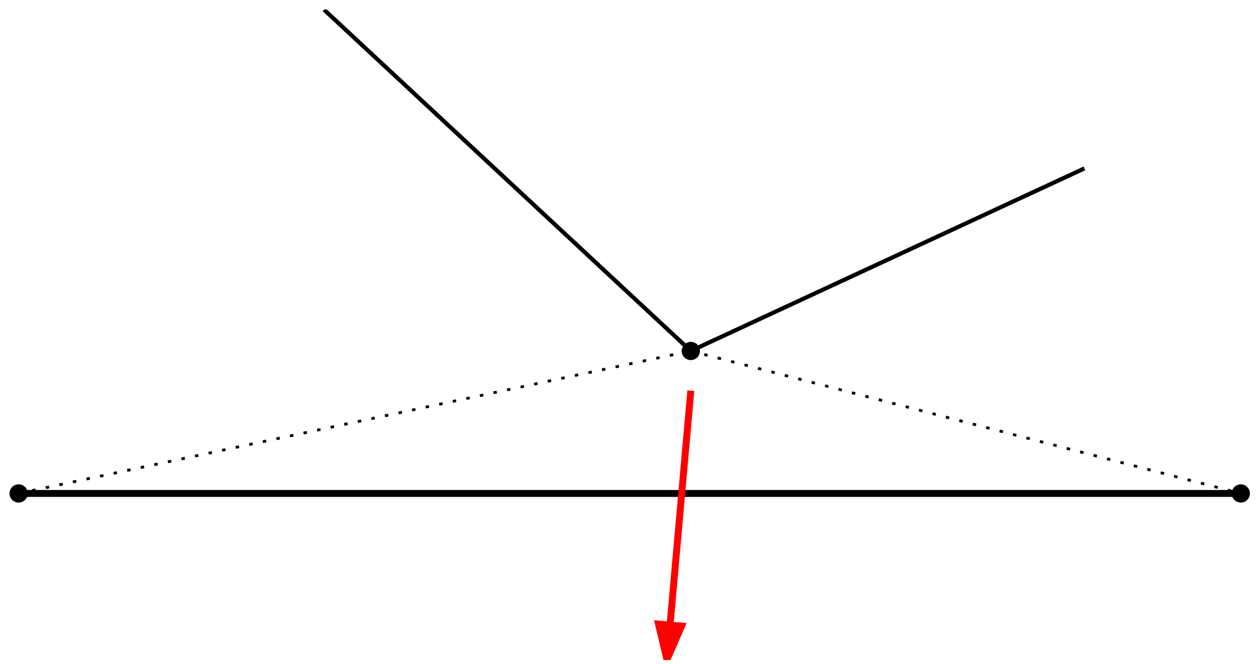
   Velocity vectors $v_1, \ldots, v_n$.

3. constraints:
   $|p_i(t) - p_j(t)|$ is constant for every edge (bar) ij.

Expansive Motions

No distance between any pair of vertices decreases.

Expansive motions cannot overlap.



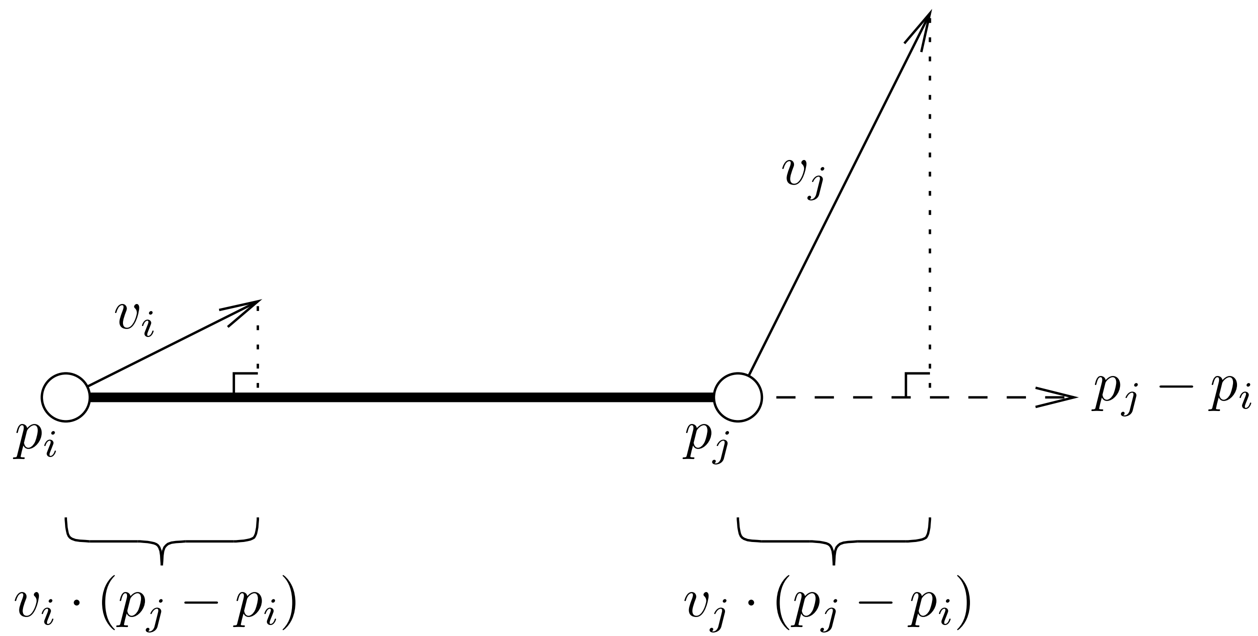
Expansive Mechanisms

A *framework* is a set of movable joints (vertices) connected by rigid bars (edges) of fixed length.

Pseudotriangulations with one convex hull edge removed are *expansive mechanisms*: The have one degree of freedom, and their motion is expansive.

Expansion

$$\tfrac{1}{2} \cdot \frac{d}{dt} |p_i(t) - p_j(t)|^2 = \langle v_i - v_j, p_i - p_j \rangle =: \exp_{ij}$$



expansion (or *strain*) $\exp_{ij}$ of the segment ij

$\exp_{ij} < 0$: "*compression*"

The rigidity map

of a framework $((V,E),(p_1,\dots,p_n))$:

$$M\colon (v_1,\dots,v_n)\mapsto (\mathrm{exp}_{ij})_{ij\in E}$$

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of a framework $((V, E), (p_1, \ldots, p_n))$:

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The rigidity matrix:

$$M = \underbrace{\begin{pmatrix} \text{the} \\ \text{rigidity} \\ \text{matrix} \end{pmatrix}}_{2|V|} \Bigg\} E$$

Infinitesimally rigid frameworks

A framework is *infinitesimally rigid* if

$$M(v) = 0$$

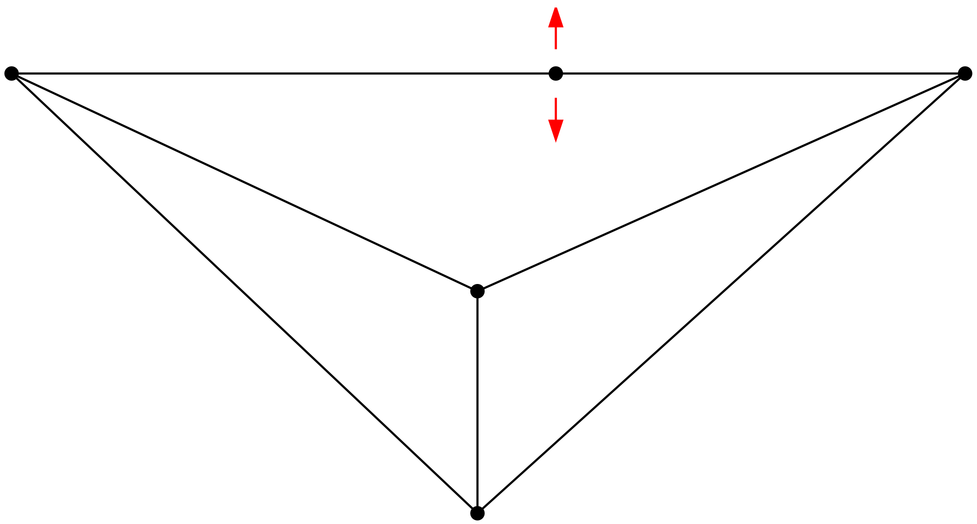
has only the trivial solutions: translations and rotations of the framework as a whole.

Rigid frameworks

A framework is *rigid* if it allows only translations and rotations of the framework as a whole.

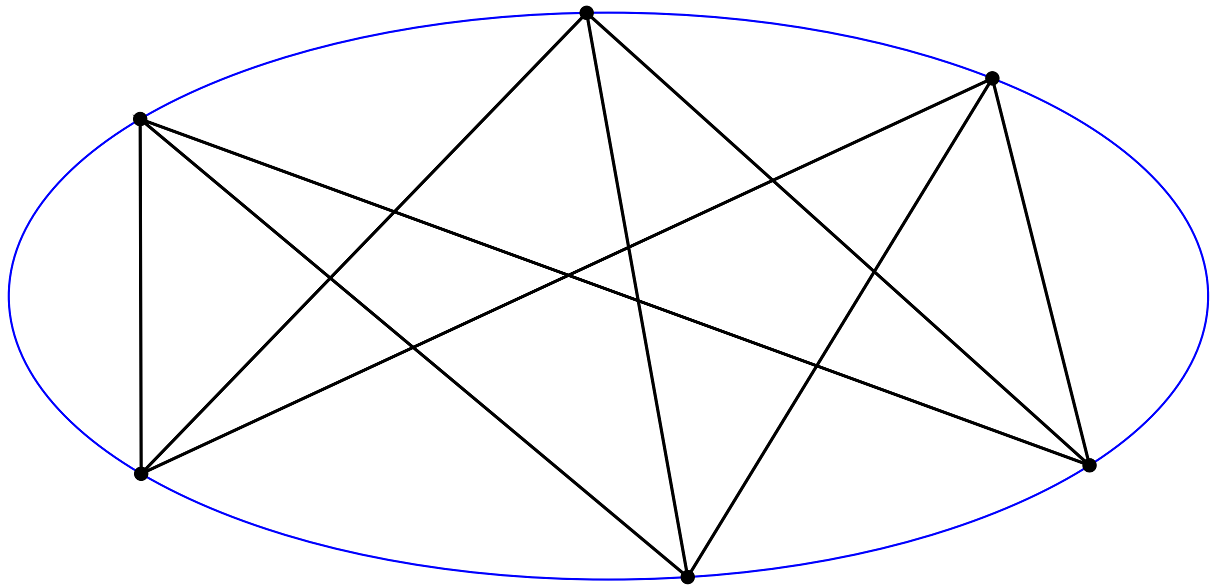
An infinitesimally rigid framework is rigid.

This framework is rigid, but not infinitesimally rigid:



Generically rigid frameworks

A given graph can be rigid in most embeddings, but it may have special non-rigid embeddings:



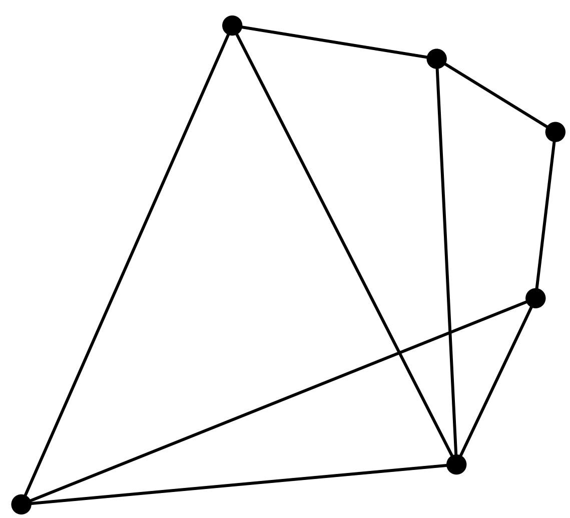
A graph is *generically rigid* if it is infinitesimally rigid in almost all embeddings.

This is a *combinatorial property* of the graph.

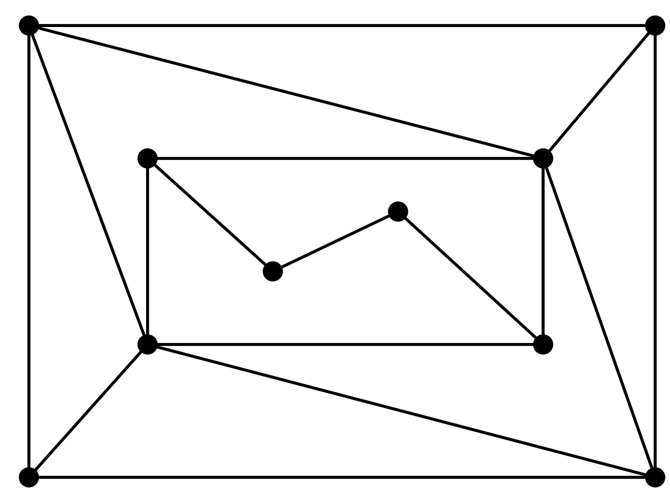
Minimally rigid frameworks

A graph with n vertices is generically *minimally rigid* in the plane (with respect to $\subseteq$) iff it has the *Laman property*:

- It has $2n-3$ edges.
- Every subset of $k \geq 2$ vertices spans at most $2k-3$ edges.



$n=6,\ e=9$



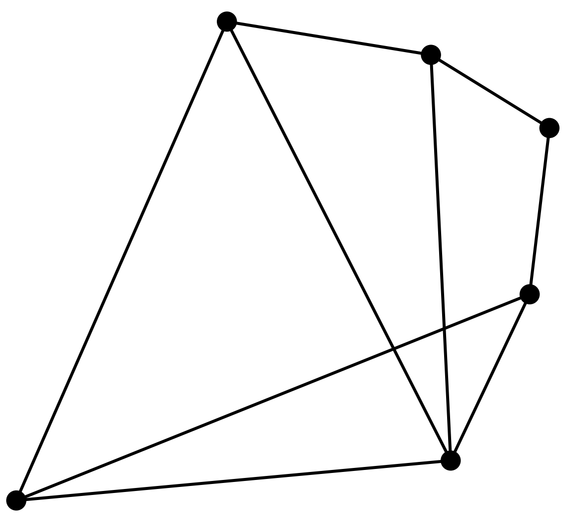
$n=10,\ e=17$

[Laman 1961]

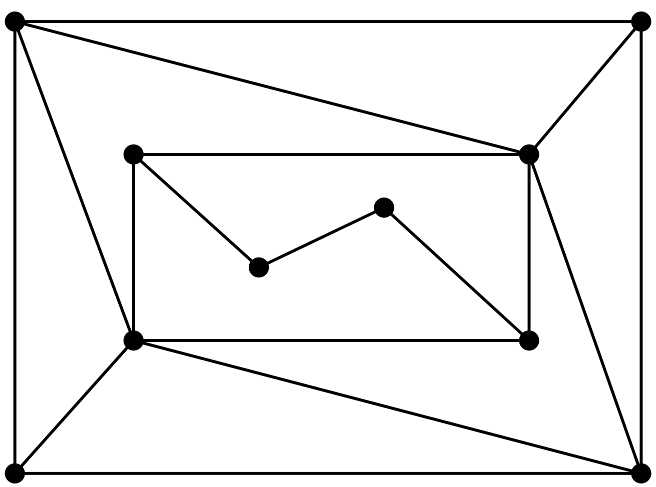
Pointed pseudotriangulations are Laman graphs

Theorem. [Streinu 2001] *Every pointed pseudotriangulation has the* Laman property*:*

It has $2n-3$ edges.
Every subset of $k \geq 2$ vertices spans at most $2k-3$ edges.



$n = 6,\ e = 9$



$n = 10,\ e = 17$

Proof: Every subgraph is pointed.

The Laman condition

The Laman property:

- It has $2n - 3$ edges.
- Every subset S of $k \geq 2$ vertices spans at most $2k-3$ edges.

The second condition can be rephrased:

- Every subset $\bar{S}$ of $k \leq n - 2$ vertices is incident to at least $2k$ edges.

3. Every planar Laman graph is a pointed pseudotriangulation

Theorem. *Every pointed pseudotriangulation is a Laman graph.*

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Theorem. *Every planar Laman graph has a realization as a pointed pseudotriangulation. The outer face can be chosen arbitrarily.*

Proof I: Induction, using *Henneberg constructions*
Proof II: via Tutte embeddings for directed graphs

[Haas, Rote, Santos, B. Servatius, H. Servatius, Streinu, Whiteley 2003]

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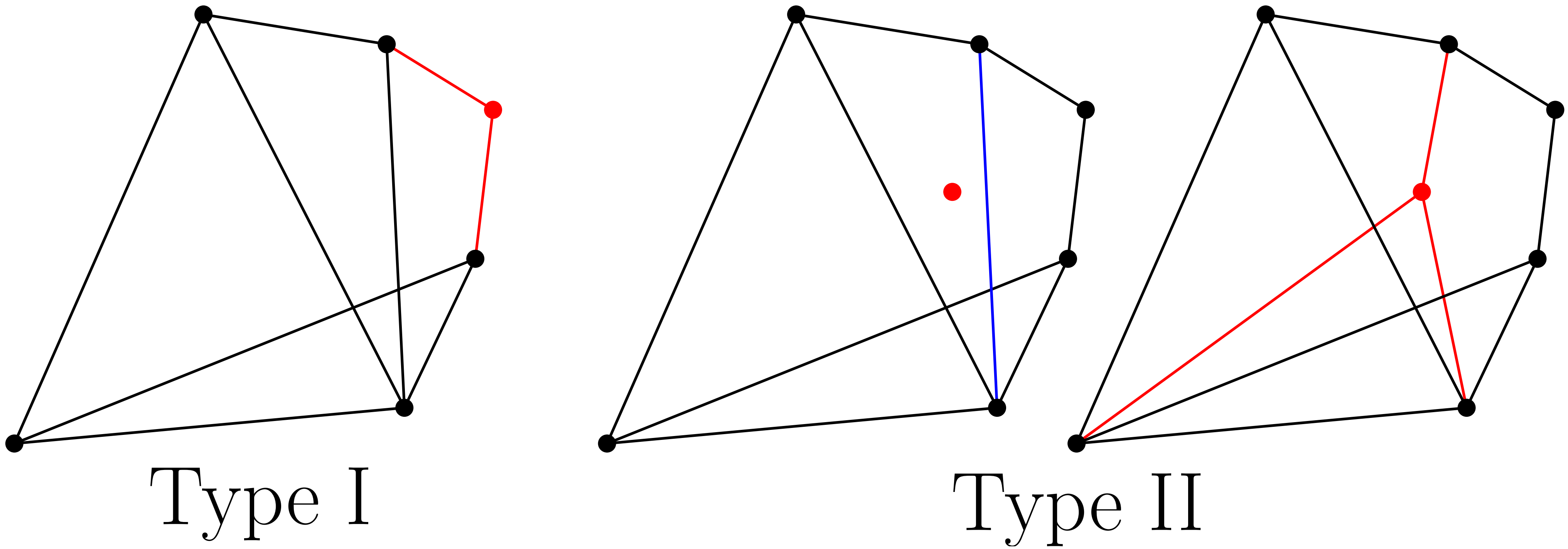
Proof I: Induction, using *Henneberg constructions*
Proof II: via Tutte embeddings for directed graphs

[Haas, Rote, Santos, B. Servatius, H. Servatius, Streinu, Whiteley 2003]

Theorem. *Every rigid planar graph has a realization as a pseudotriangulation.*

[Orden, Santos, B. Servatius, H. Servatius 2003]

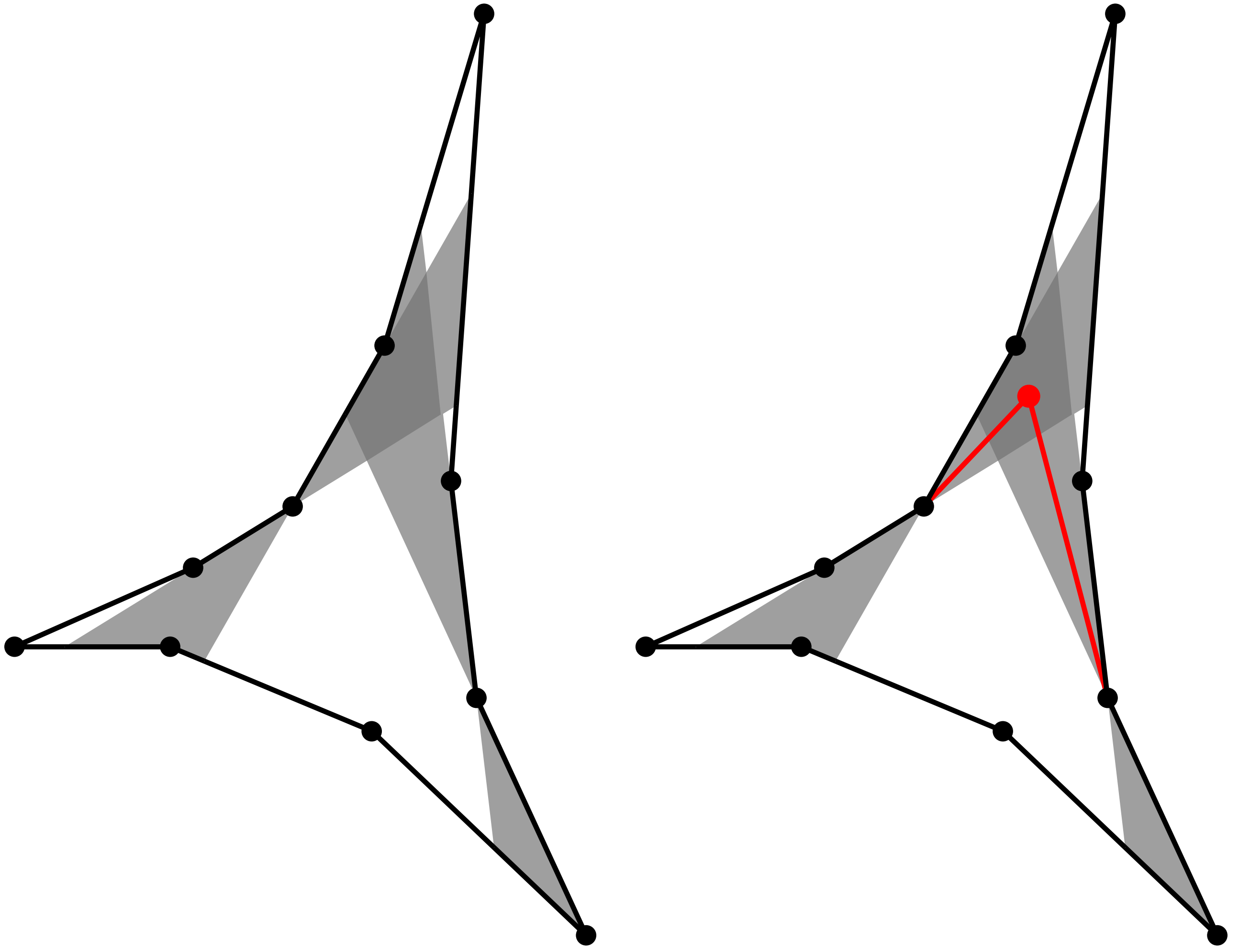
Henneberg constructions



Every Laman graph can be built up by a sequence of Henneberg construction steps, starting from a single edge.

(Exercises 14 and 15)

Proof I: Henneberg constructions



Planarity can be maintained during the Henneberg construction.

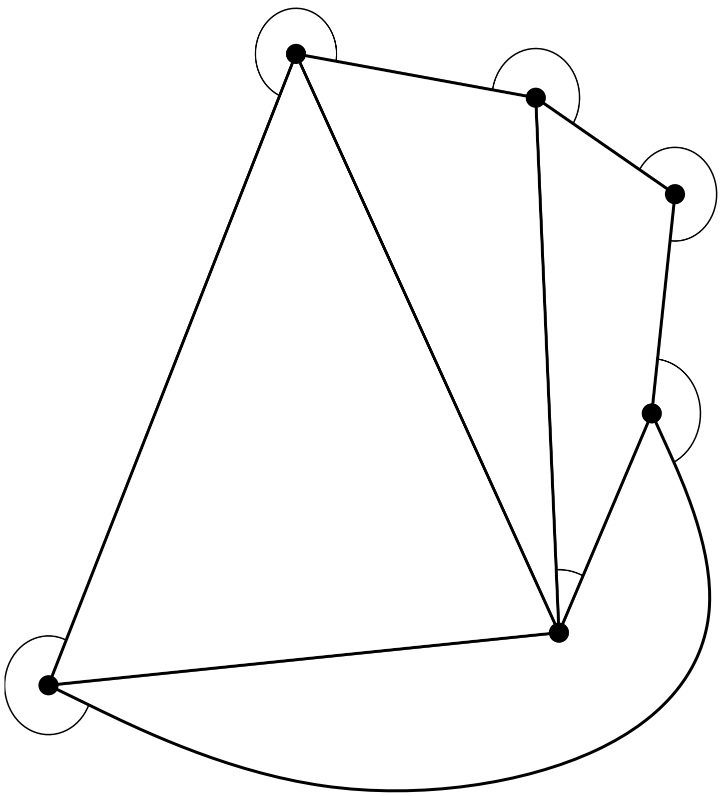
Proof II: embedding Laman graphs via directed Tutte embeddings

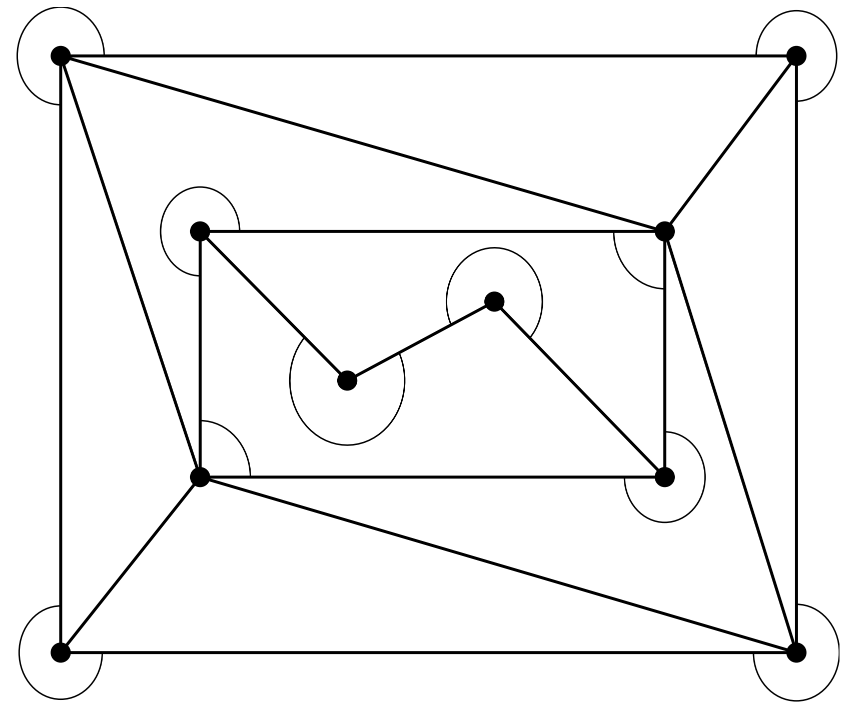
Step 1: Find a *combinatorial pseudotriangulation* (CPT):
Mark every angle of the embedding either as *small* or *big*.

- Every interior face has 3 small angles.
- The outer face has no small angles.
- Every vertex is incident to one big angle.

Step 2: Find a geometric realization of the CPT.

4. COMBINATORIAL PSEUDOTRIANGULATIONS



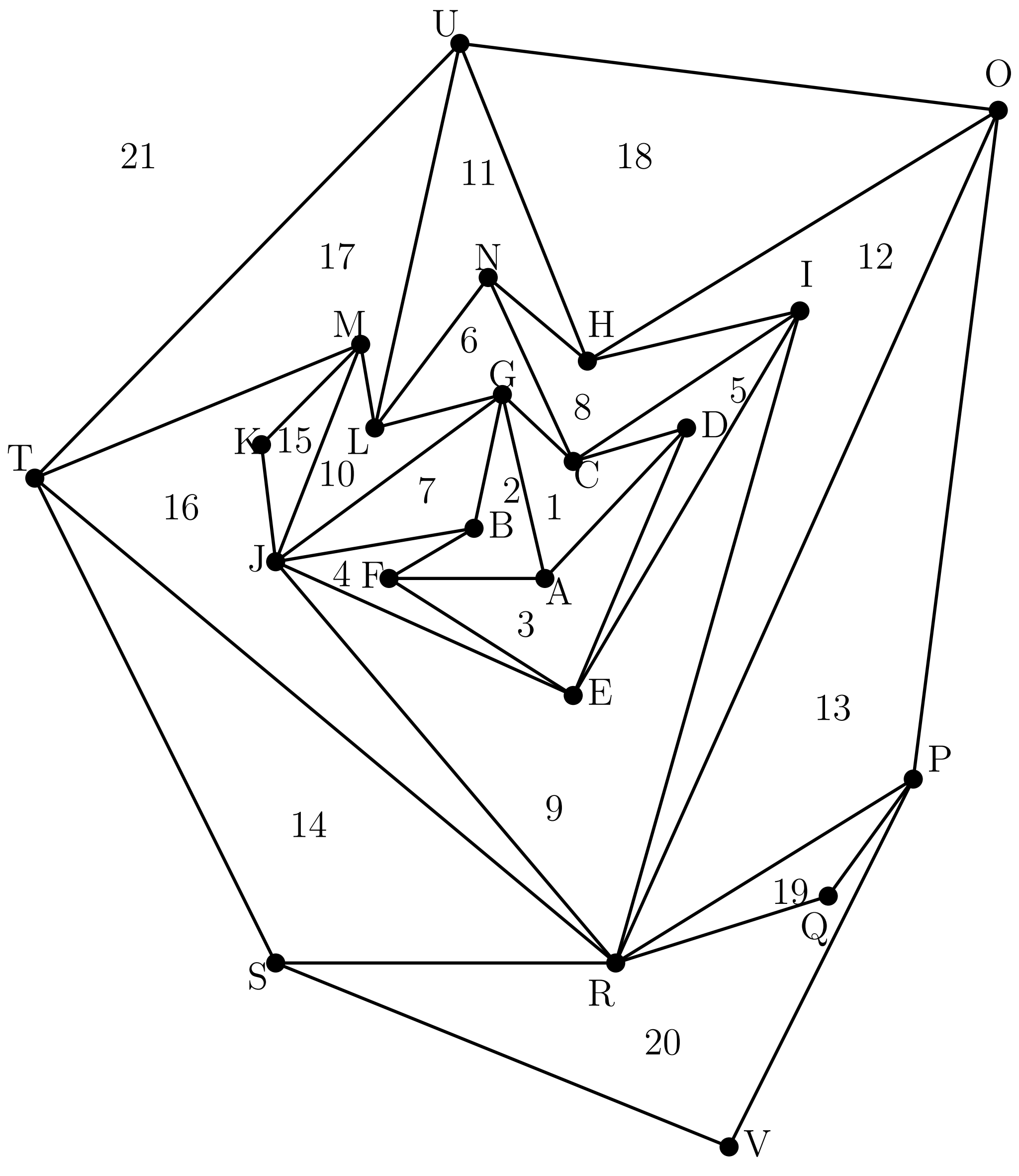


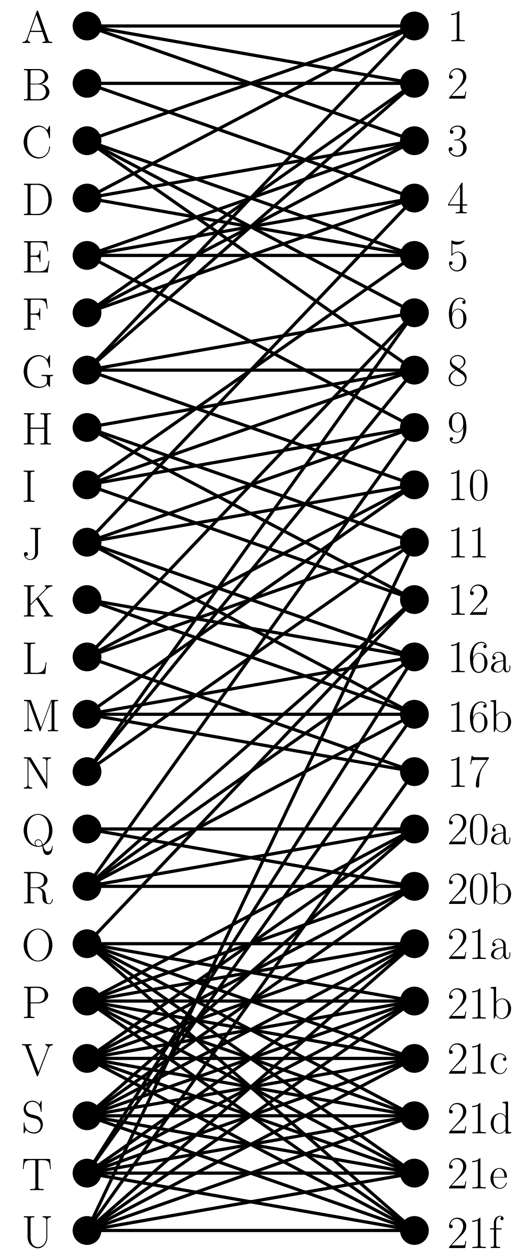
Step 1: Find a combinatorial pseudotriangulation

Bipartite network flow model:
sources = vertices: supply $= 1$.
sinks = faces: demand $= k - 3$ for a k-sided face
arcs = angles: capacity 1. flow=1 $\iff$ angle is big.

Prove that the max-flow min-cut condition is satisfied.

Step 1: Find a combinatorial pseudotriangulation





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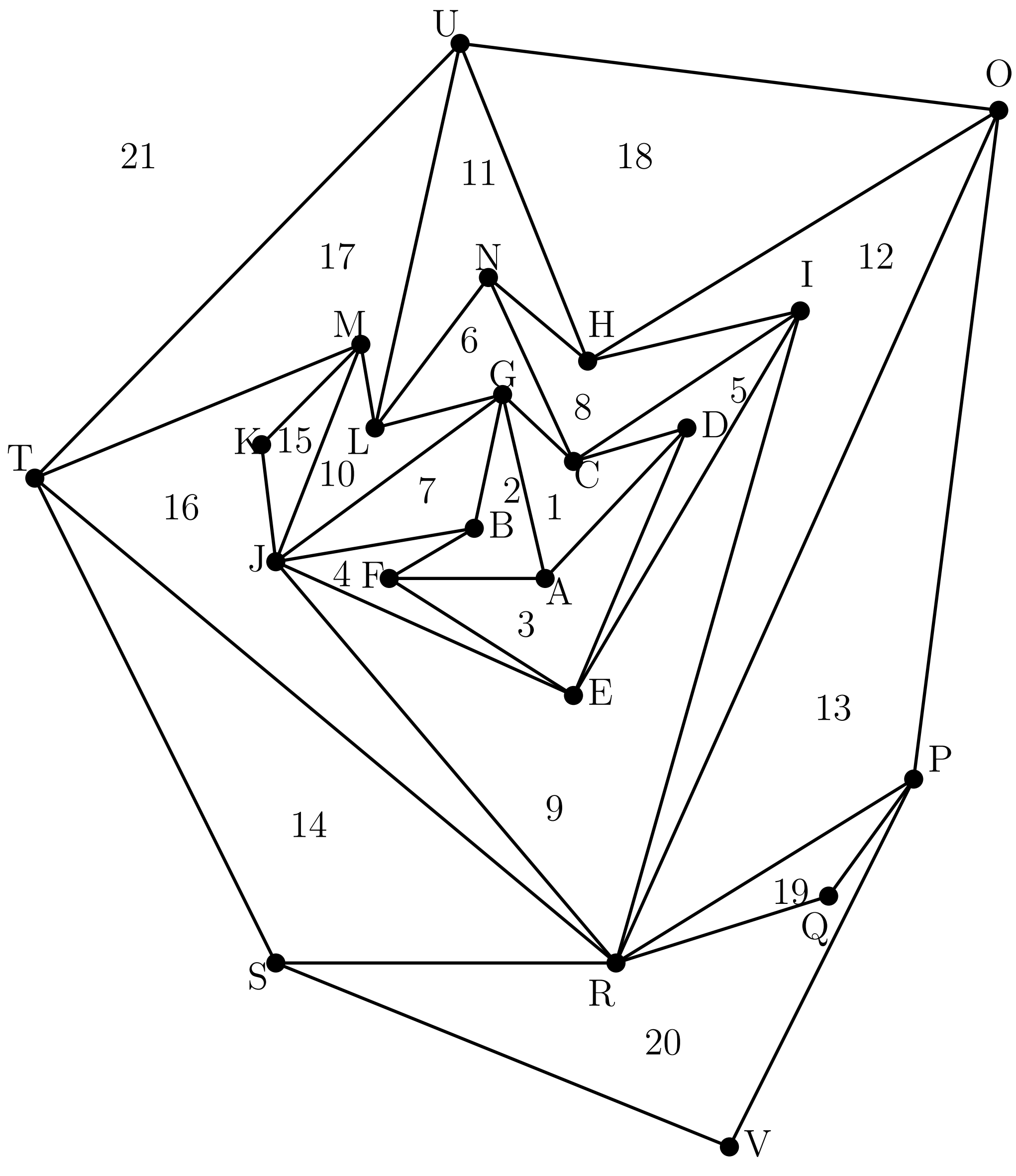
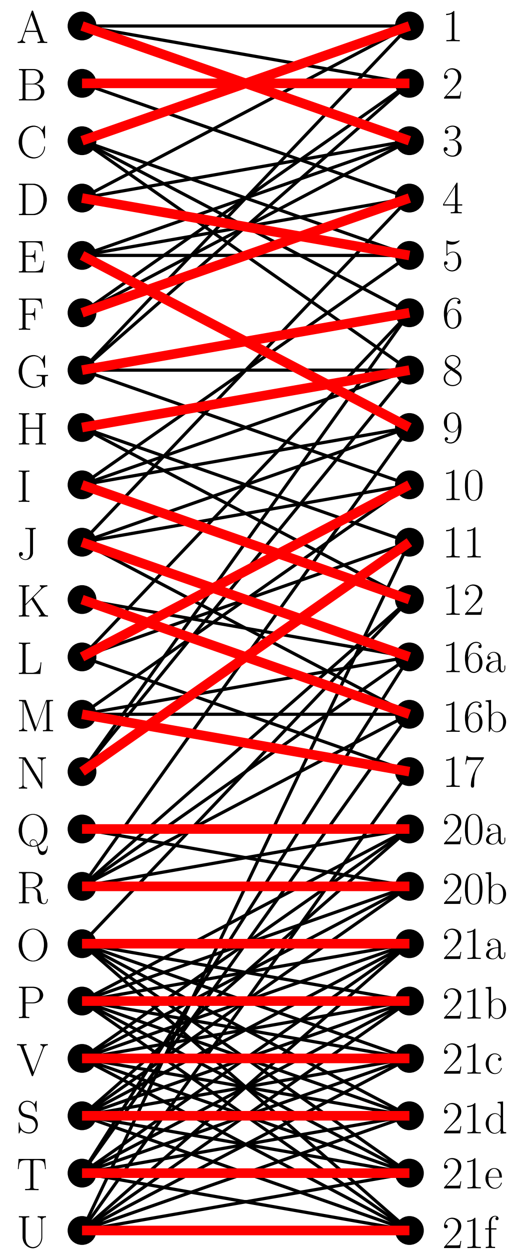




Step 2—Tutte's barycenter method

Fix the vertices of the outer face in convex position. Every interior vertex p_i should lie at the barycenter of its neighbors.

$$\sum_{(i,j)\in E} \omega_{ij}(p_j - p_i) = 0, \quad \text{for every vertex } i$$

$\omega_{ij} \geq 0$, but ω need not be symmetric.

Theorem. *If every interior vertex has three vertex disjoint paths to the outer boundary, using arcs with $\omega_{ij} > 0$, the solution is a planar embedding.*

[Tutte 1961, 1964], [Floater and Gotsman 1999],
[Colin de Verdière, Pocchiola, Vegter 2003]

5. TUTTE'S BARYCENTER METHOD FOR 3-CONNECTED PLANAR GRAPHS

Theorem. *Every 3-connected planar graph G has a planar straight-line embedding with convex faces. The outer face of the embedding and the convex shape of this face can be chosen arbitrarily.*

Tutte used *symmetric* $\omega_{ij} = \omega_{ji} > 0$.

$\rightarrow$ animation of spider-web embedding (requires Cinderella 2.0 software)

Good embeddings

Consider a directed subgraph of G. A *good* embedding is a set of positions for the vertices with the following properties:

1. The vertices of the outer face form a strictly convex polygon.

2. Every other vertex lies in the relative interior of the convex hull of its out-neighbors.

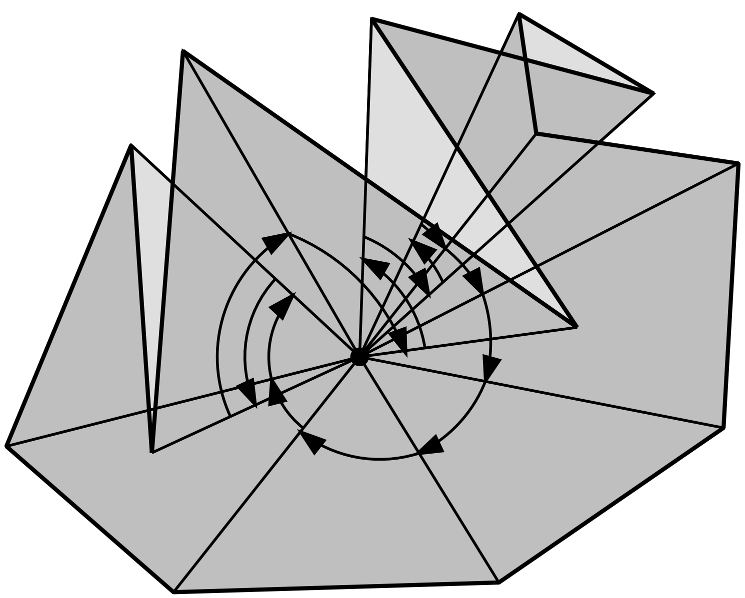
3. No vertex v is degenerate, in the sense that all out-neighbors lie on a line through v.

Lemma. *A good embedding gives rise to a planar straight-line embedding with strictly convex faces.*

Good embeddings are good

Lemma. *A good embedding is non-crossing.*

Proof: Assume that interior faces of G are triangles. (Add edges with $\omega_{ij} = 0$.)
Total angle at b boundary vertices: $\geq (b-2)\pi$.
Total angle around interior vertices: $\geq (n-b) \times 2\pi$.
$2n-b-2$ triangles generate an angle sum of $(2n-b-2)\pi$.



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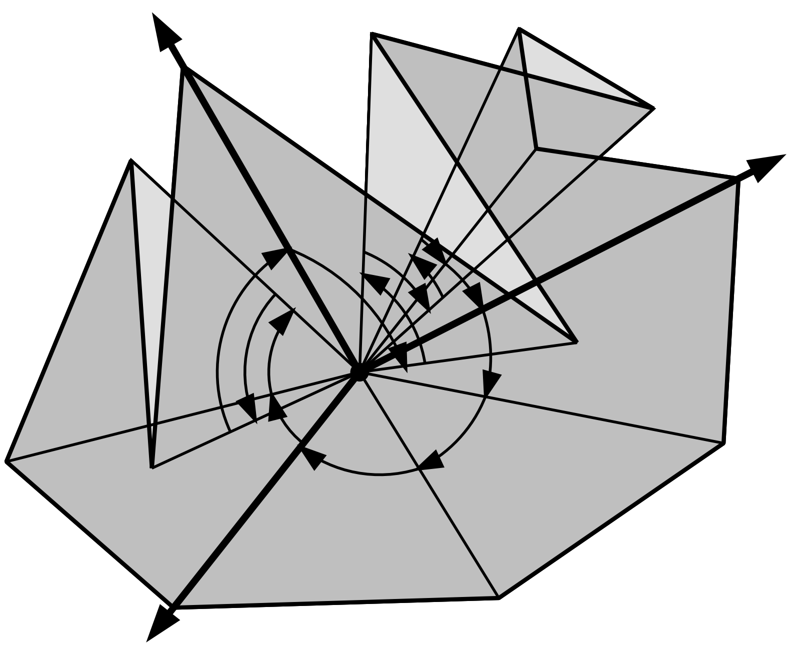

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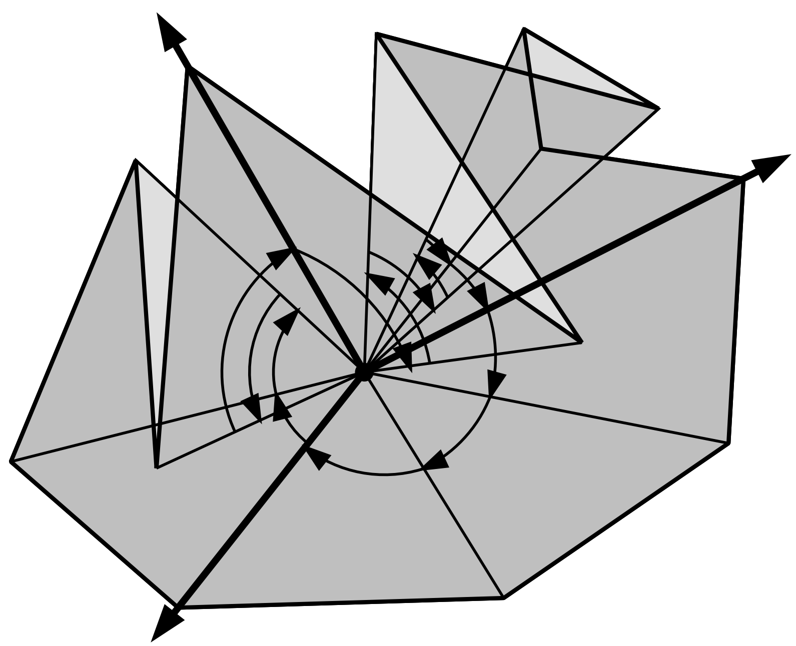
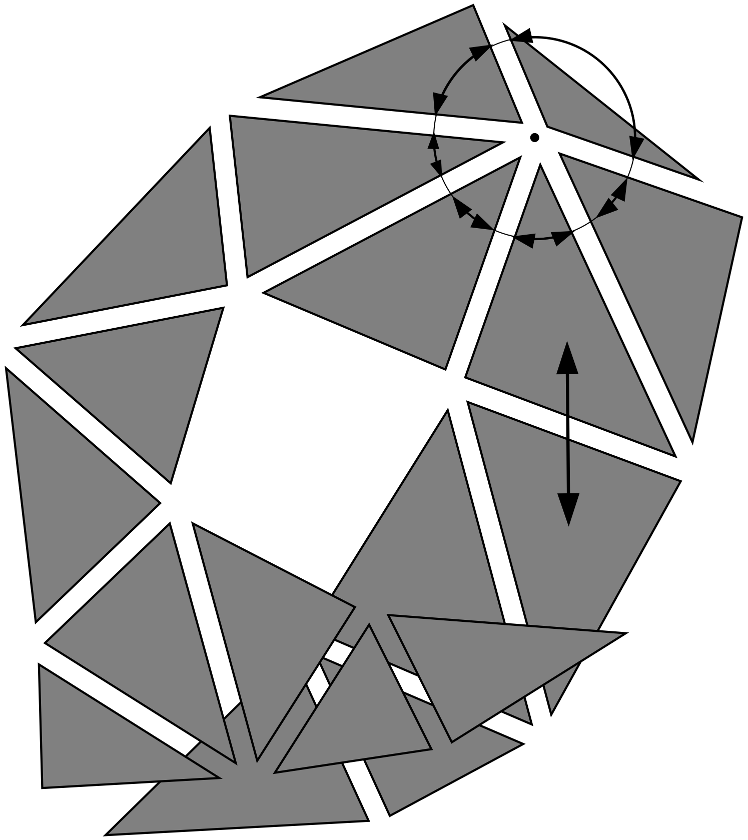


$\rightarrow$ all triangles must be oriented consistently.

Good embeddings are good

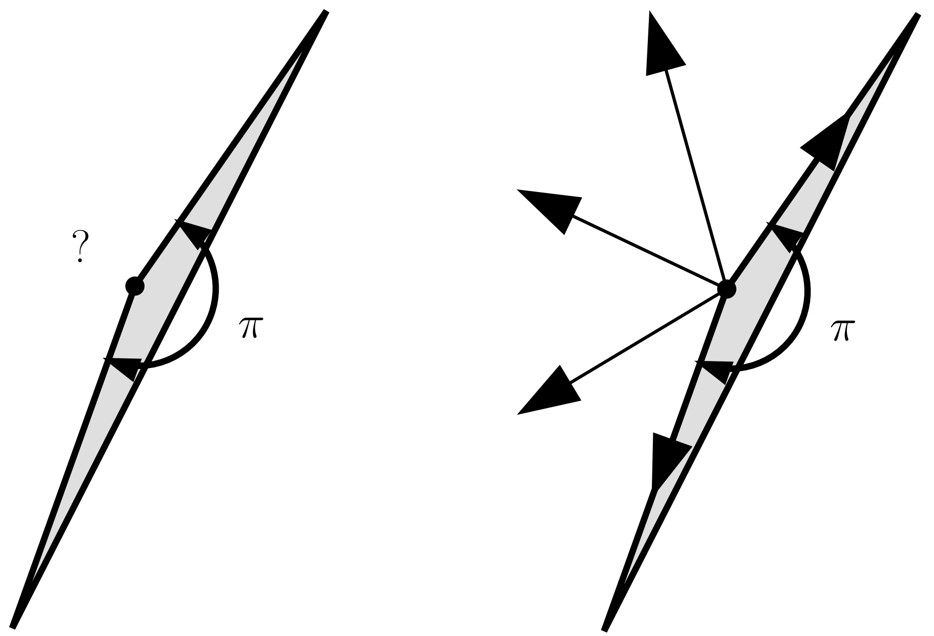
Triangles fit together locally.



equal covering number on both sides of every edge.

Good embeddings are good

There is no space for triangles with 180° angles.



no equilibrium

Equilibrium implies good embedding

The system

$$\sum_{(i,j)\in E} \omega_{ij}(p_j - p_i) = 0, \quad \text{for every interior vertex } i \qquad (*)$$

has a unique solution. (Exercise 11)

We have to show that the solution gives rise to a good embedding. The out-neighbors of a vertex i in the directed subgraph are the vertices j with $\omega_{ij} > 0$.

Equilibrium implies good embedding

(i) The vertices of the outer face form a convex polygon.

(ii) Every other vertex lies in the relative interior of the convex hull of its out-neighbors.

(iii) No vertex p_i is degenerate, in the sense that all out-neighbors p_j lie on a line through p_j.

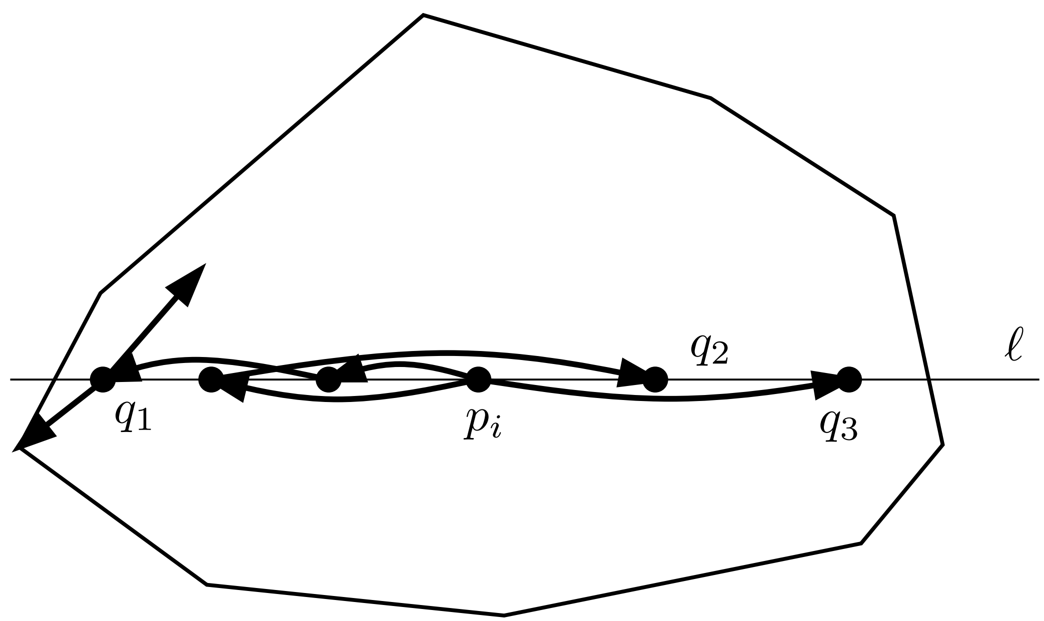
We have (i) by construction. (ii) follows directly from the system (see Exercise 13)

$$\sum_{(i,j)\in E} \omega_{ij}(p_j - p_i) = 0, \quad \text{for every interior vertex } i \qquad (*)$$

We will need 3-connectedness and planarity for (iii).

The equilibrium embedding is nondegenerate

Assume that all neighbors of p_i lie on a horizontal line ℓ.
We have 3 *vertex-disjoint paths* from p_i to the boundary.
$q_1, q_2, q_3 =$ last vertex on each path that lies on ℓ.
By *equilibrium*, q_k must have a neighbor above ℓ and below ℓ.



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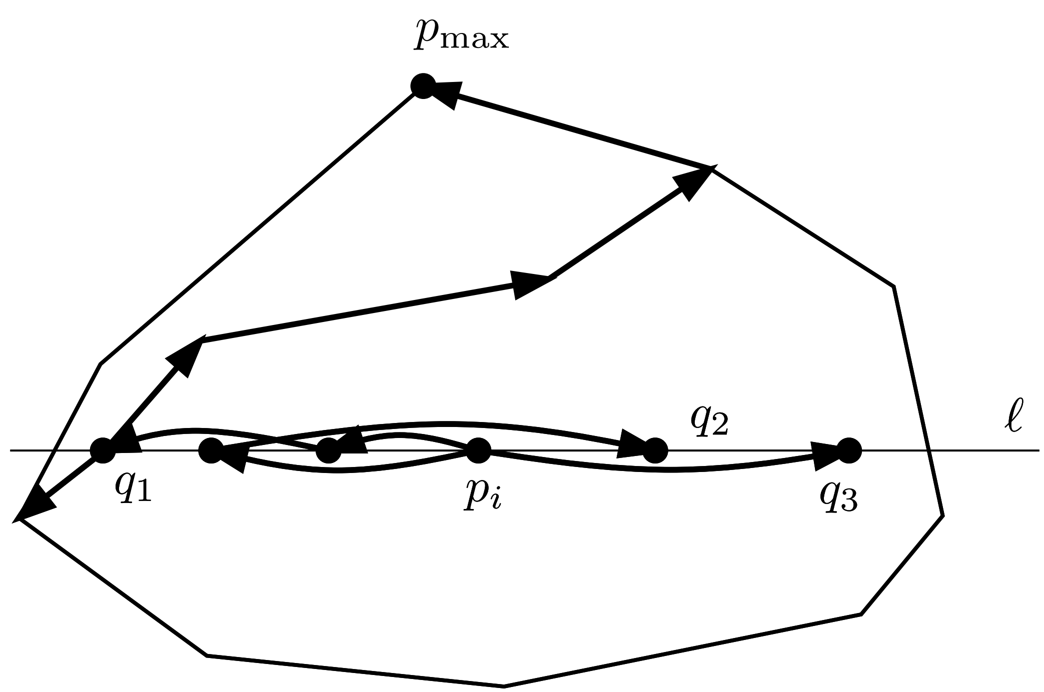
Continue upwards to the boundary and along the boundary to the highest vertex $p_{\max}$



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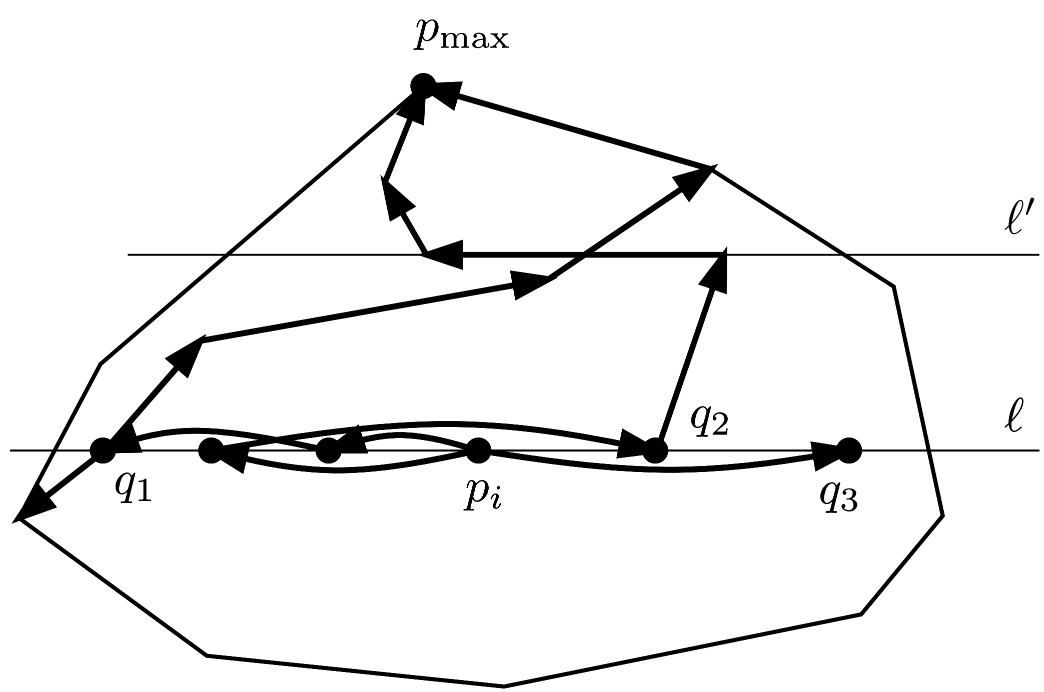

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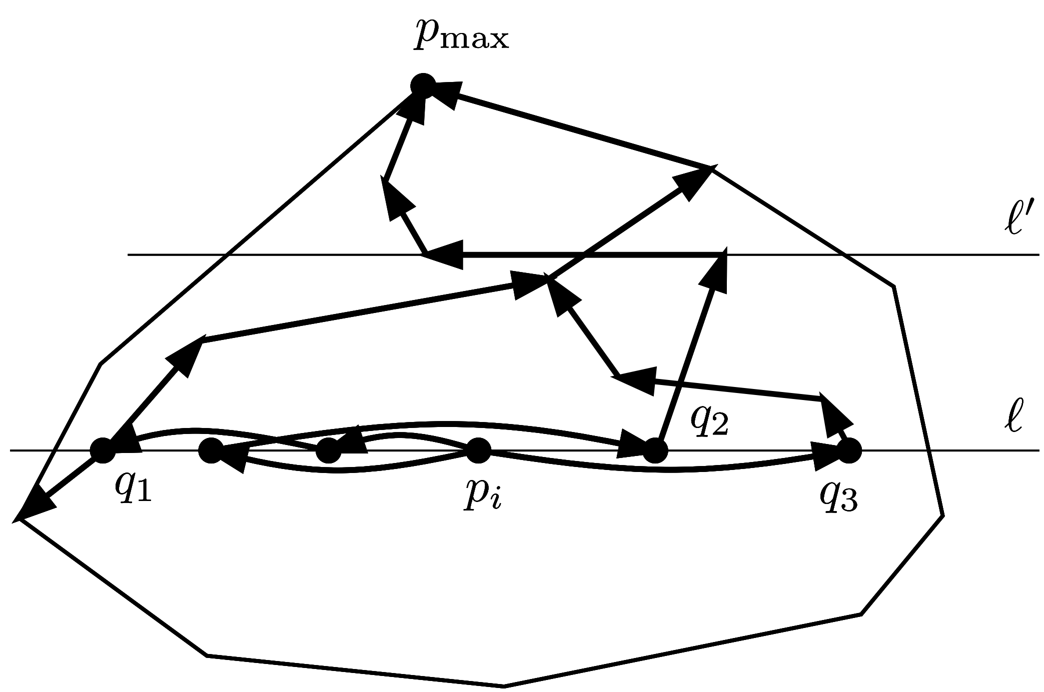

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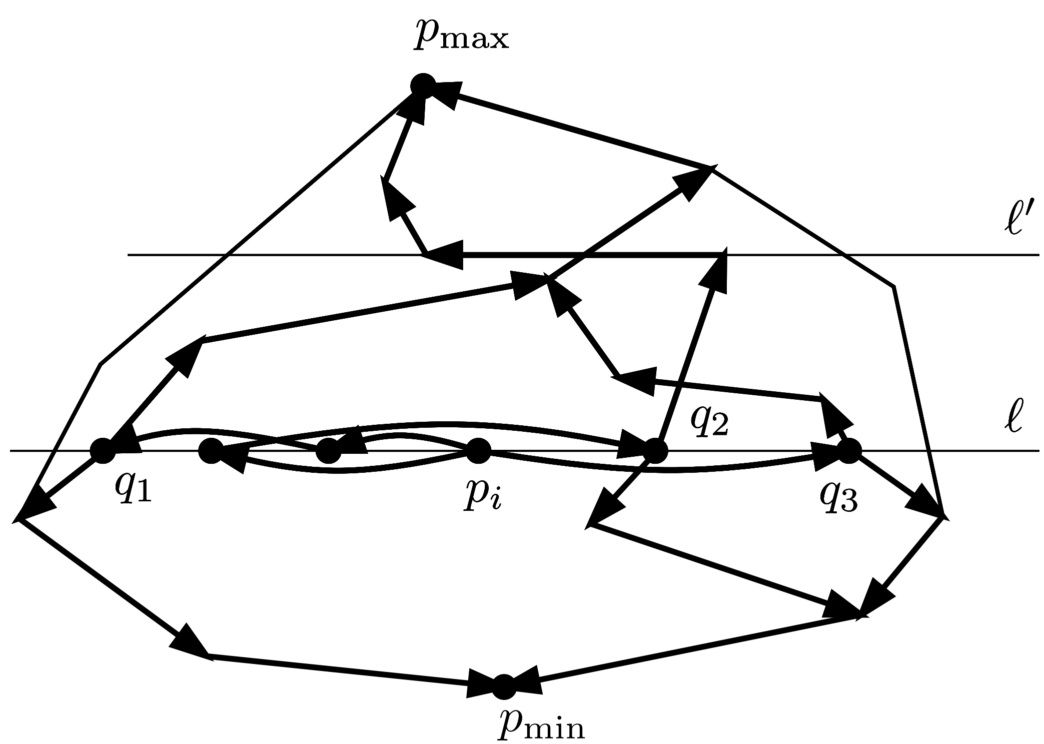
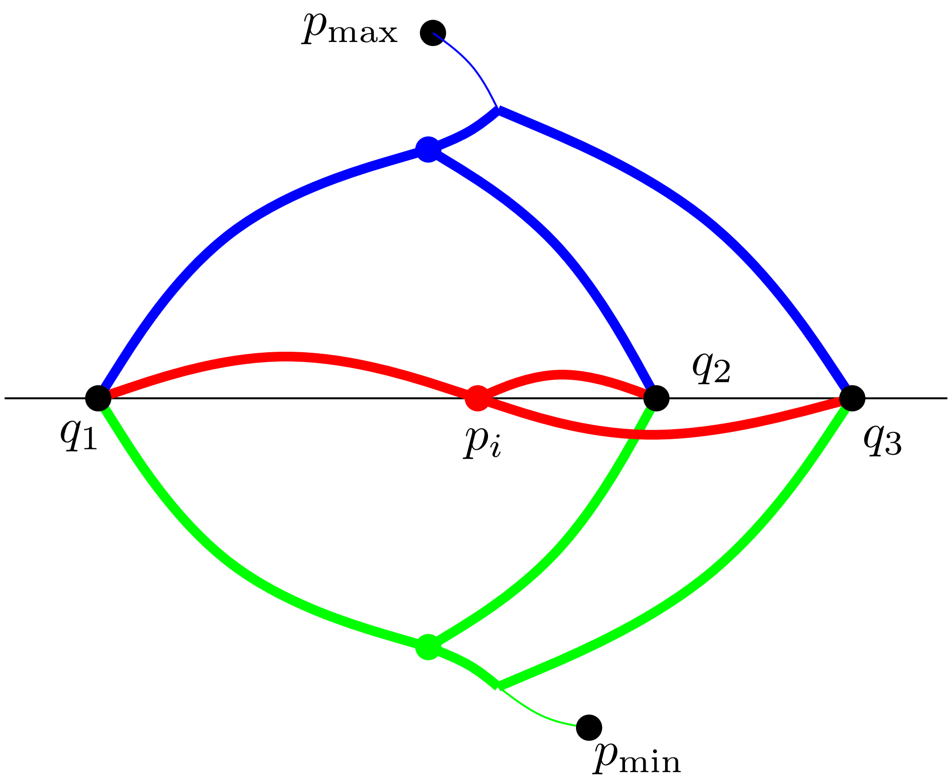
By *equilibrium*, q_k must have a neighbor above ℓ and below ℓ.
Continue upwards to the boundary and along the boundary to the highest vertex $p_{\max}$, and similarly to the lowest vertex.



Using planarity



Three paths from three different vertices q_1, q_2, q_3 to a common vertex $p_{\max}$ always contain three vertex-disjoint paths from q_1, q_2, q_3 to a common vertex (the "Y-lemma").
Together with the three paths from p_i to q_1, q_2, q_3 we get a subdivision of $K_{3,3}$.

Tutte's barycenter method for *directed* planar graphs

Theorem. *Let D be a partially directed subgraph of a planar graph G with specified outer face.*

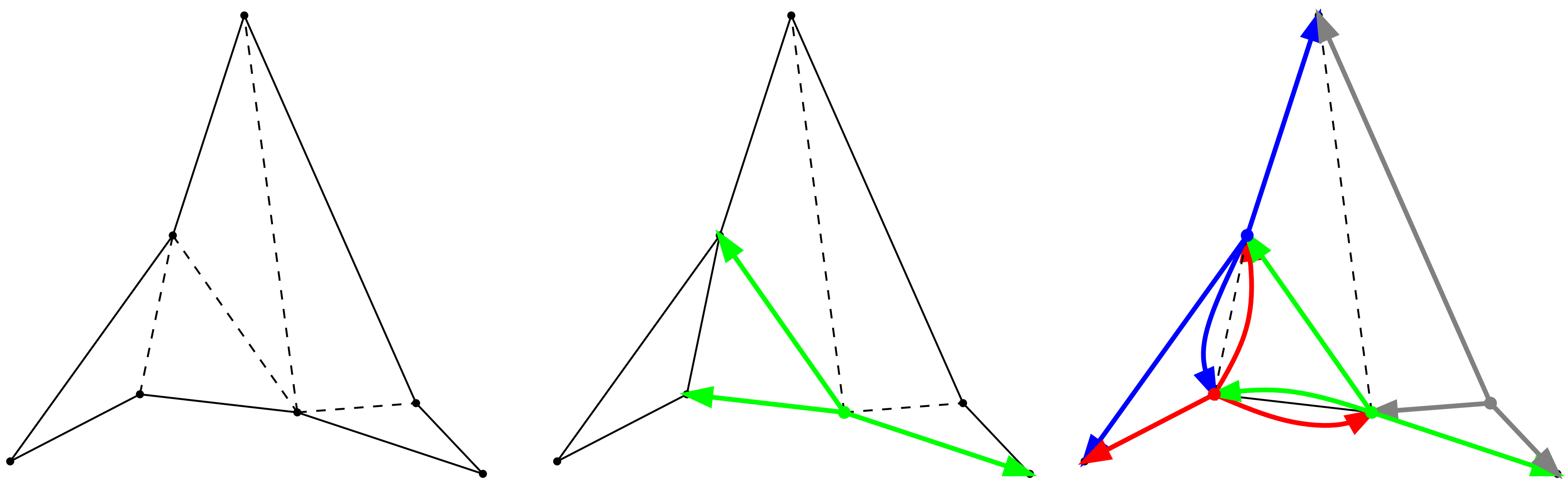
If every interior vertex has three vertex disjoint paths to the outer face, there is a planar embedding where every interior vertex lies in the interior of its out-neighbors. $\square$

Selection of outgoing arcs

3 outgoing arcs for every interior vertex:

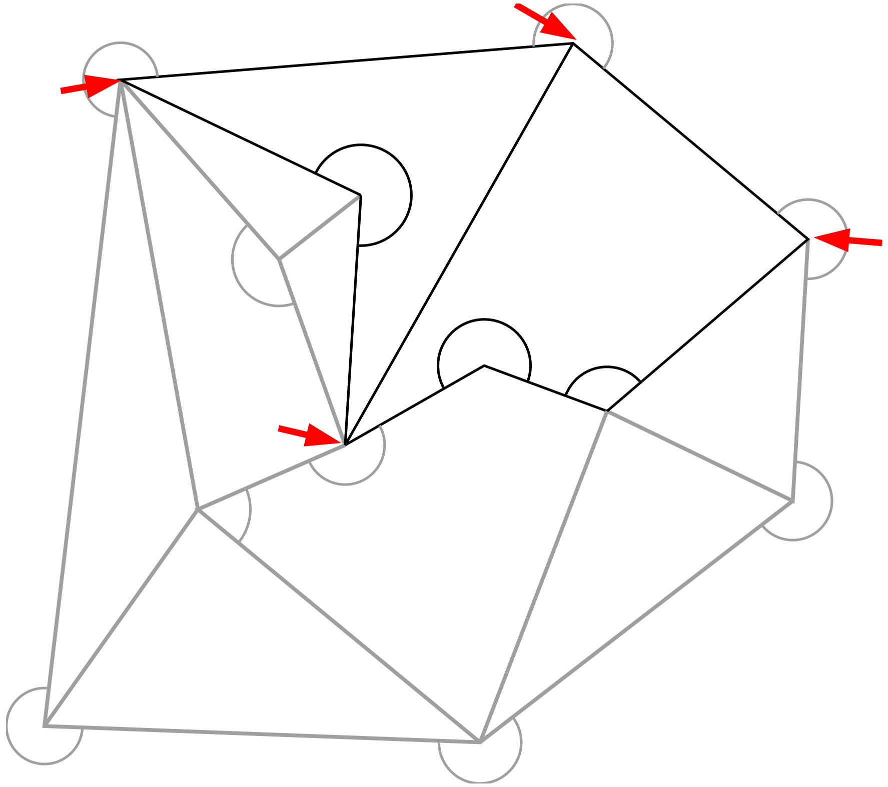
Triangulate each pseudotriangle arbitrarily.
For each reflex vertex, select

- the two incident boundary edges
- an interior edge of the pseudotriangulation



3-connectedness—geometric version

Lemma. *Every induced subgraph of a planar Laman graph with a CPT has at least 3 outside "corners".*



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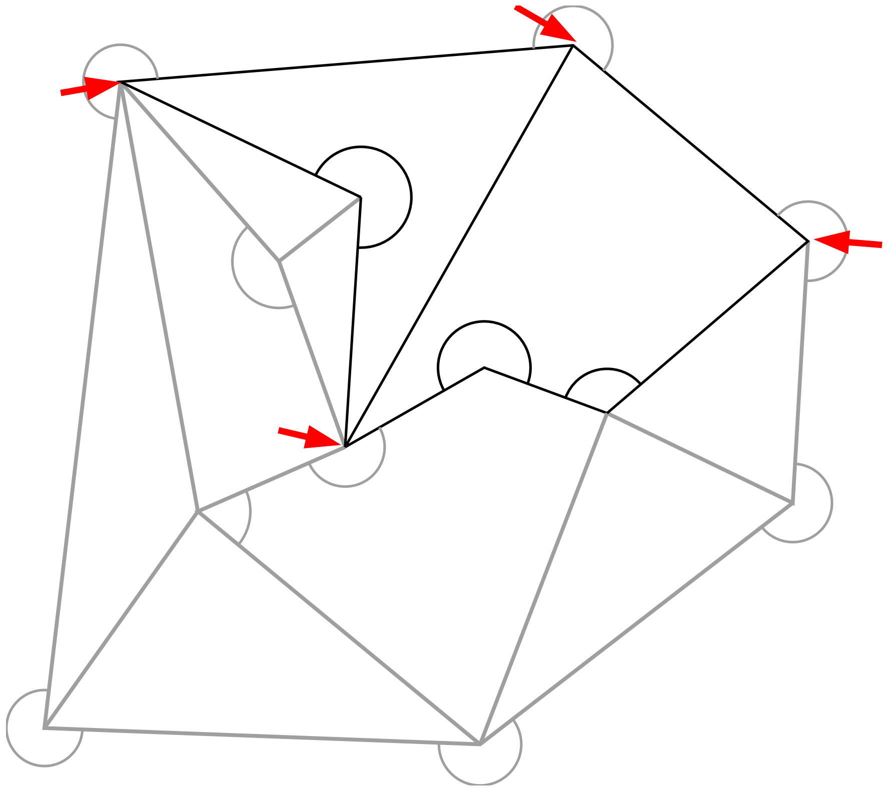
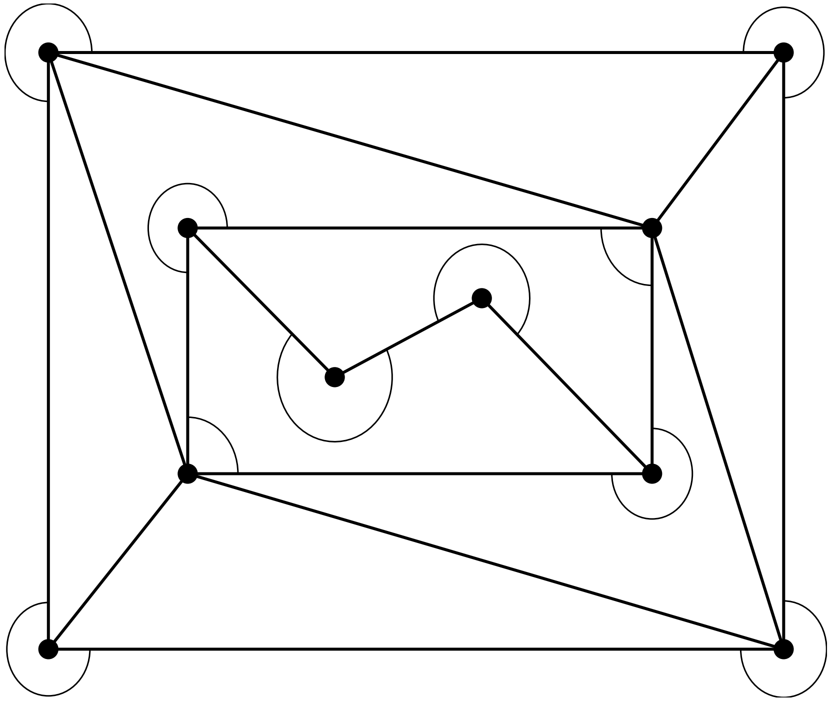
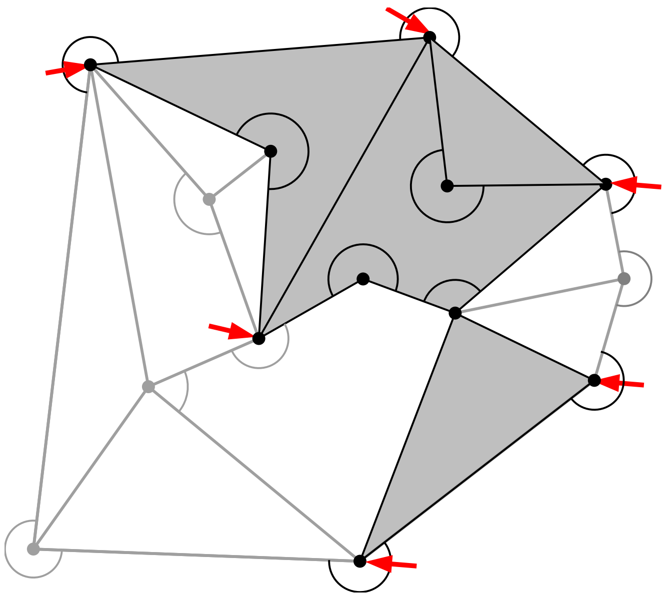




Every subgraph has at least 3 corners



b boundary edges, $b_0 \le b$ boundary vertices, with c corners.

\# interior angles $= 2e - b$

\# interior small angles $= 3f$

\# interior big angles $= n - c$

Euler: $e + 2 = n + (f + 1)$

$$\implies e = 2n - 3 - (b - c)$$

interior edges and vertices: $e_{\text{int}} = e - b$, $v_{\text{int}} = n - b_0$

Laman: $e_{\text{int}} \ge 2v_{\text{int}}$

$$\implies c \ge 3$$

3-connectedness in the graph

Need to show: Every interior vertex a has three vertex disjoint paths to the outer face.

Apply Menger's theorem: After removing two "blocking vertices" b_1, b_2, there is still a path $a \longrightarrow$ boundary.

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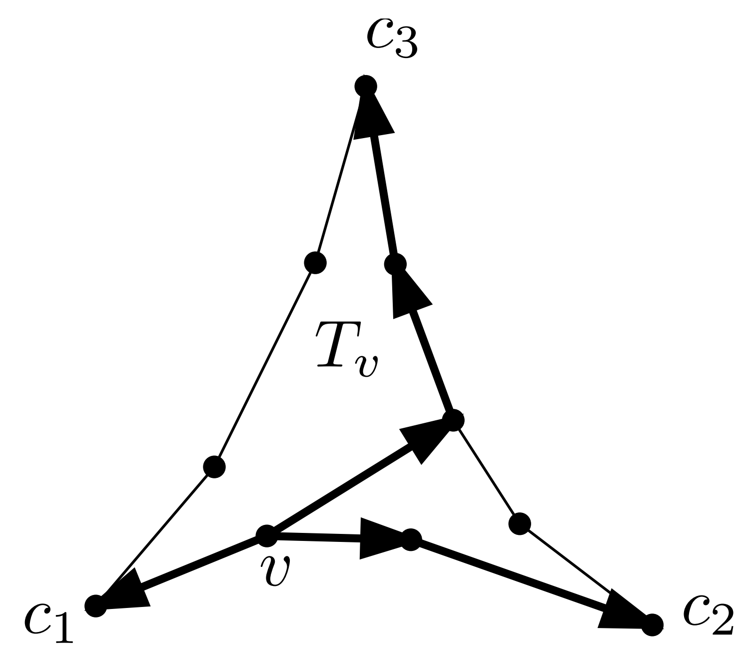
Apply Menger's theorem: After removing two "blocking vertices" b_1, b_2, there is still a path $a \longrightarrow$ boundary.

Lemma. *An interior vertex v has its big angle in a unique pseudotriangle T_v.*
There are three vertex-disjoint paths $v \longrightarrow c_1$, $v \longrightarrow c_2$, $v \longrightarrow c_3$ to the three corners c_1, c_2, c_3 of T_v.



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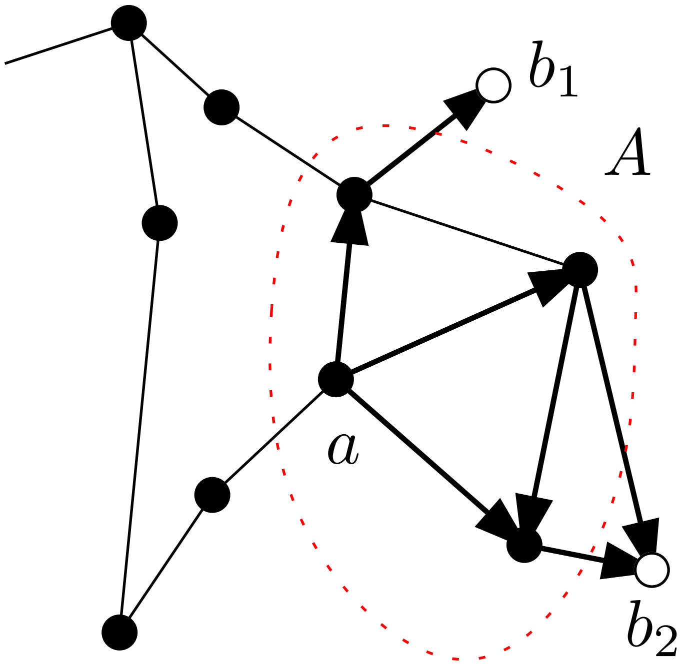

i

S

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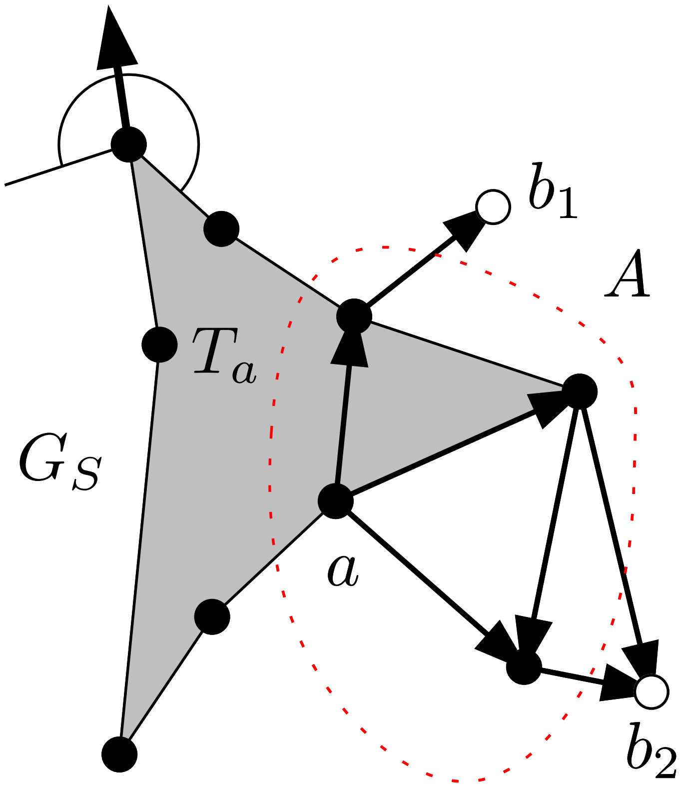
$G_S := \cup\{\, T_v : v \in A \,\}$



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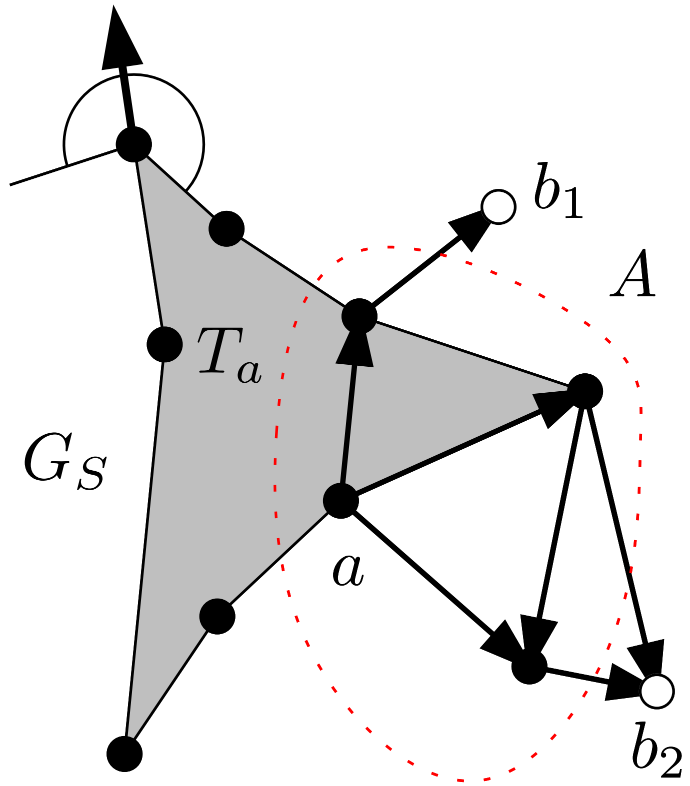
G_S has at least three corners c_1, c_2, c_3.
Find v_1, v_2, v_3 with $c_i \in T_{v_i}$ and paths
$v_1 \longrightarrow c_1$, $v_2 \longrightarrow c_2$, $v_3 \longrightarrow c_3$.



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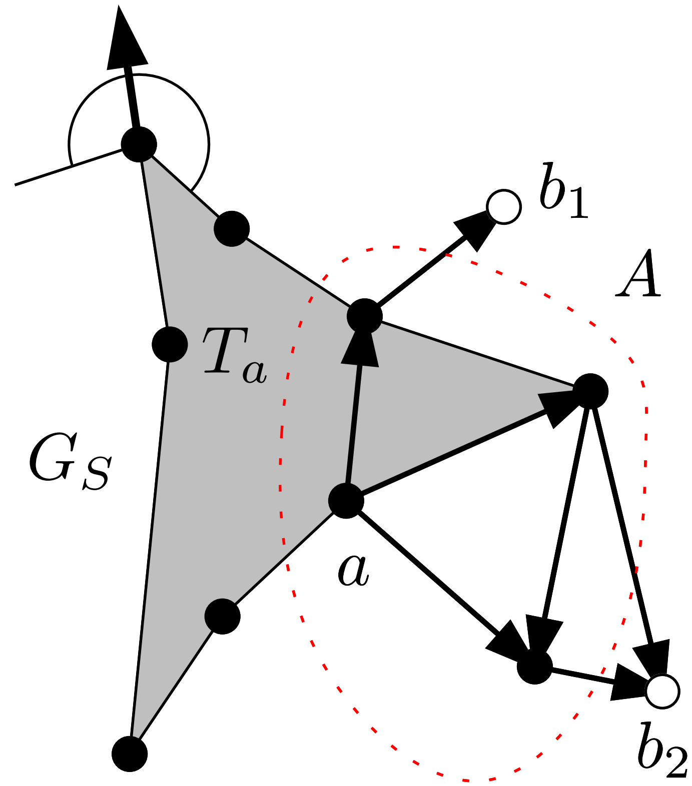
A blocking vertex b_1, b_2 can block only one of these paths. $\Longrightarrow$ some $c_i \in A$.



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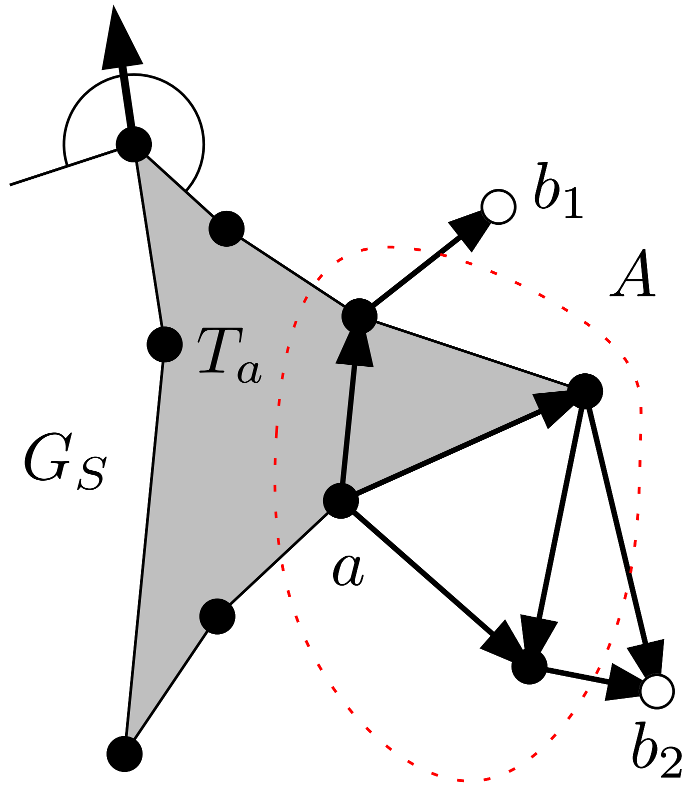
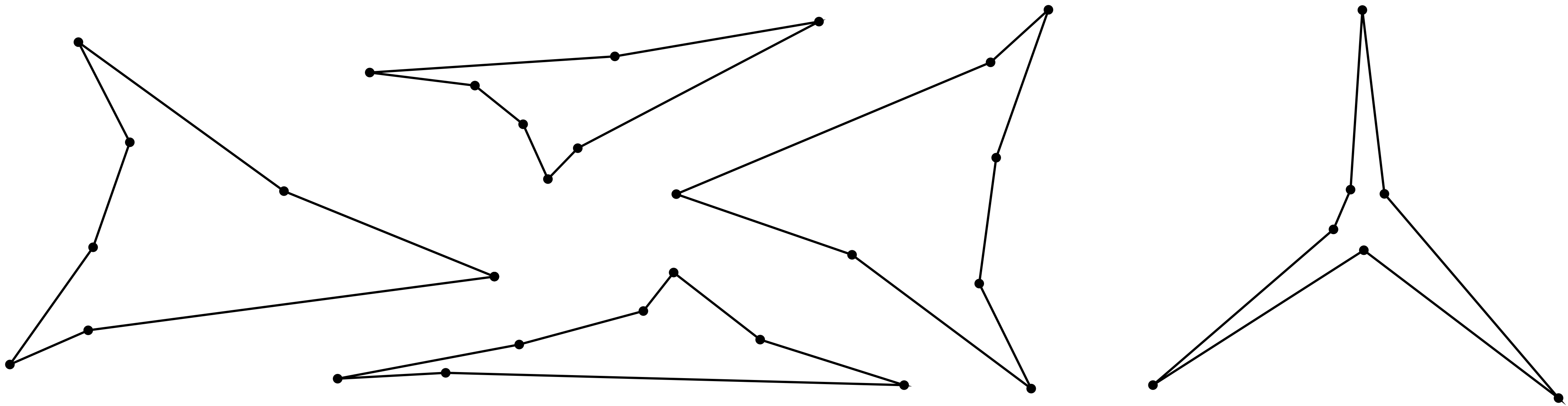


Either c_i lies on the boundary or one can jump out of G_S.

Specifying the shape of pseudotriangles

The shape of every pseudotriangle (and the outer face) can be arbitrarily specified up to affine transformations.



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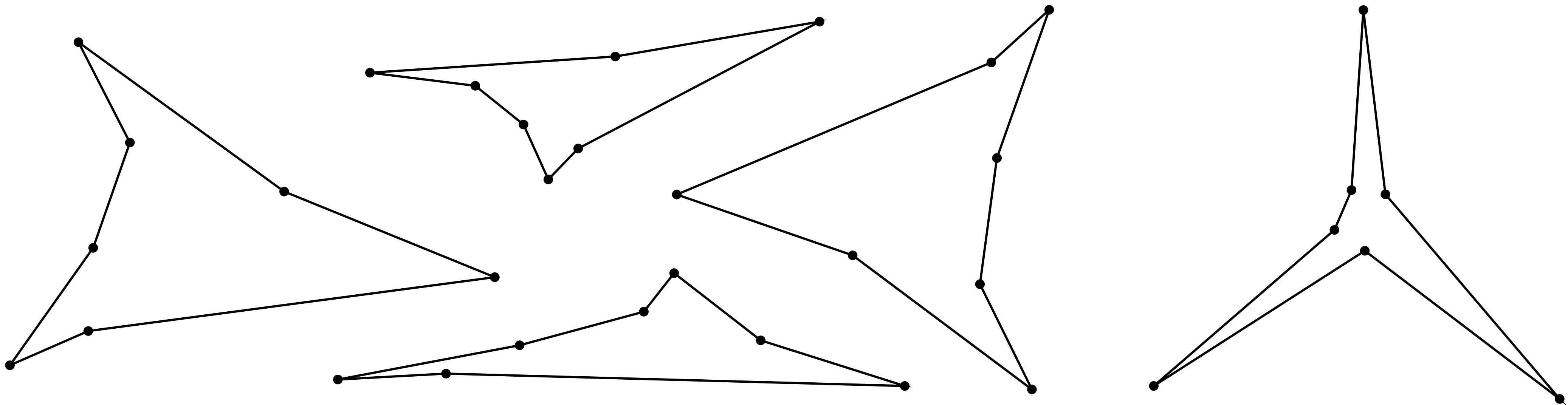


The Tutte embedding with all $\omega_{ij} = 1$ yields rational coordinates with a common denominator which is at most $12^{n/2}$, i. e. with $O(n)$ bits.

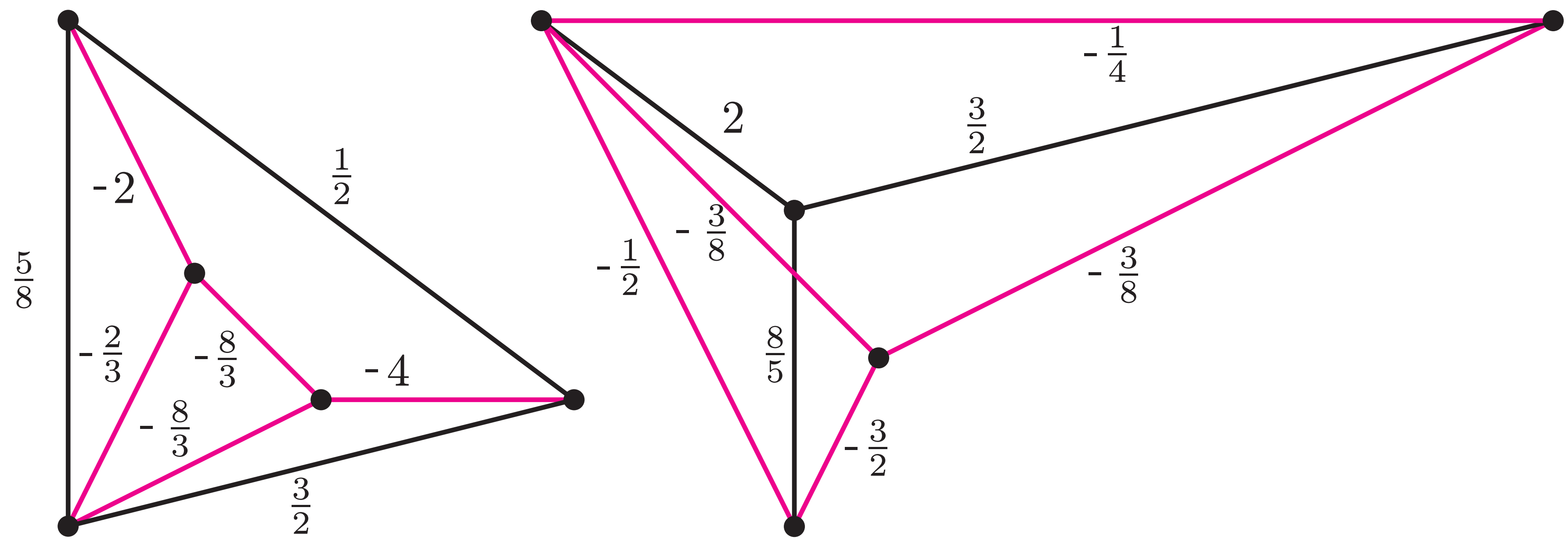
OPEN PROBLEM: Can every pseudotriangulation be embedded on a polynomial size grid? On an $O(n^{3/2}) \times O(n^{3/2})$ grid?

6. STRESSES AND RECIPROCALS

Reciprocal frameworks

Given: A plane graph G and its planar dual G^*.

A framework (G, p) is *reciprocal* to (G^*, p^*) if corresponding edges are parallel.

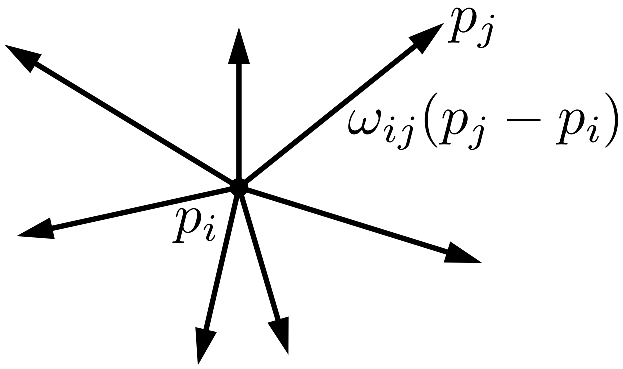


a) b)

→ dynamic animation of reciprocal diagrams with *Cinderella*

Self-stresses

A *self-stress* in a framework is given by a set of internal forces (compressions and tensions) on the edges in *equilibrium* at every vertex i:



$$\sum_{j:(i,j)\in E} \omega_{ij}(p_j - p_i) = 0$$

The force of edge (i, j) on vertex i is

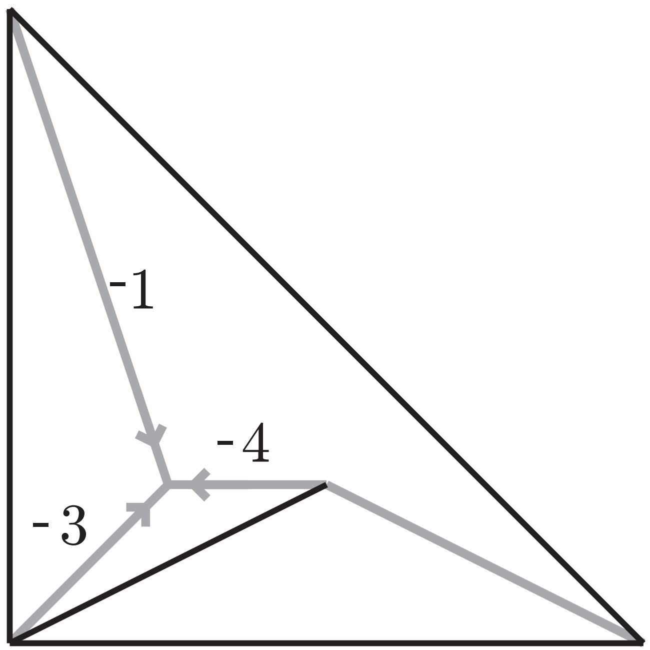
$$\omega_{ij}(p_j - p_i).$$

The force of edge (i, j) on vertex j is

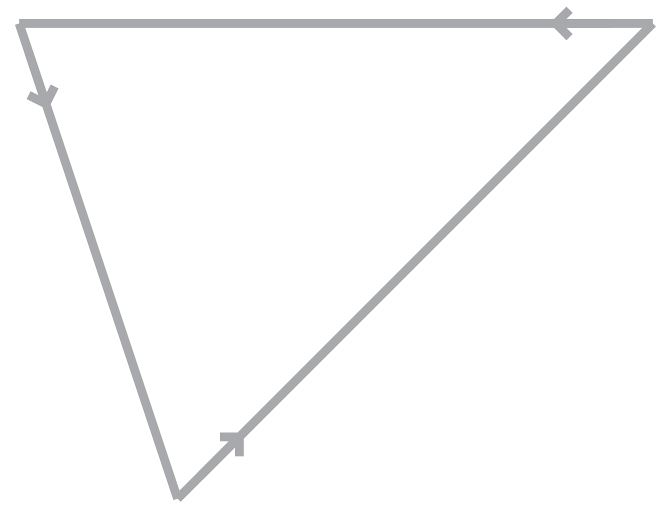
$$\omega_{ji}(p_i - p_j) = -\omega_{ij}(p_j - p_i). \qquad (\omega_{ij} = \omega_{ji})$$

Self-stresses and reciprocal frameworks

An equilibrium at a vertex gives rise to a polygon of forces:



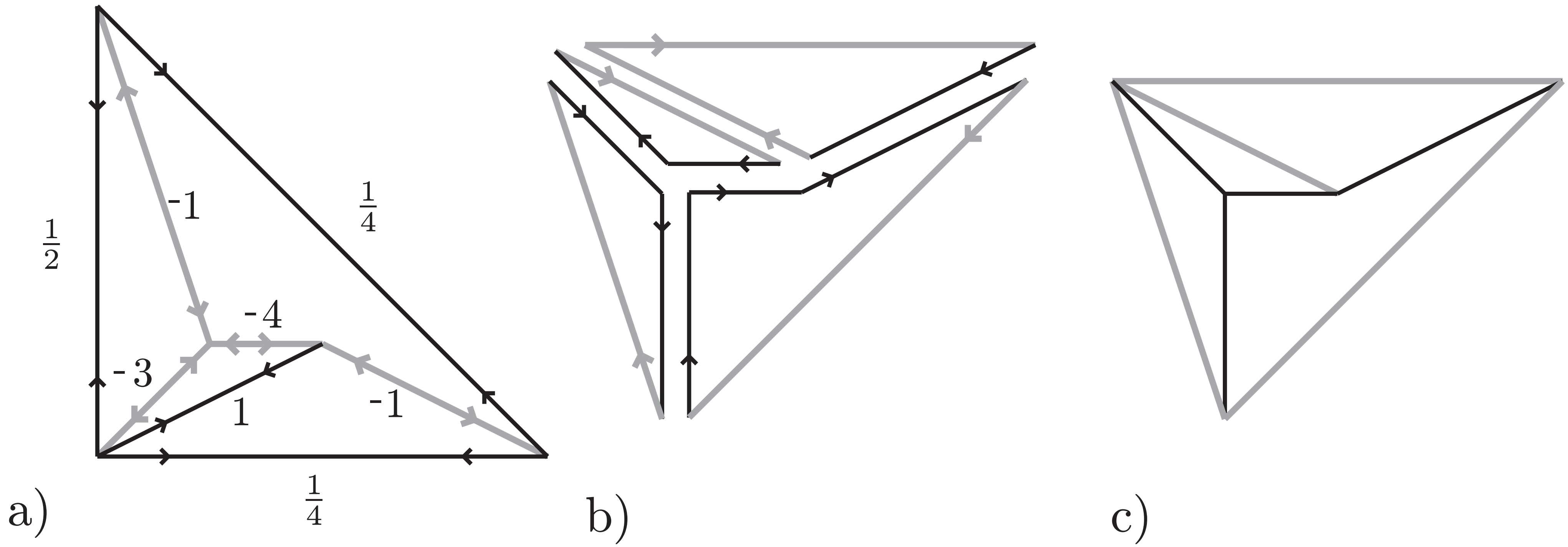
a)



b)

These polygons can be assembled to the reciprocal diagram.

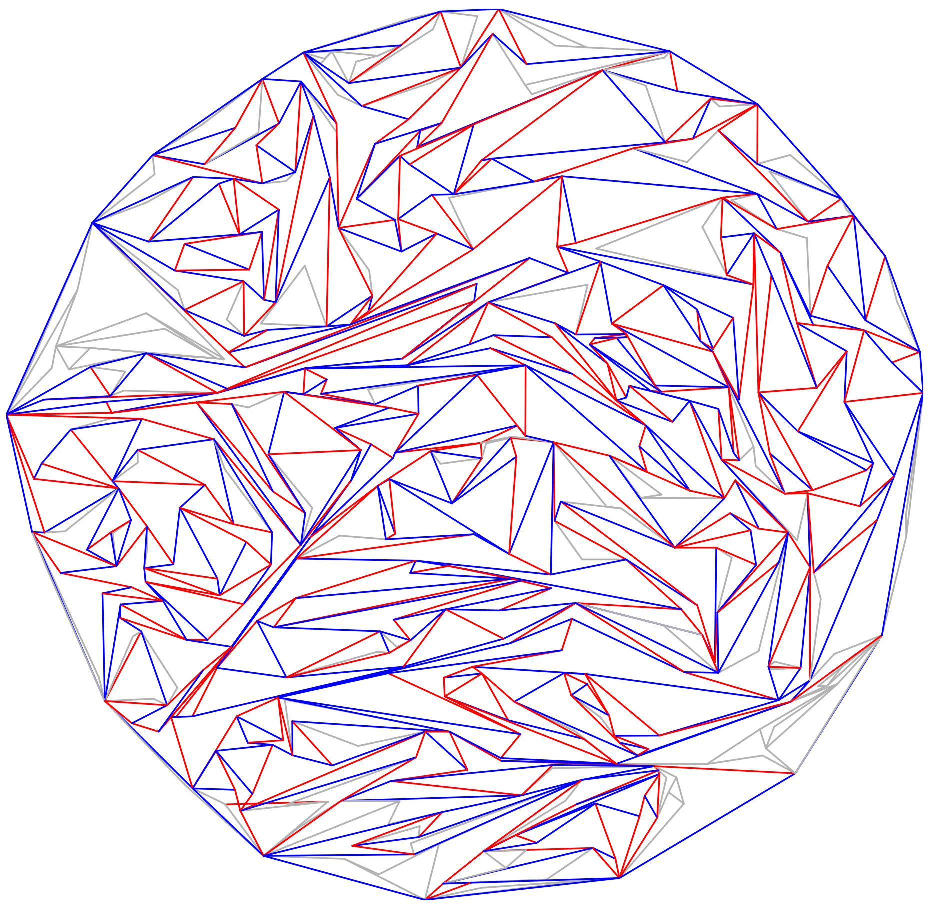
Assembling the reciprocal framework



$\omega^*_{ij} := 1/\omega_{ij}$ defines a self-stress on the reciprocal.

Minimally dependent graphs (rigidity circuits)

A Laman graph plus one edge has a unique self-stress (up to scalar multiplication).



→ It has a unique reciprocal (up to scaling).

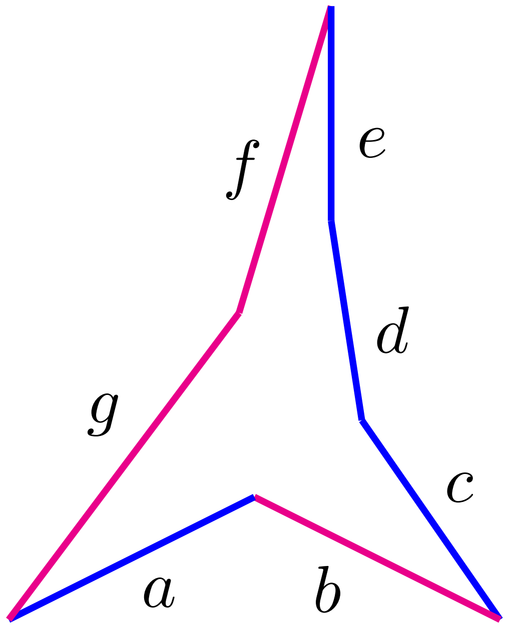
Planar frameworks with planar reciprocals

Theorem. *Let G be a pseudotriangulation with $2n-2$ edges (and hence with a single nonpointed vertex). Then G^* is non-crossing.*
Moreover, if the stress on G is nonzero on all edges, G^* is also a pseudotriangulation with $2n-2$ edges.

[Orden, Rote, Santos, B. Servatius, H. Servatius, Whiteley 2003]

Constructing the reciprocal

Walk around the face counterclockwise.
Take negative edges in the reverse direction and positive edges in the forward direction.



Constructing the reciprocal

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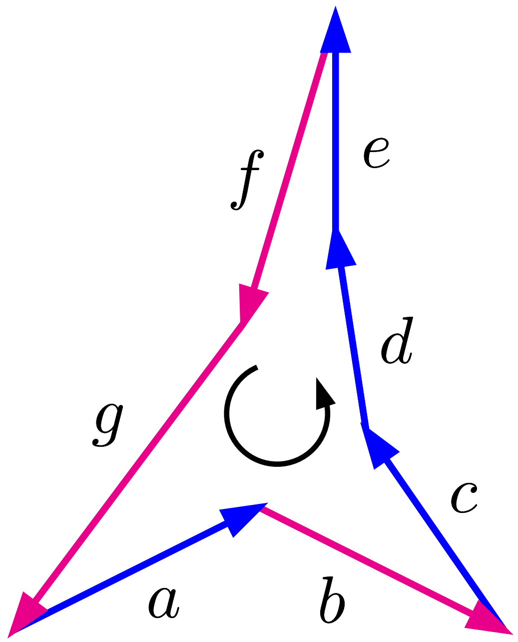

Constructing the reciprocal

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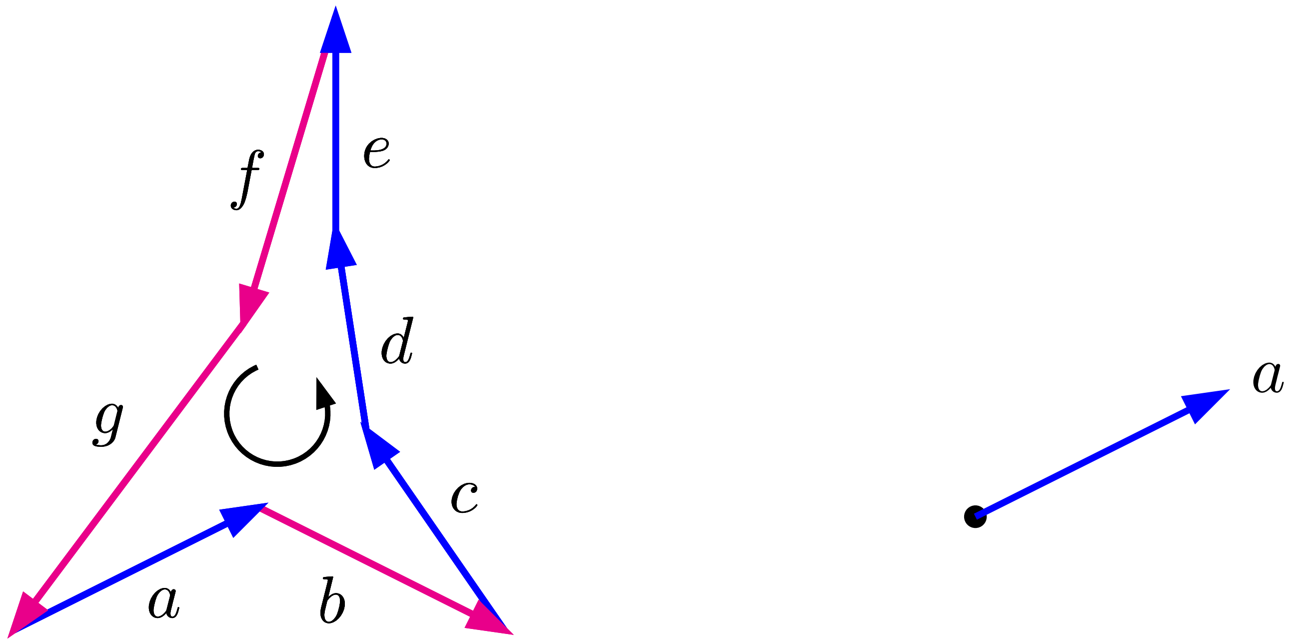

Constructing the reciprocal

Walk around the face counterclockwise.

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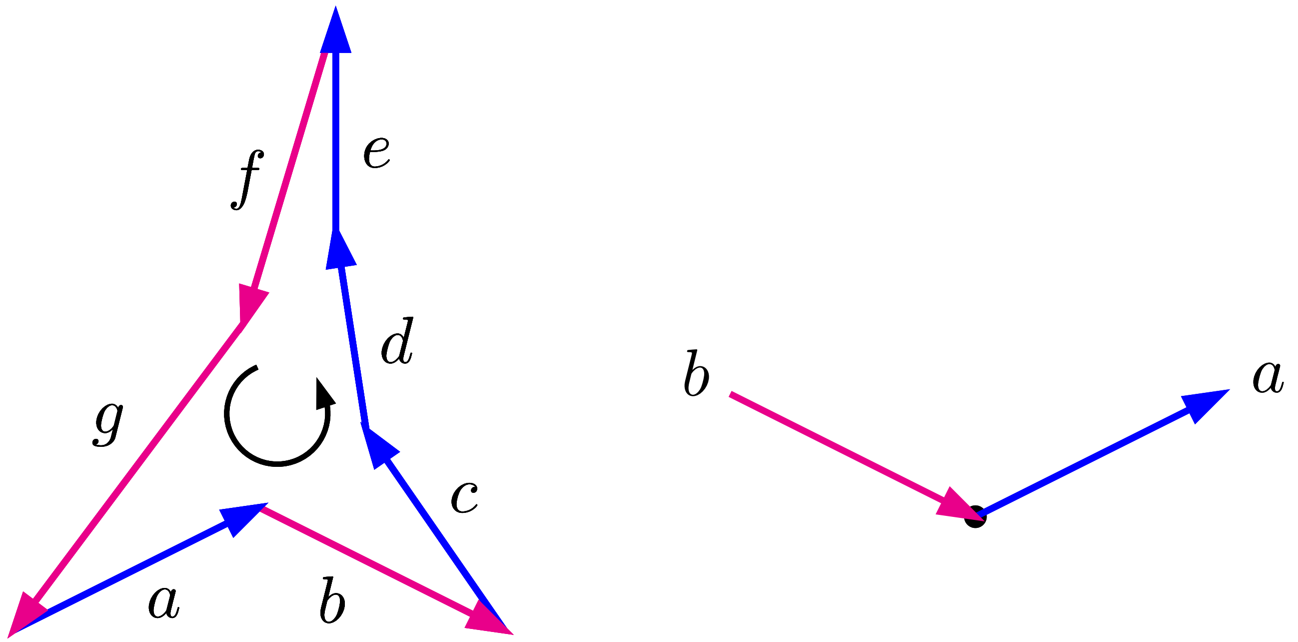

Constructing the reciprocal

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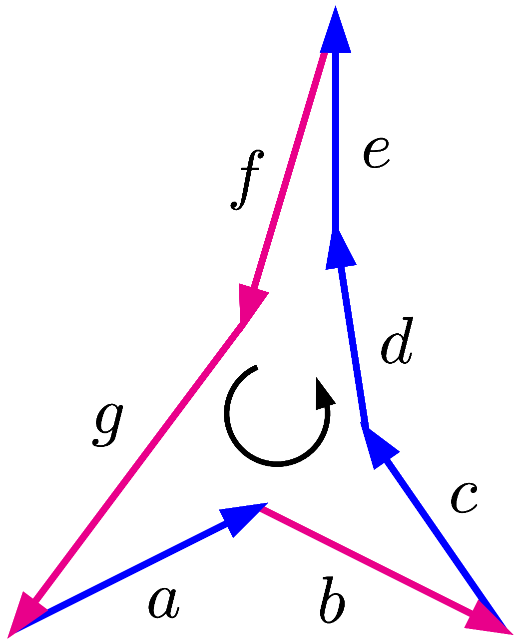

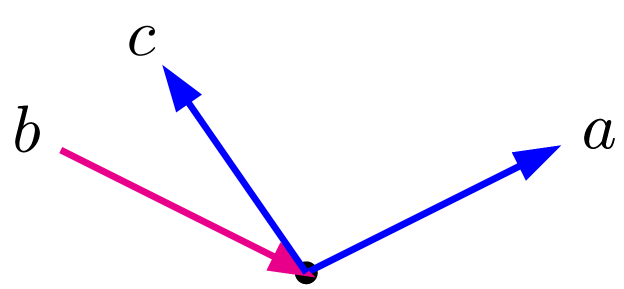

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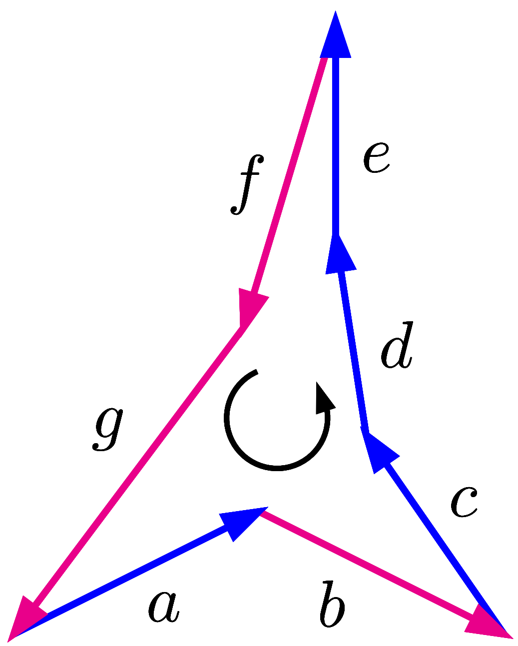

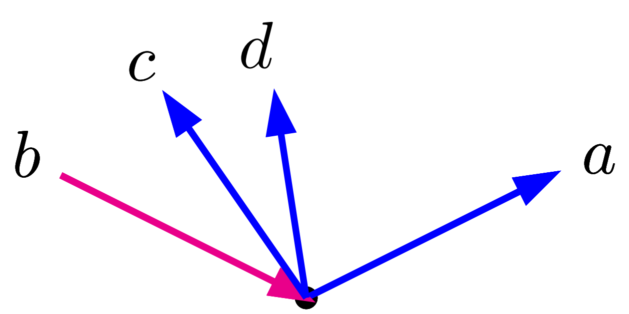

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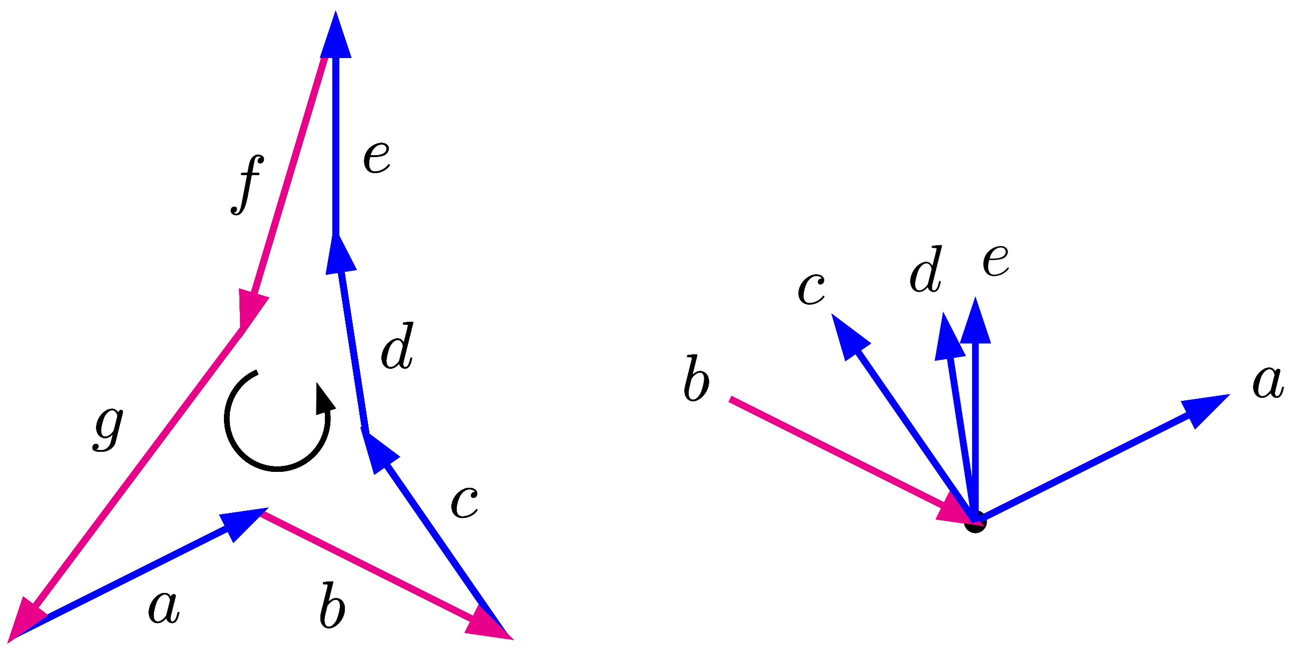

Constructing the reciprocal

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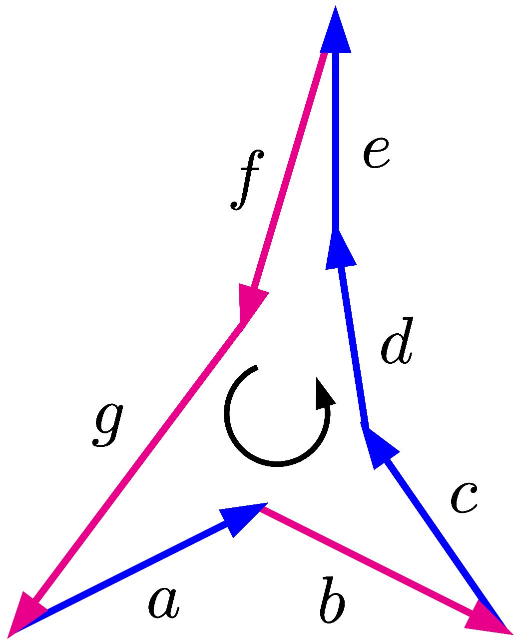

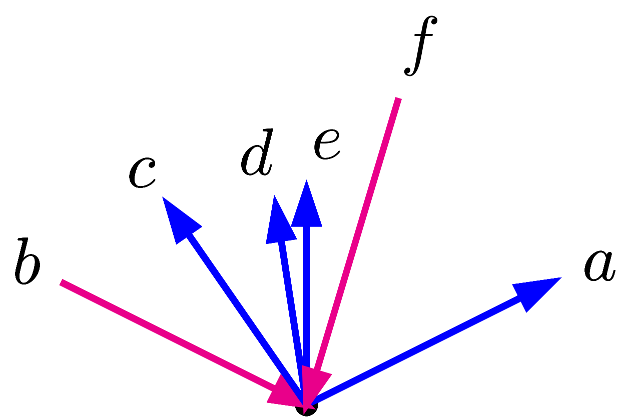

Constructing the reciprocal

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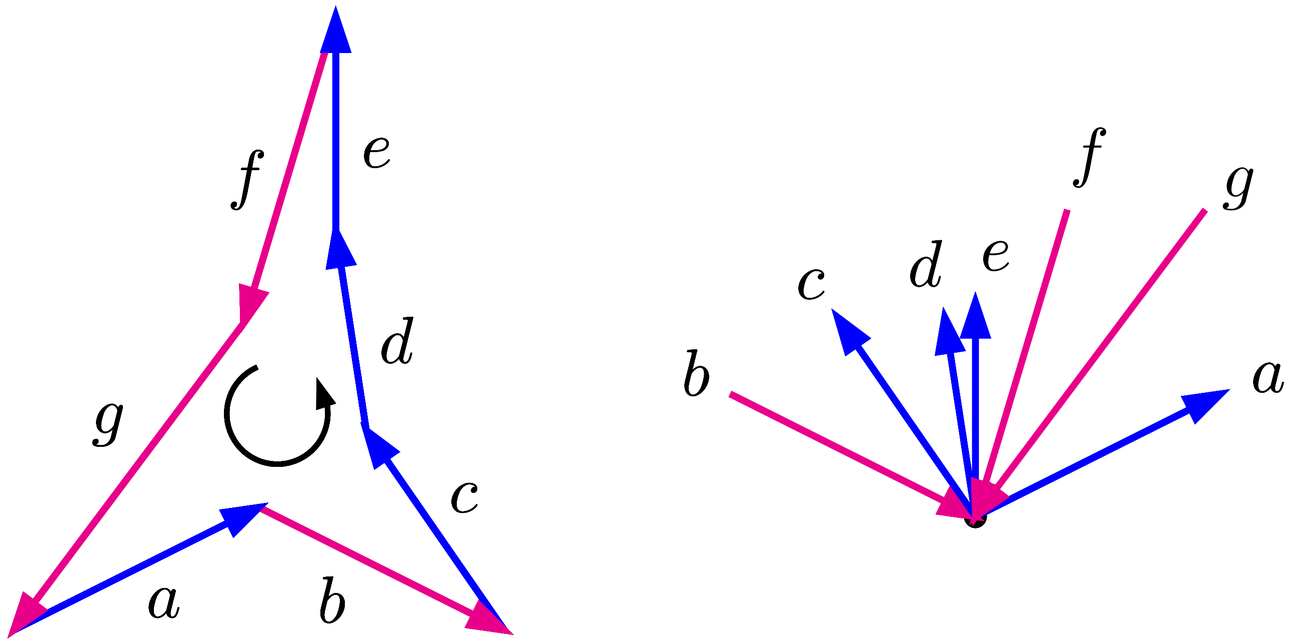
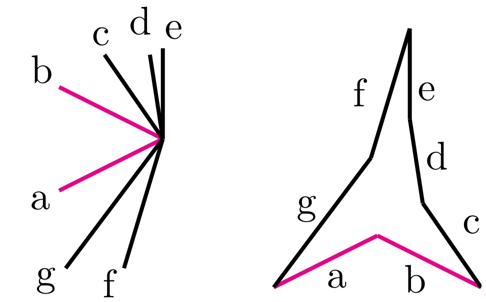
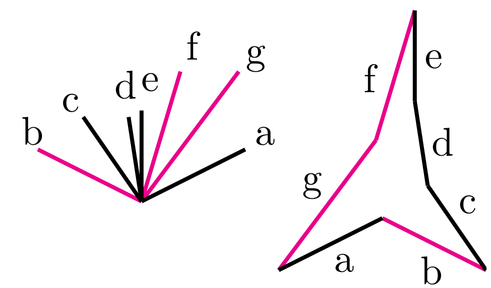


Possible sign patterns around vertices

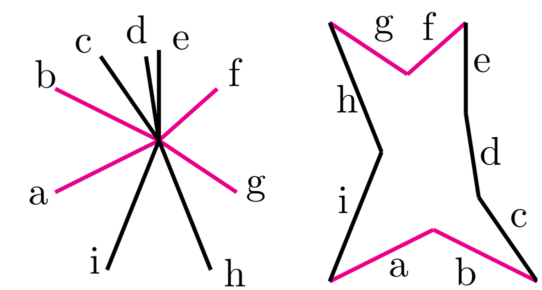
pointed, with two sign changes
(none at the big angle)



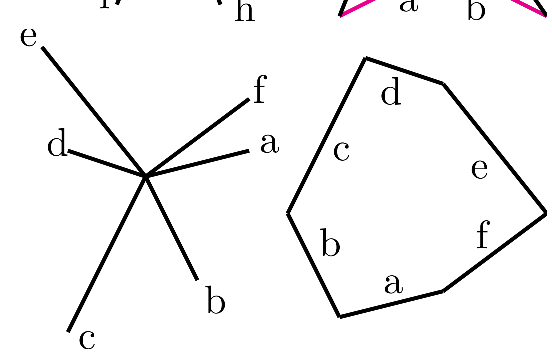
pointed, with four sign changes
(including one at the big angle)



nonpointed, with four sign changes



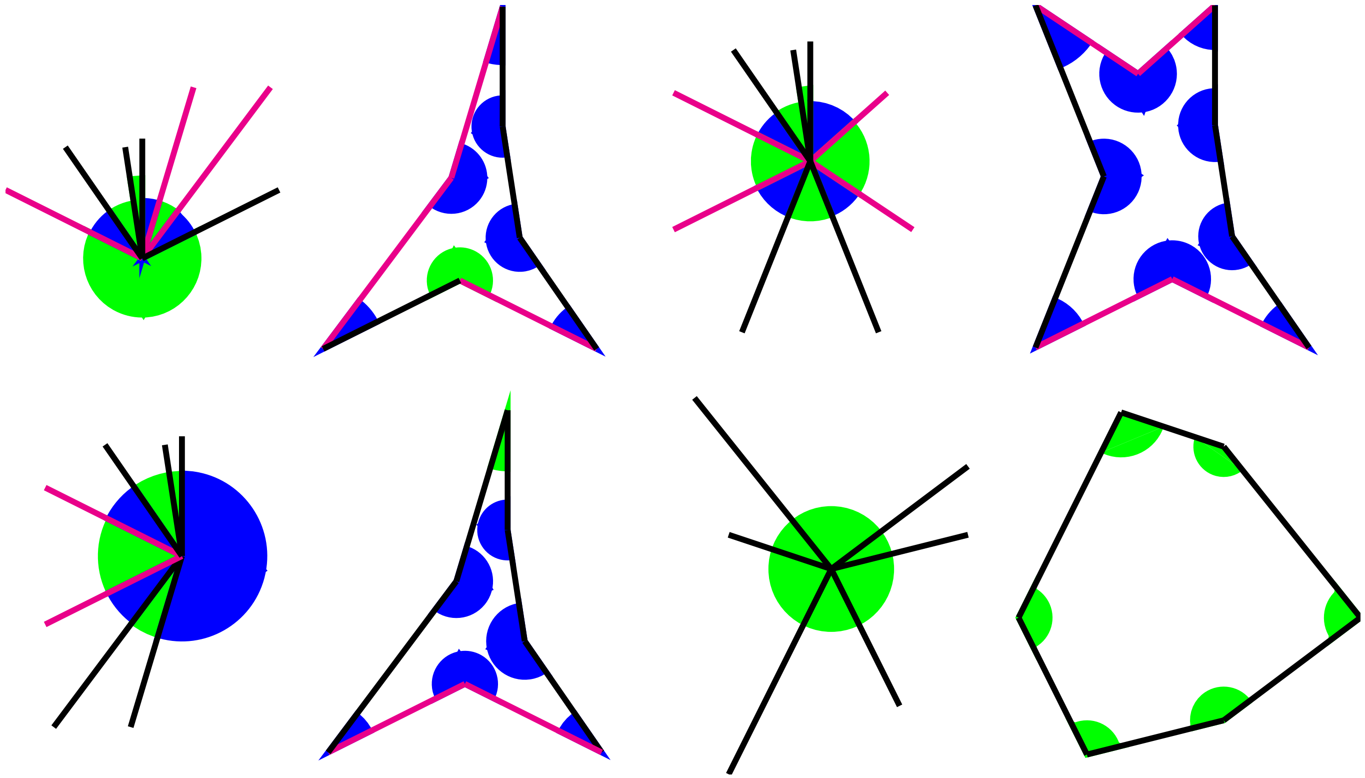
nonpointed, with no sign changes



Vertex-proper and Face-proper angles

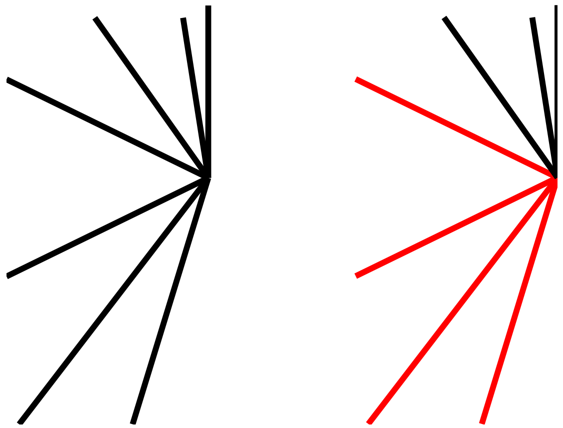
A face-proper angle is a big angle with equal signs or a small angle with a sign change.

A vertex-proper angle is a small angle with equal signs or a big angle with a sign change.



Counting angles

Lemma. *At every pointed vertex, there are at least* 3 *face-proper angles in a self-stress.*



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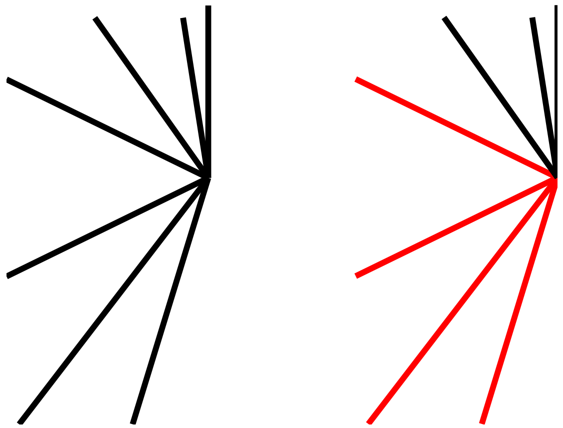
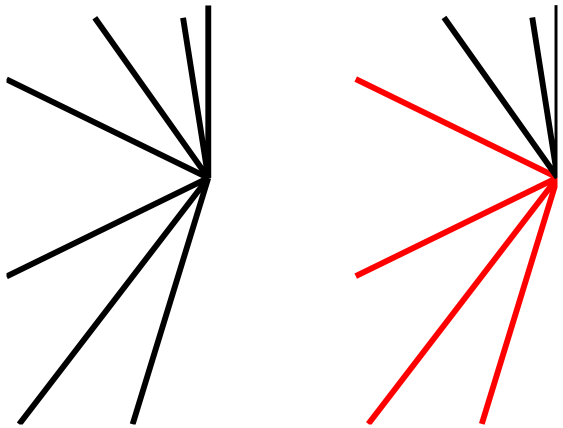
Lemma. *At every pointed vertex, there are at least* 3 *face-proper angles in a self-stress.*



Lemma. *In every pseudotriangle, there is at least* 1 *vertex-proper angle.*

Counting angles

Lemma. *At every pointed vertex, there are at least* 3 *face-proper angles in a self-stress.*



Lemma. *In every pseudotriangle, there is at least* 1 *vertex-proper angle.*

$$2e = \#\text{angles} \geq 3(n-1) + (n-1) = 2(2n-2) = 2e$$

$\longrightarrow$ equality throughout!

Counting angles—conclusion

Every pointed vertex has exactly 3 face-proper angles.
$\longrightarrow$ reciprocal face is a pseudotriangle.

The non-pointed vertex has no face-proper angles.
$\longrightarrow$ reciprocal face is convex $=$ the outer face.

Every pseudotriangle has exactly 1 vertex-proper angle.
$\longrightarrow$ reciprocal vertex is pointed.

The outer face has no vertex-proper angles.
$\longrightarrow$ reciprocal vertex is nonpointed.

Counting angles—conclusion

Every pointed vertex has exactly 3 face-proper angles.
$\longrightarrow$ reciprocal face is a pseudotriangle.

The non-pointed vertex has no face-proper angles.
$\longrightarrow$ reciprocal face is convex $=$ the outer face.

Every pseudotriangle has exactly 1 vertex-proper angle.
$\longrightarrow$ reciprocal vertex is pointed.

The outer face has no vertex-proper angles.
$\longrightarrow$ reciprocal vertex is nonpointed.

If some edges have zero stress, the reciprocal can have more than one non-pointed vertex.

General pairs of non-crossing reciprocal frameworks

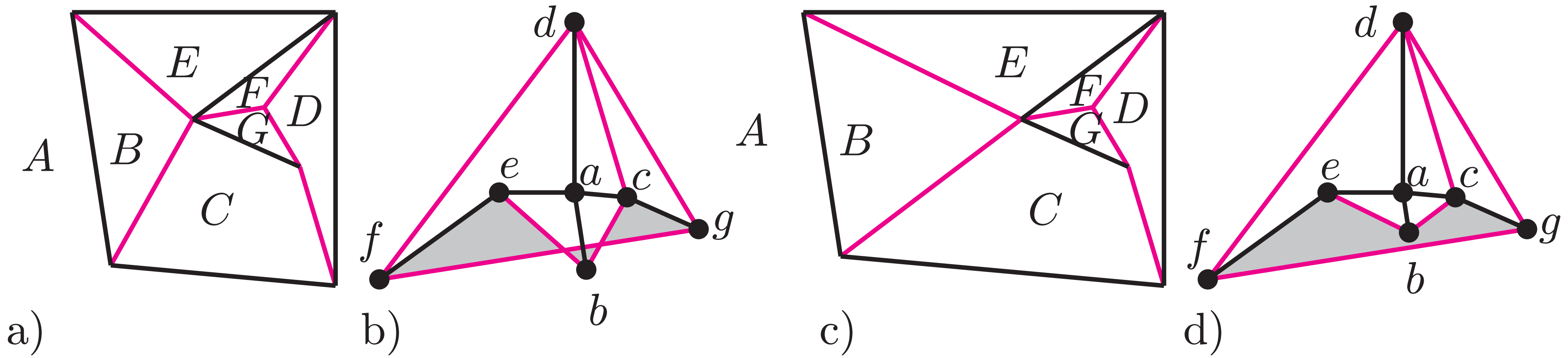
G and G^* can have more than one non-pointed vertex and can contain *pseudoquadrangles*.

Necessary conditions:

- Vertices must be as above, with a unique non-pointed vertex that has no sign changes.
- All other non-pointed vertices must have 4 sign changes.
- Analogous *face conditions*.

General pairs of non-crossing reciprocals

These combinatorial vertex conditions are also sufficient for a non-crossing reciprocal, except possibly for “self-crossing” pseudoquadrangles.



7. UNFOLDING OF FRAMEWORKS

Theorem. *Let G be a* pointed *bar-and-joint framework that does not contain all convex hull edges.*

Then G has an expansive infinitesimal motion.

Case 1: G is a path or polygon (not convex).

[Connelly, Demaine, Rote 2001]

Case 2: G is a pseudotriangulation with one convex hull edge removed. [Streinu 2001]

Expansive Motions

$\exp_{ij} = 0$ for all *bars* ij

(preservation of length)

$\exp_{ij} \geq 0$ for all other pairs (*struts*) ij

(expansiveness)

[$\exp_{ij} := \frac{1}{2} \cdot \frac{d}{dt}|p_i(t) - p_j(t)|^2 = \langle v_i - v_j, p_i - p_j \rangle$]

Proof Outline

Existence of an expansive motion

$\Updownarrow$ (duality)

Self-stresses (rigidity)
Self-stresses on planar frameworks

$\Updownarrow$ (Maxwell-Cremona correspondence)

polyhedral terrains

[Connelly, Demaine, Rote 2000]

The expansion cone

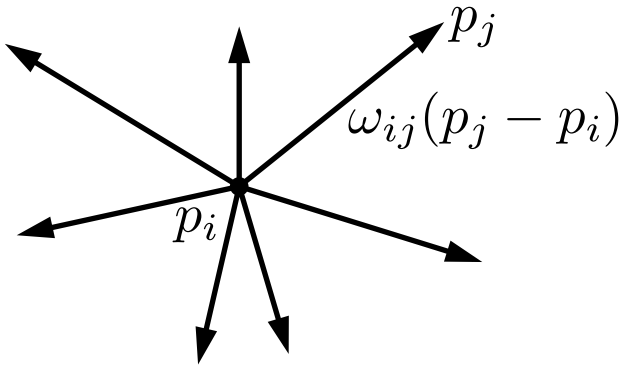
The set of expansive motions forms a convex polyhedral cone $\bar{X}_0$ in $\mathbb{R}^{2n}$, defined by homogeneous linear equations and inequalities of the form

$$\langle v_i - v_j, p_i - p_j \rangle \left\{ \begin{array}{c} = \\ \geq \end{array} \right\} 0$$

Bars, struts, frameworks, stresses

Assign a *stress* $\omega_{ij} = \omega_{ji} \in \mathbb{R}$ to each edge.

Equilibrium of forces in vertex i:



$$\sum_j \omega_{ij}(p_j - p_i) = 0$$

$\omega_{ij} \le 0$ for struts: Struts can only push.
$\omega_{ij} \in \mathbb{R}$ for bars: Bars can push or pull.

Motions and stresses

Linear Programming duality:

There is a strictly expansive motion if and only if there is no non-zero stress.

$$\langle v_i - v_j, p_i - p_j \rangle \begin{cases} = 0 \\ > 0 \end{cases}$$

$$\sum_j \omega_{ij}(p_j - p_i) = 0, \text{ for all } i$$

$$\begin{aligned} \omega_{ij} &\in \mathbb{R}, && \text{for a bar } ij \\ \omega_{ij} &\leq 0, && \text{for a strut } ij \end{aligned}$$

Motions and stresses

Linear Programming duality:

There is a strictly expansive motion if and only if there is no non-zero stress.

$$\langle v_i - v_j, p_i - p_j \rangle \begin{cases} = 0 \\ > 0 \end{cases}$$

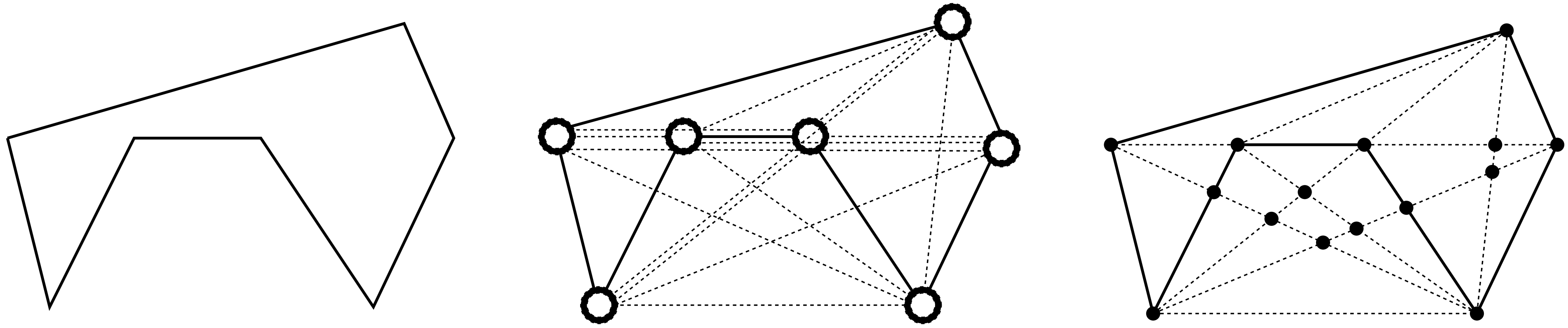
$$\left[\; Mv \begin{cases} = 0 \\ > 0 \end{cases} \; \right]$$

$$\sum_j \omega_{ij}(p_j - p_i) = 0, \text{ for all } i$$

$$[\; M^{\mathrm{T}}\omega = 0 \;]$$

$$\begin{aligned} \omega_{ij} &\in \mathbb{R}, && \text{for a bar } ij \\ \omega_{ij} &\leq 0, && \text{for a strut } ij \end{aligned}$$

Making the framework planar



- subdivide edges at intersection points
- collapse multiple edges

The Maxwell-Cremona Correspondence [1864/1872]

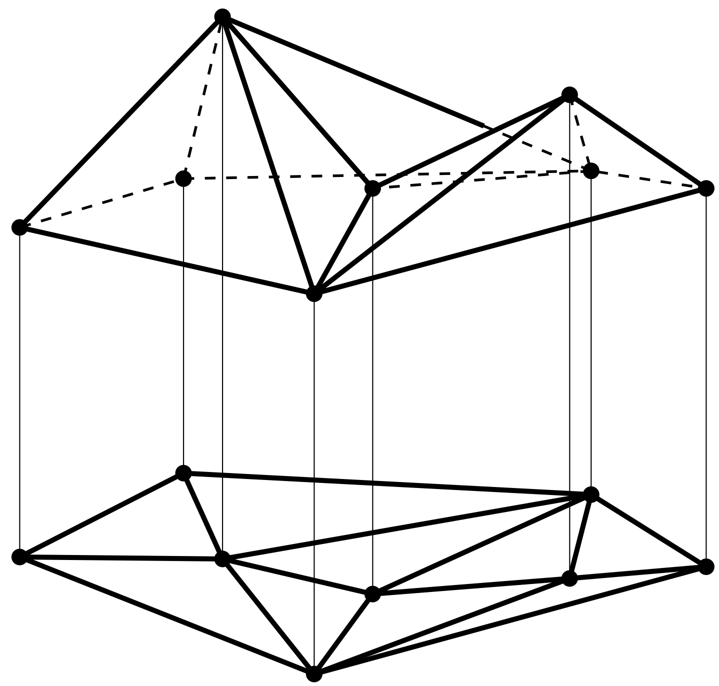
self-stresses on a planar framework

$\Updownarrow$ one-to-one correspondence

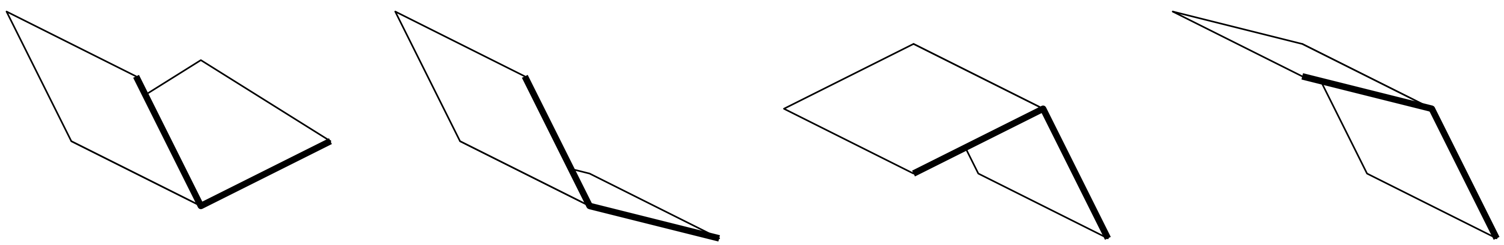
reciprocal diagram

$\Updownarrow$ one-to-one correspondence

3-d lifting (polyhedral terrain)

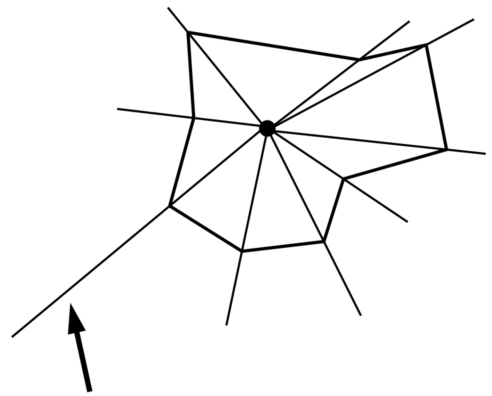


Valley and mountain folds



$\omega_{ij} > 0$	$\omega_{ij} < 0$
valley	mountain
bar or strut	bar

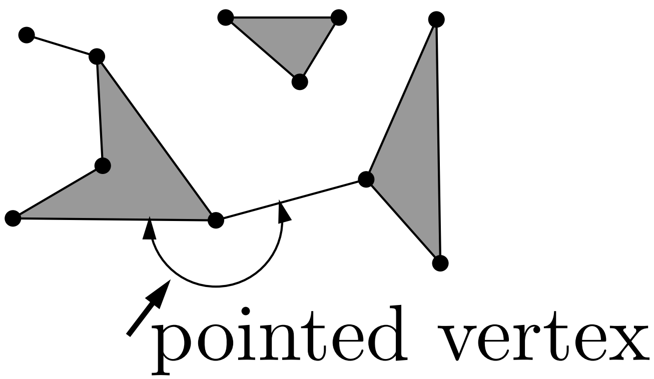
Look a the highest peak!



Angle between adjacent mountains $< \pi$.

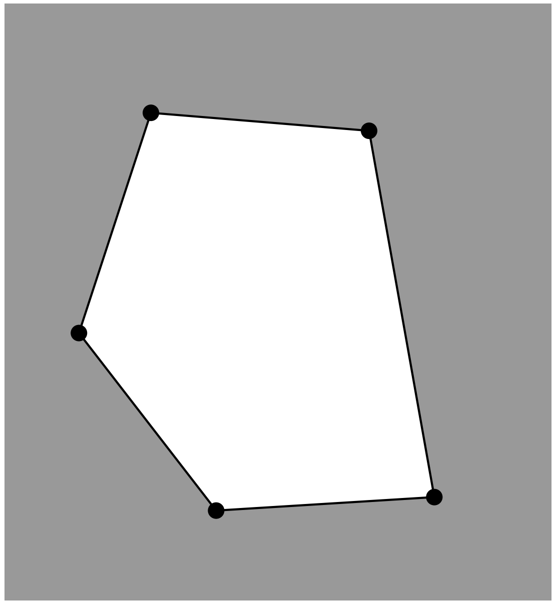
$\Longrightarrow$ bars cannot be pointed $\Longrightarrow$ contradiction.

The general case



There is at least one vertex with angle $> \pi$.

The only remaining possibility



a convex polygon

□

The Maxwell-Cremona Correspondence [1864/1872]

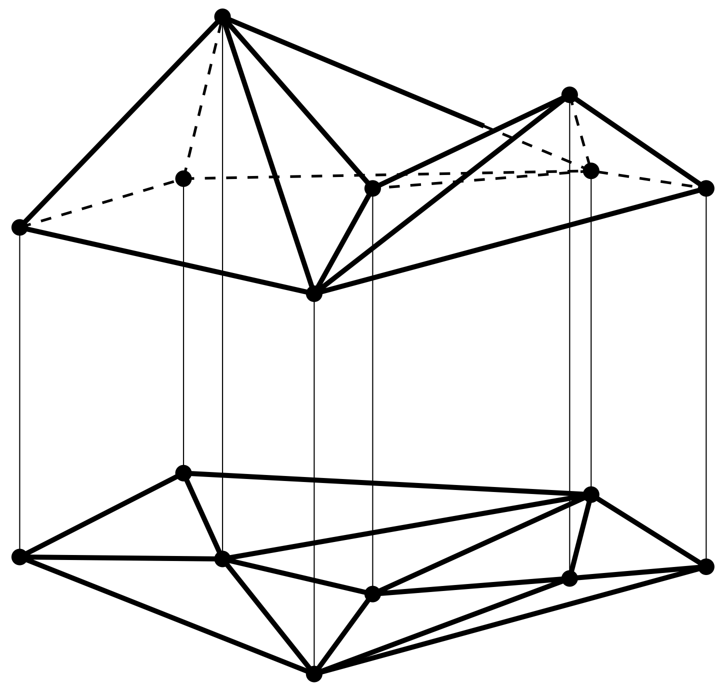
self-stresses on a planar framework

$\Updownarrow$ one-to-one correspondence

reciprocal diagram

$\Updownarrow$ one-to-one correspondence

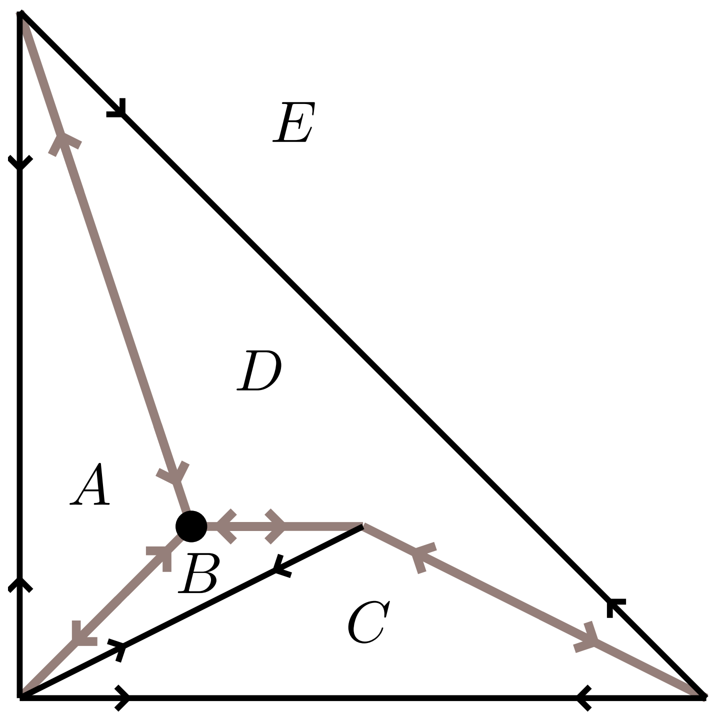
3-d lifting (polyhedral terrain)

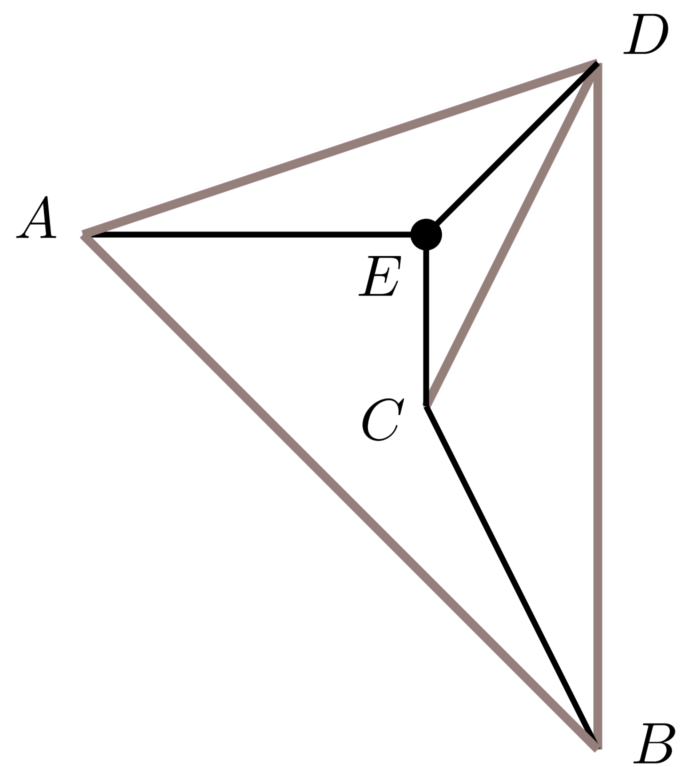


Proof:

The Maxwell reciprocal

In the *Maxwell reciprocal*, corresponding edges of the two frameworks (G, p) and (G^*, p^*) are *perpendicular*.





Interpret vertices (vectors) of (G^*, p^*) as *gradients* of faces in the lifted framework (G, p) (and vice versa).

The Maxwell reciprocal

Face f:

$$z = \langle f^*, \begin{pmatrix} x \\ y \end{pmatrix} \rangle + c_f$$

Need to determine scalars c_f (vertical shifts) so that lifted faces share common edges.

Lifted faces f and g in G with gradients f^* and g^*
$\rightarrow$ the intersection of the planes f and g (the lifted edge) is perpendicular to the dual edge f^*g^*.

$f\colon z = \langle f^*, \binom{x}{y} \rangle + c_f$
$g\colon z = \langle g^*, \binom{x}{y} \rangle + c_g$
$f \cap g\colon \langle f^* - g^*, \binom{x}{y} \rangle = c_g - c_f$

Constructing a global motion

[Connelly, Demaine, Rote 2000]:

- Define a point $v := v(p)$ in the *interior* of the expansion cone, by a suitable non-linear convex objective function.
- $v(p)$ depends smoothly on p.
- Solve the differential equation $\dot{p} = v(p)$

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Alternative: Select an *extreme ray* of the expansion cone.

[Streinu 2000]: Extreme rays correspond to pseudotriangulations.

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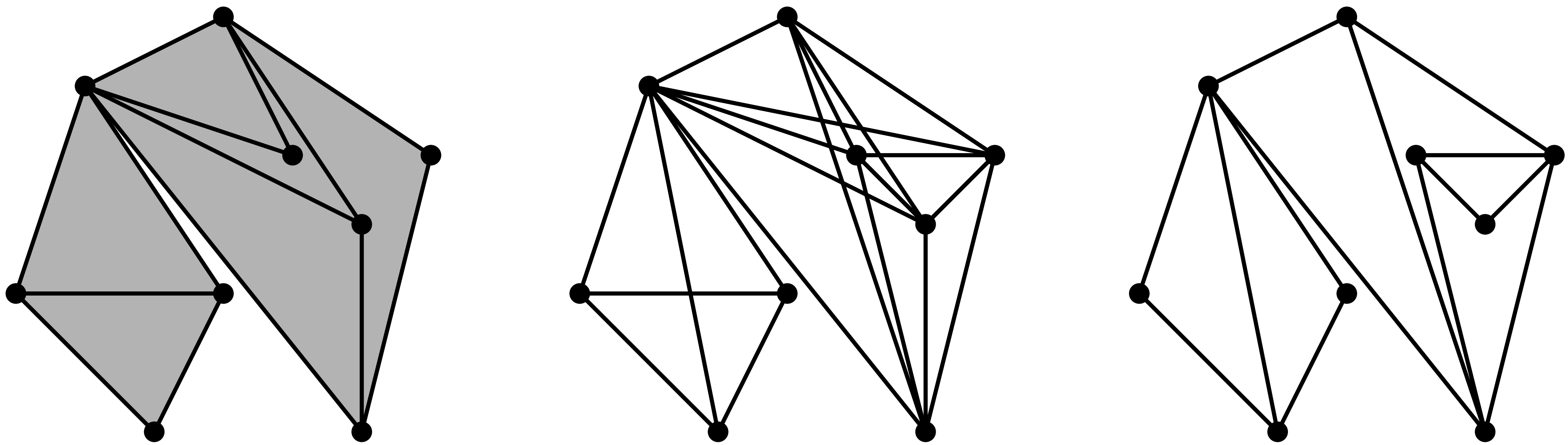
Alternative: Select an *extreme ray* of the expansion cone.

[Streinu 2000]: Extreme rays correspond to pseudotriangulations.

[Cantarella, Demaine, Iben, O'Brian 2004]:
An energy-based approach

Extreme rays of the expansion cone

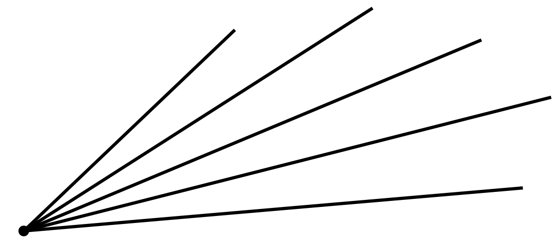
Pseudotriangulations with one convex hull edge removed yield expansive mechanisms. [Streinu 2000]
Rigid substructures can be identified.



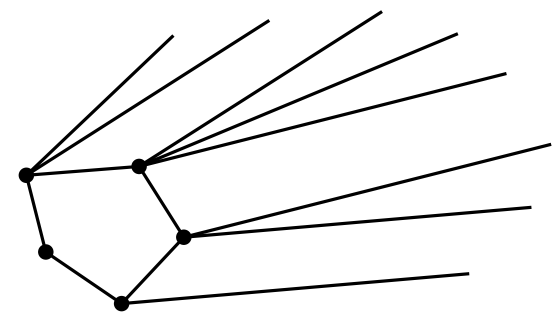
Cones and polytopes

[Rote, Santos, Streinu 2002]

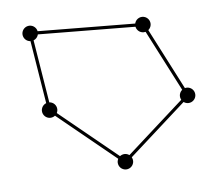
- The *expansion cone*
  $\bar{X}_0 = \{ \exp_{ij} \geq 0 \}$



- The *perturbed expansion cone*
  $=$ the *PPT polyhedron*
  $\bar{X}_f = \{ \exp_{ij} \geq f_{ij} \}$



- The *PPT polytope*
  $X_f = \{ \exp_{ij} \geq f_{ij},$
  $\exp_{ij} = f_{ij}$ for ij on boundary $\}$



A polyhedron for pseudotriangulations

With a suitable perturbation of the constraints "$\exp_{ij} \geq 0$" to "$\exp_{ij} \geq f_{ij}$", the vertices are in 1-1 correspondence with the pointed pseudotriangulations.

$\longrightarrow$ the PPT-polyhedron

$\longrightarrow$ an independent proof that expansive motions exist

The PPT polytope

Set $\exp_{ij} = f_{ij}$ for convex hull edges ij:

Theorem. *For every set S of points in general position, there is a convex $(2n-3)$-dimensional polytope whose vertices correspond to the pointed pseudotriangulations of S.*

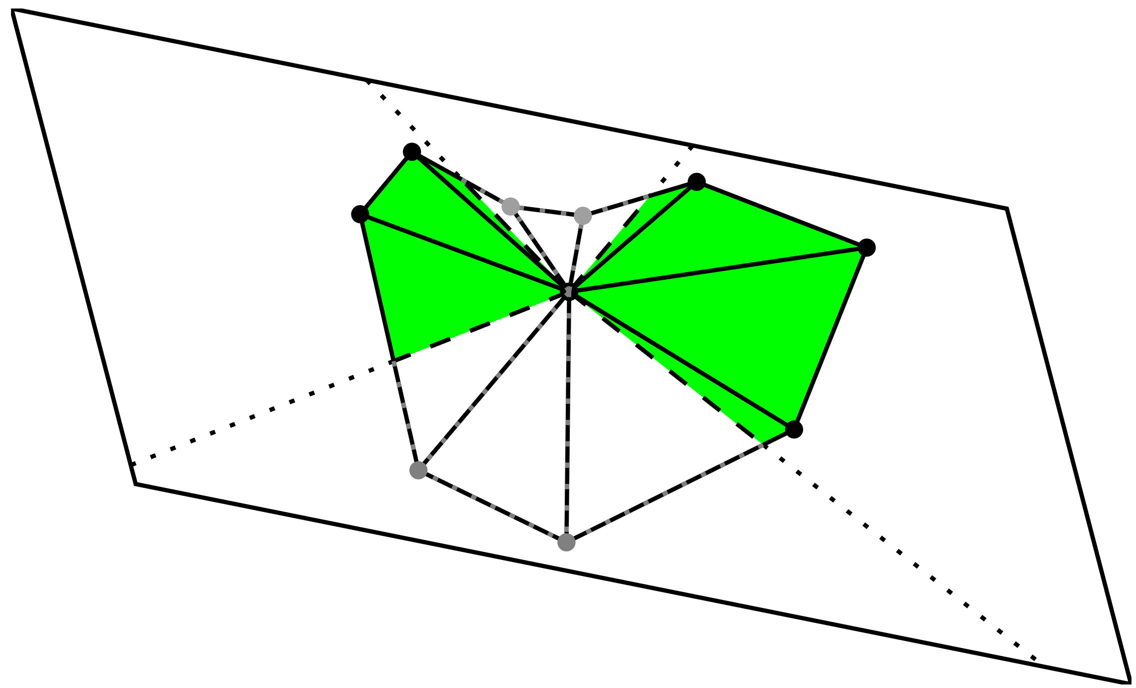
8. LIFTINGS AND SURFACES

8a. Liftings of non-crossing reciprocals

8b. Locally convex liftings

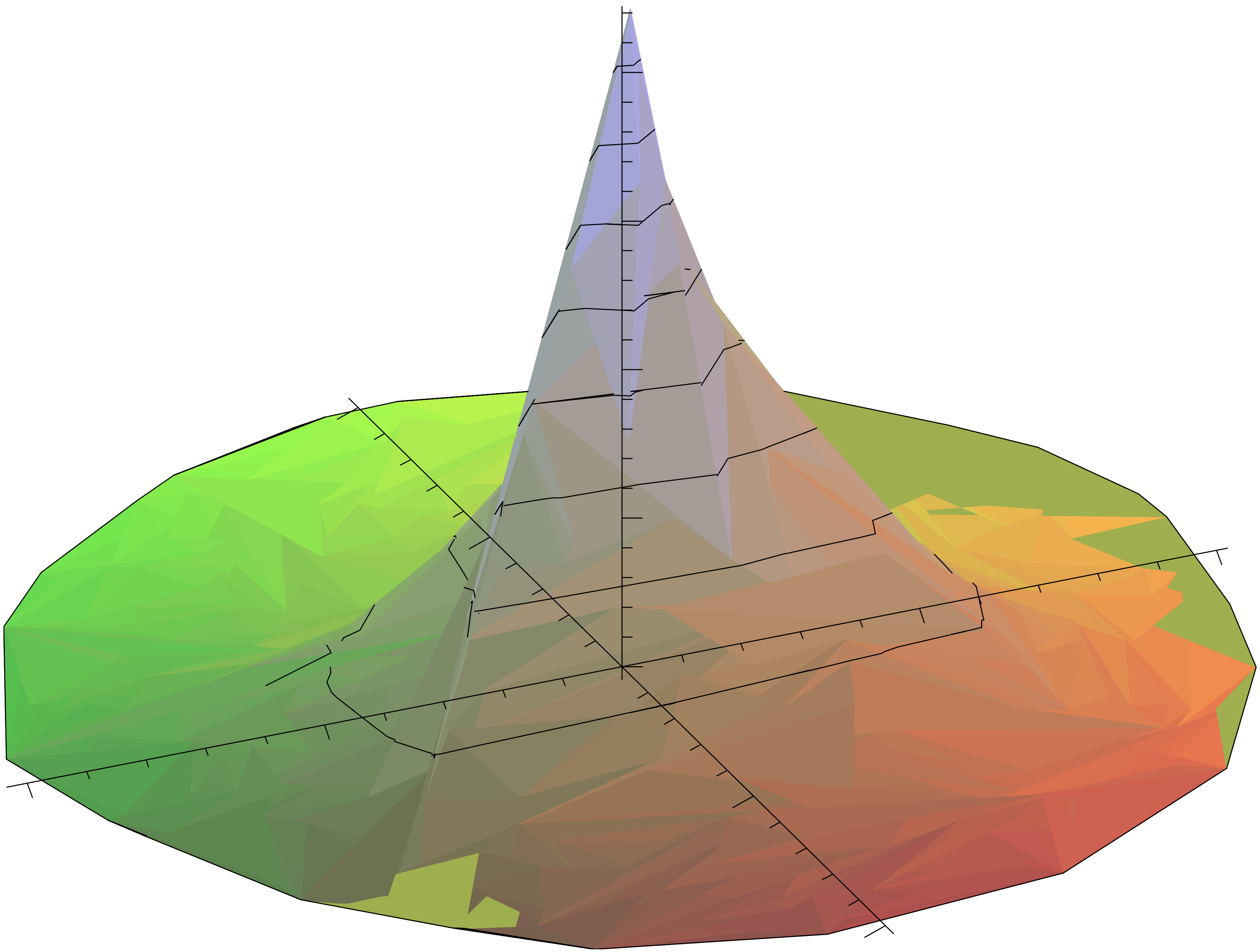
8a. Liftings of non-crossing reciprocals

Theorem. *If G and G^* are non-crossing reciprocals, the lifting has a unique maximum. There are no other critical points. Every other point p is a "twisted saddle": Its neighborhood is cut into four pieces by some plane through v (but not more).*

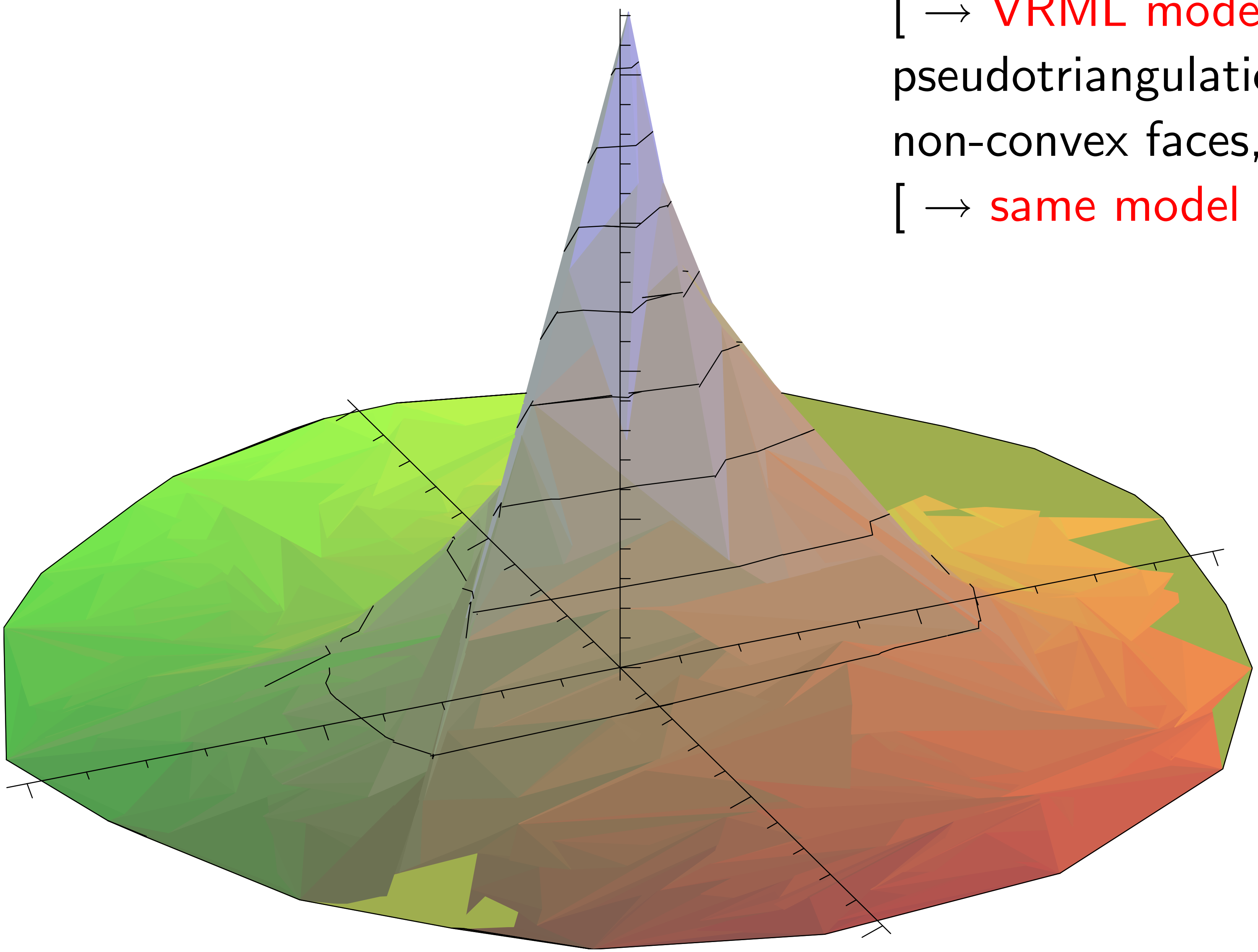


"Negative curvature" everywhere except at the peak!

Liftings of non-crossing reciprocals



Liftings of non-crossing reciprocals



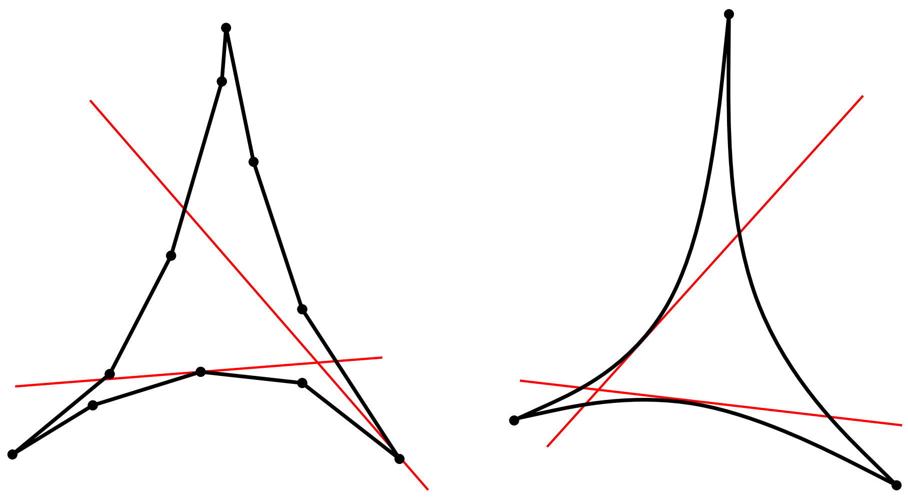
[→ VRML model of a different pseudotriangulation (with non-convex faces, too!)]

[→ same model without light]

Tangent planes of lifted pseudotriangulations

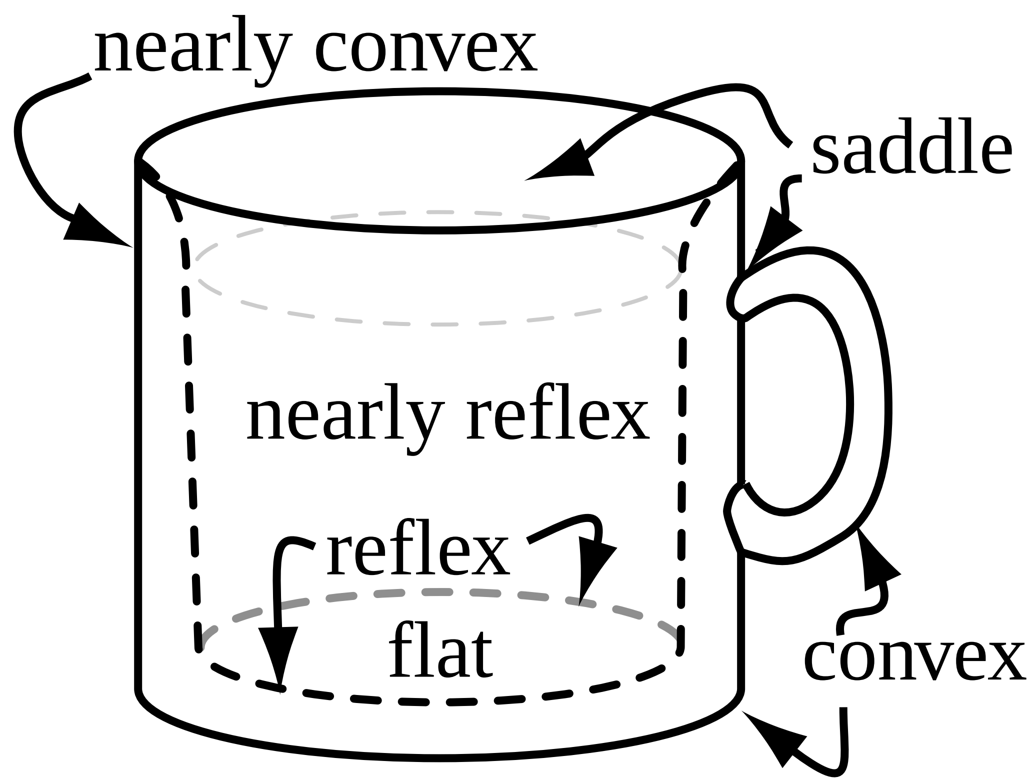
For every plane which touches the peak from above, there is a unique parallel plane which cuts a vertex like a saddle (a "tangent plane").

Remember: In a pseudotriangle, for every direction, there is a unique line which is "tangent" at a reflex vertex or "cuts through" a corner.



8b. LOCALLY CONVEX LIFTINGS

The reflex-free hull

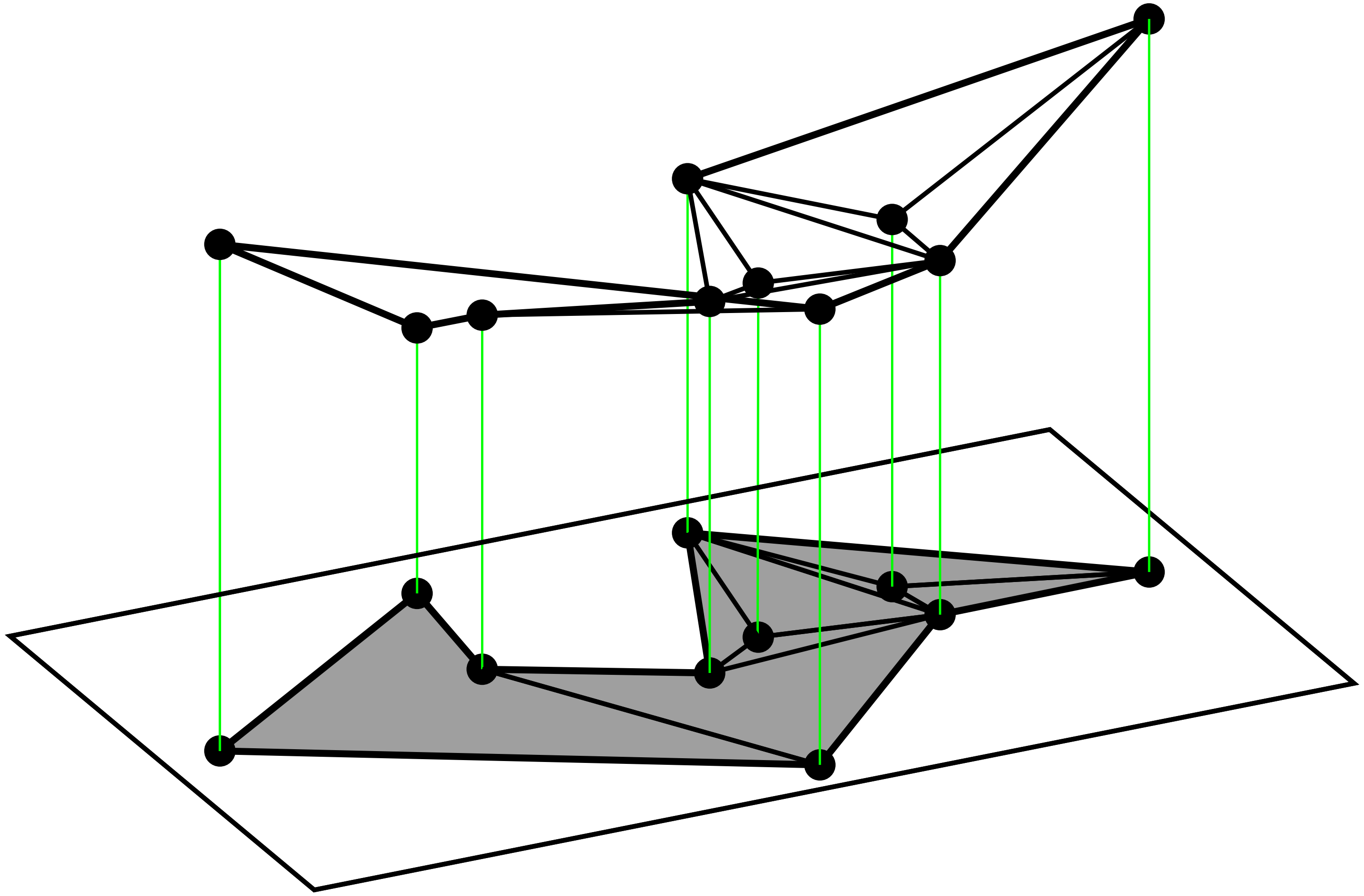


an approach for recognizing pockets in biomolecules

[Ahn, Cheng, Cheong, Snoeyink 2002]

Locally convex surfaces

A function over a polygonal domain P is *locally convex* if it is convex on every segment in P.



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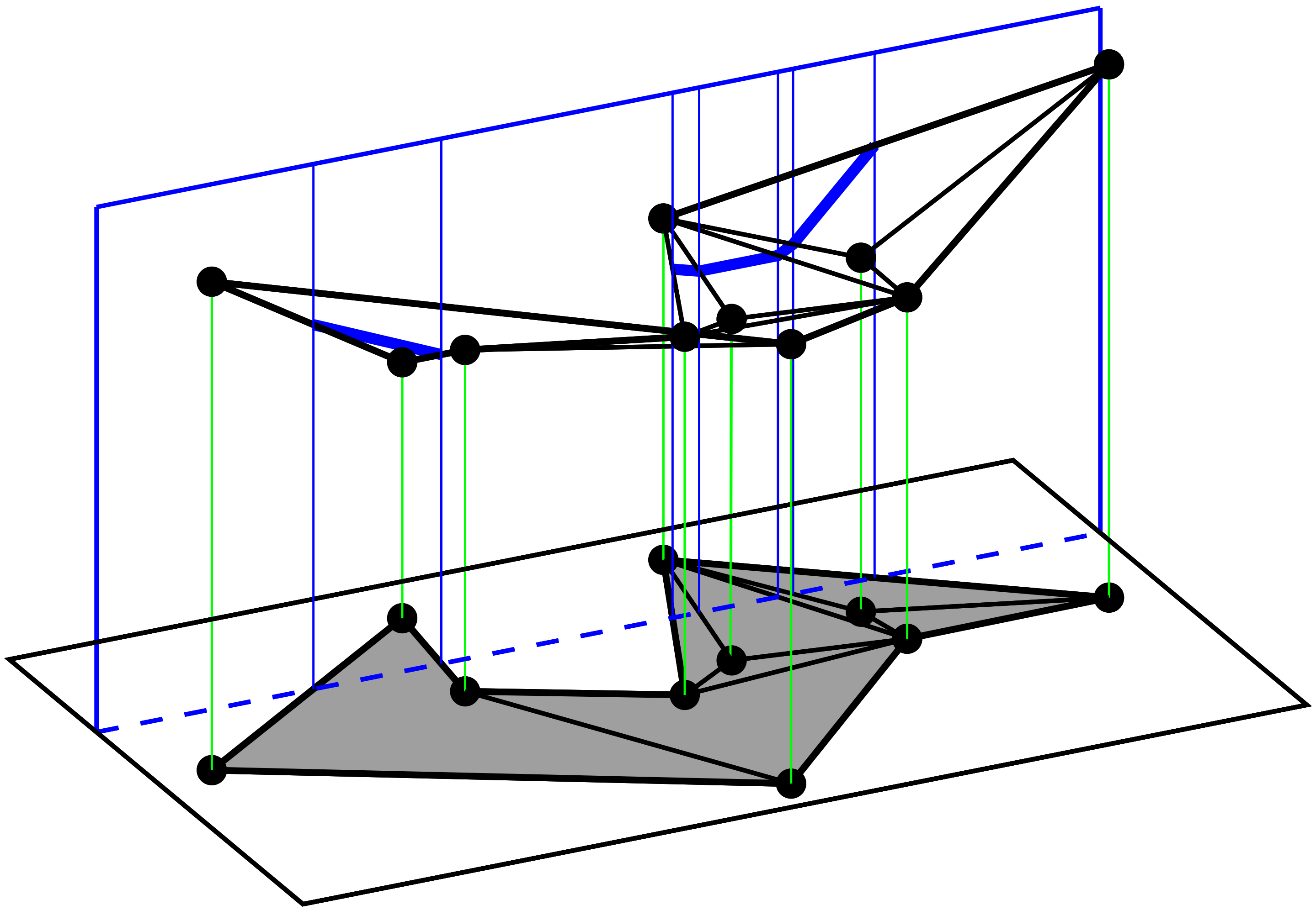


Locally convex functions on a poipogon

A *poipogon* (P, S) is a simple polygon P with some additional vertices inside.

Given a poipogon and a height value h_i for each $p_i \in S$, find the highest locally convex function $f\colon P \to \mathbb{R}$ with $f(p_i) \leq h_i$.

If P is convex, this is the lower convex hull of the three-dimensional point set (p_i, h_i).

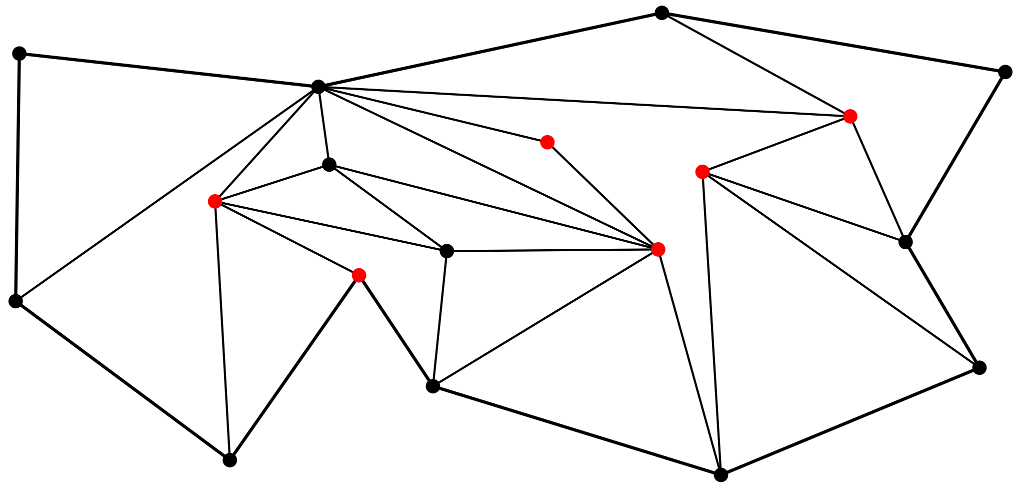
In general, the result is a piecewise linear function defined on a pseudotriangulation of (P, S). (Interior vertices may be missing.)

$\longrightarrow$ *regular pseudotriangulations*

[Aichholzer, Aurenhammer, Braß, Krasser 2003]

The surface theorem

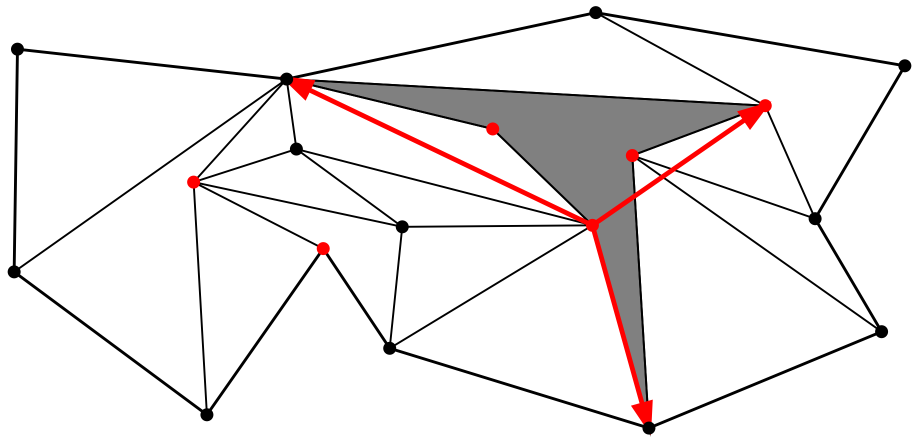
In a pseudotriangulation T of (P, S), a vertex is *complete* if it is a corner in all pseudotriangulations to which it belongs.



Theorem. *For any given set of heights h_i for the complete vertices, there is a unique piecewise linear function on the pseudotriangulation with the complete vertices. The function depends monotonically on the given heights.*

In a triangulation, all vertices are complete.

Proof of the surface theorem



Each incomplete vertex p_i is a convex combination of the three corners of the pseudotriangle in which its large angle lies:

$$p_i = \alpha p_j + \beta p_k + \gamma p_l, \text{ with } \alpha + \beta + \gamma = 1,\ \alpha, \beta, \gamma > 0.$$

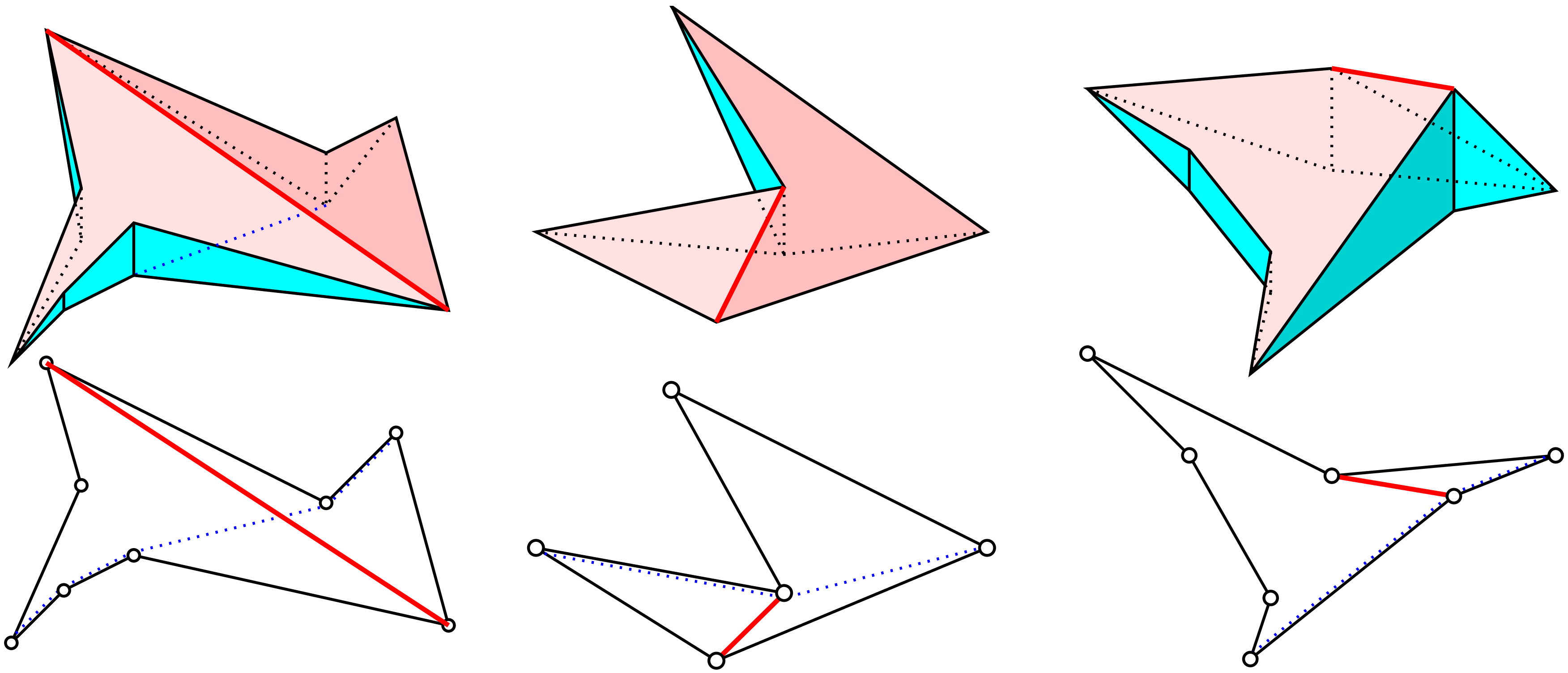
$\rightarrow h_i = \alpha h_j + \beta h_k + \gamma h_l$

The coefficient matrix of this mapping $F\colon (h_1, \ldots, h_n) \mapsto (h'_1, \ldots, h'_n)$ is nonnegative, with row sums 1.

$\Longrightarrow$ there is always a unique solution.

Flipping to optimality

Find an edge where convexity is violated, and flip it.



convexifying flips

a planarizing flip

A flip has a non-local effect on the whole surface.
The surface moves down monotonically.

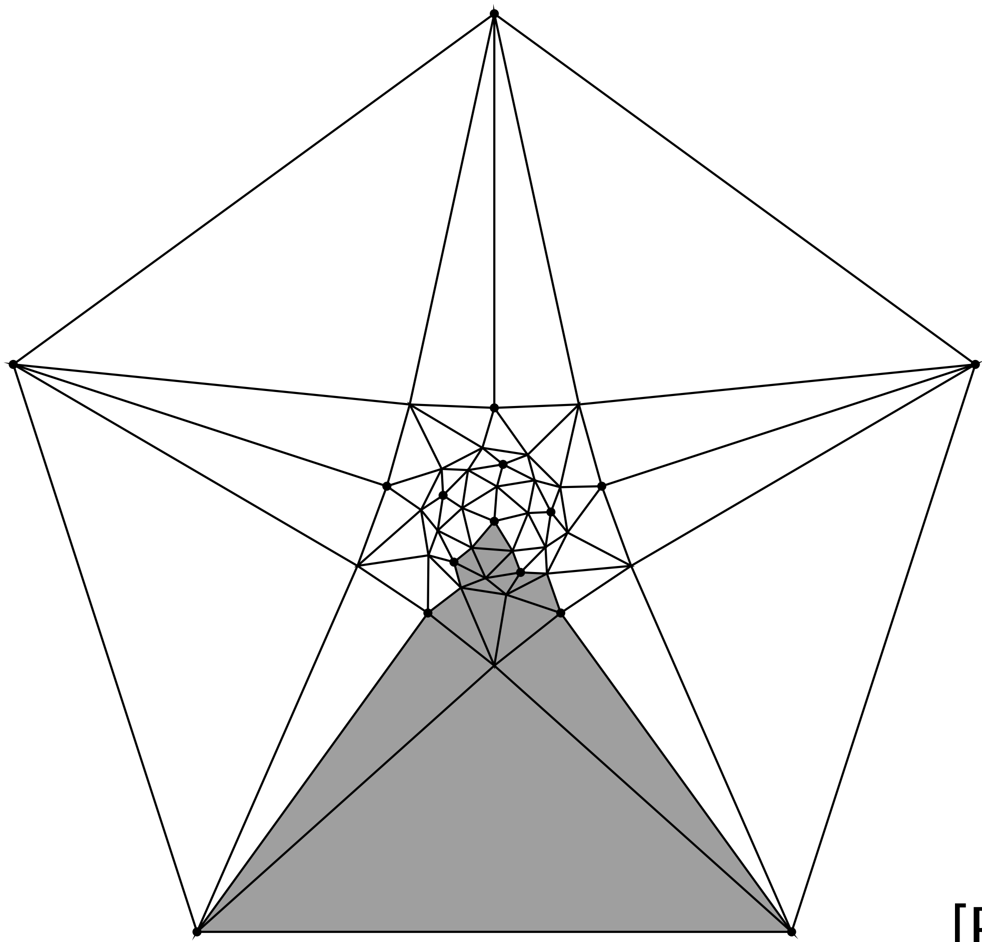
Realization as a polytope

There exists a convex polytope whose vertices are in one-to-one correspondence with the regular pseudotriangulations of a poipogon, and whose edges represent flips.

For a simple polygon (without interior points), all pseudotriangulations are regular.

9. Minimal pseudotriangulations

Minimal pseudotriangulations (w.r.t. $\subseteq$) are not necessarily minimum-cardinality pseudotriangulations.



A minimal pseudotriangulation has at most $3n - 8$ edges, and this is tight for infinitely many values of n.

[Rote, C. A. Wang, L. Wang, Xu 2003]

Open Questions

1. Pseudotriangulations on a small grid. $O(n^{3/2}) \times O(n^{3/2})$?

2. Pseudotriangulations in 3-space

(a) locally convex functions

(b) the expansion cone