**ADFOCS 2008 - Primal-Dual Algorithms for Online Optimization: Exercise Set 1**

1. Provide a primal-dual proof that the deterministic algorithm for the ski problem presented in class is 2-competitive.

2. Routers in a network receive packets and forward them to their destinations. Routers are equipped with buffers in case the arrival rate is too high. Each packet is associated with a value and an expiration time, and the goal is to maximize the sum of the values of the packets sent by the router prior to their expiration time.

   Formally, packets arrive in non-integral times. For packet $p$:

   - arrival time: $a(p)$
   - expiration time: $e(p)$
   - value: $w(p)$
   - it can be sent at any integral time $t$, provided $t \leq e(p)$

   (a) Write a linear program that finds an optimal solution to the objective function. Hint: define a variable $y(p,t)$ indicating that packet $p$ is sent at time $t$.

   (b) Formulate the dual of this linear program.

   (c) The *greedy algorithm* picks at every time $t$ the packet having the maximum value among non-expired packets and forwards it. Show that the greedy algorithm is 2-competitive by defining an appropriate dual solution.

3. In the multicut problem we are given an undirected graph with non-negative capacities on the edges and a set of source-sink pairs. The goal is to find a minimum capacity set of edges that disconnects each source-sink pair. Show that the multicut problem on trees is a special case of the set cover problem.

4. Given is an undirected graph $G = (V, E)$, a cost function $c : E \to \mathbb{R}^+$, and a requirement function $f$. The requirement function $f$ is a set of demands of the form $D = (S, T)$, where $S$ and $T$ are subsets of vertices in the graph, such that $S \cap T = \emptyset$. A demand is satisfied by picking a path from a vertex in $S$ to a vertex in $T$. Edges picked in the graph can be used to satisfy multiple demands. The goal is to find a minimum cost subgraph that satisfies the requirement function $f$.

   Our model is online; that is, the requirement function is not known in advance and it is given "demand by demand" to the algorithm, while the input graph is known in advance.

   (a) Show that online set cover is a special case of the above problem.

   (b) Formulate the above problem as a linear program and derive its dual.

   (c) Define an online primal-dual algorithm for the problem. What is the competitive factor obtained?

   (d) Suppose the input graph is a set of disjoint trees $T_1, \ldots, T_m$. The edges of the trees have non-negative costs. There is a set of *terminals* and each leaf in a tree is associated with a terminal. A terminal can be associated with several leaves, where each leaf belongs to a different tree. Each request is to a terminal $t$ and a feasible solution is a path from one of the leaves associated with $t$ to the root of the tree. The cost of the solution is the cost of the path picked.

i. Convince yourself that this problem falls into the model defined in this question.
ii. How would you apply online randomized rounding to a fractional solution generated by your primal-dual online algorithm for this problem?

5. Same setting as the previous one. Now, a feasible solution is a set of edges that separates for each demand $D = (S, T)$, any two vertices $s \in S$ and $t \in T$. The goal is to find a minimum cost subgraph that satisfies the requirement function $f$.

   (a) Formulate the above problem as a linear program and derive its dual.
   (b) Define an online primal-dual algorithm for the problem. What is the competitive factor obtained?