Cost Sharing and Approximation Algorithms

— Lecture 1 —

Guido Schäfer
CWI Amsterdam / VU University Amsterdam
g.schaefer@cwi.nl

ADFOCS 2010
11th Max Planck Advanced Course on the Foundations of Computer Science
August 2–6, 2010, Saarbrücken, Germany

Motivation: Auction

- suppose we want to auction off a single item to one of n potential buyers in U
- every bidder $i \in U$ has a valuation v_i for receiving the item
- valuation is only known to i and not to the auctioneer
- every bidder i announces a bid b_i

Mechanism: protocol that based on the bids determines a winner of the auction and a selling price p

Motivation: Auction

Selfishness: every player wants to maximize his net gain $(v_i - p)q_i$, where $q_i = 1$ if i is the winner and $q_i = 0$ otherwise.

Goal: economic efficiency, i.e., sell the item to the buyer with maximum valuation.

Question: Can efficiency be achieved although valuations are private?

Vickrey's Truthful Mechanism

First-Price Auction: sell the item to the buyer with the highest bid and charge his bid

- buyers have an incentive to underbid

Second-Price Auction (Vickrey Auction '61): sell the item to the buyer with the highest bid and charge the second-highest bid

- buyers bid their valuations truthfully, i.e., $b_i = v_i$
- economic efficiency is achieved

Vickrey's Truthful Mechanism

First-Price Auction: sell the item to the buyer with the highest bid and charge his bid

- buyers have an incentive to underbid

Second-Price Auction (Vickrey Auction '61): sell the item to the buyer with the highest bid and charge the second-highest bid

- buyers bid their valuations truthfully, i.e., $b_i = v_i$
- economic efficiency is achieved

Vickrey's Truthful Mechanism

First-Price Auction: sell the item to the buyer with the highest bid and charge his bid

- buyers have an incentive to underbid

Second-Price Auction (Vickrey Auction '61): sell the item to the buyer with the highest bid and charge the second-highest bid

- buyers bid their valuations truthfully, i.e., $b_i = v_i$
- economic efficiency is achieved

Vickrey's Truthful Mechanism

First-Price Auction: sell the item to the buyer with the highest bid and charge his bid

- buyers have an incentive to underbid

Second-Price Auction (Vickrey Auction '61): sell the item to the buyer with the highest bid and charge the second-highest bid

- buyers bid their valuations truthfully, i.e., $b_i = v_i$
- economic efficiency is achieved

Group-Strategyproof Cost Sharing Mechanisms

Cooperative Cost Sharing

Setting:

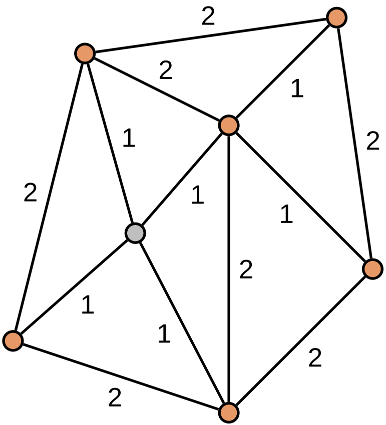
- set of players are interested in receiving some service
- provision of service incurs a (player-set dependent) cost that needs to be shared among the players
- players act strategically: aim at receiving service at low individual price
- players can coordinate their strategies

Applications: sharing the cost of public investments, access to network, etc.

Goal: design selection and payment scheme such that

- it is in every player's self-interest to behave truthfully
- payments of selected players cover the service cost
- player selection is "socially efficient"

Motivating Example



Given:

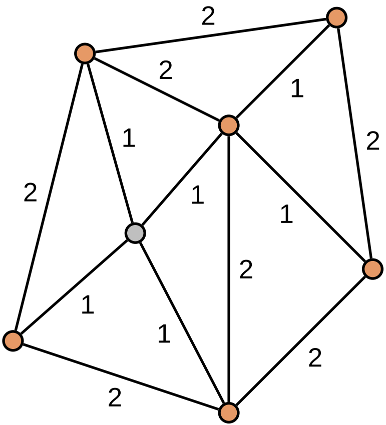
- network $N = (V, E, c)$
- set of players $U = [n]$
- player $i \in U$ requests connection between s_i, t_i

Cost Function:

$C(S) =$ min. cost to satisfy all requests of players in $S \subseteq U$

Example: $C(\{1, 3, 4\}) = 5$

Motivating Example



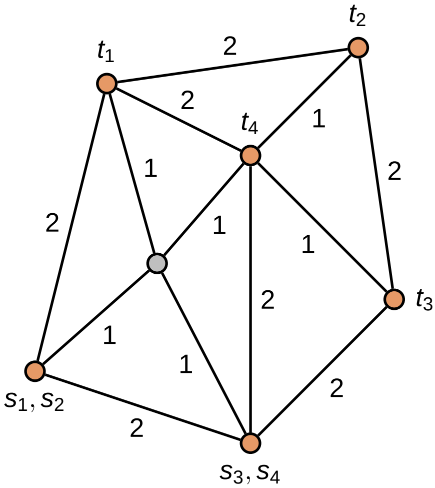
Given:

- network $N = (V, E, c)$
- set of players $U = [n]$
- player $i \in U$ requests connection between s_i, t_i

Cost Function:
$C(S) =$ min. cost to satisfy all requests of players in $S \subseteq U$

Example: $C(\{1, 3, 4\}) = 5$

Motivating Example



Given:

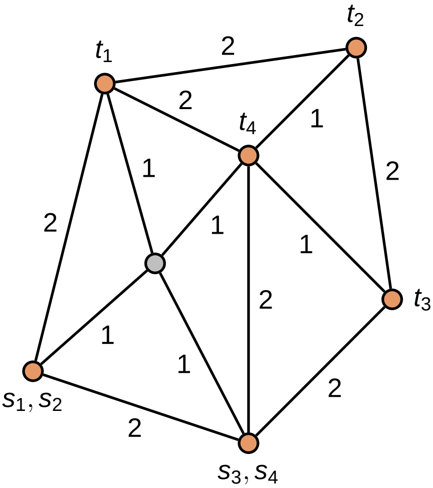
- network $N = (V, E, c)$
- set of players $U = [n]$
- player $i \in U$ requests connection between s_i, t_i

Cost Function:

$C(S)$ = min. cost to satisfy all requests of players in $S \subseteq U$

Example: $C(\{1, 3, 4\}) = 5$

Motivating Example



Given:

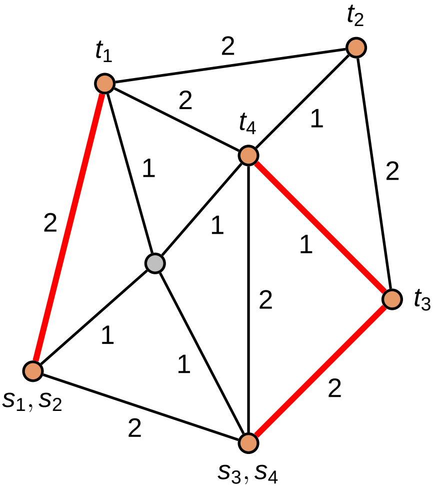
- network $N = (V, E, c)$
- set of players $U = [n]$
- player $i \in U$ requests connection between s_i, t_i

Cost Function:

$C(S) =$ min. cost to satisfy all requests of players in $S \subseteq U$

Example: $C(\{1, 3, 4\}) = 5$

Motivating Example



Given:

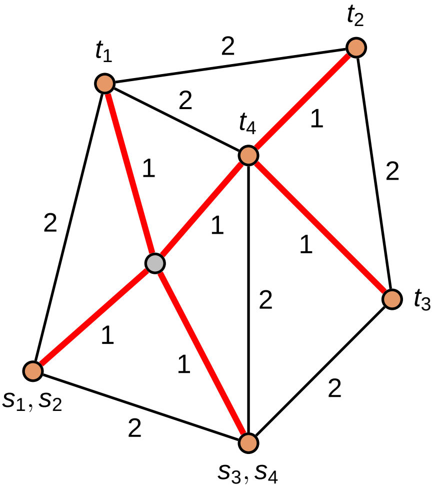
- network $N = (V, E, c)$
- set of players $U = [n]$
- player $i \in U$ requests connection between s_i, t_i

Cost Function:

$C(S) =$ min. cost to satisfy all requests of players in $S \subseteq U$

Example: $C(\{1, 3, 4\}) = 5$

Motivating Example



Given:

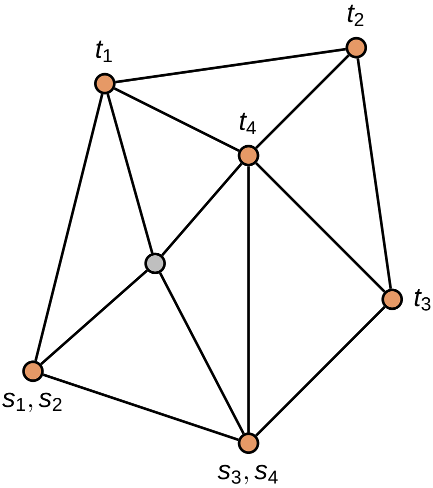
- network $N = (V, E, c)$
- set of players $U = [n]$
- player $i \in U$ requests connection between s_i, t_i

Cost Function:

$C(S) =$ min. cost to satisfy all requests of players in $S \subseteq U$

Example: $C(\{1, 2, 3, 4\}) = 6$

Motivating Example



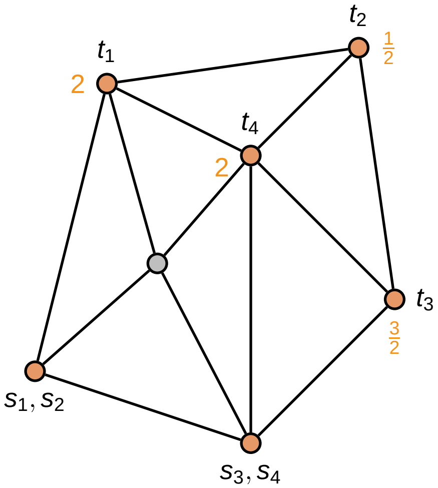
Player $i \in U$:

- valuation v_i (private!)
- bid b_i (public)
- **goal:** maximize $v_i - p_i$

Cost Sharing Mechanism:

- selects a set Q of players whose requests are satisfied
- determines a payment p_i for every $i \in Q$ to distribute the cost $C(Q)$

Motivating Example



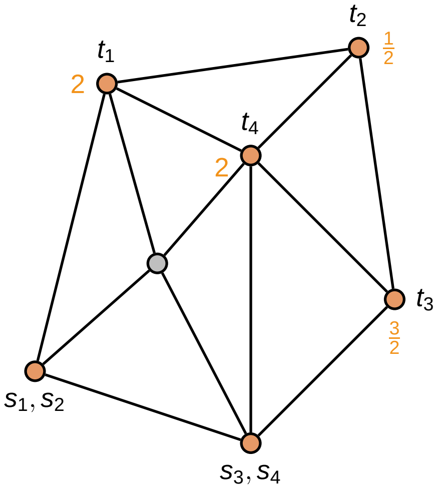
Player $i \in U$:

- valuation v_i (private!)
- bid b_i (public)
- **goal:** maximize $v_i - p_i$

Cost Sharing Mechanism:

- selects a set Q of players whose requests are satisfied
- determines a payment p_i for every $i \in Q$ to distribute the cost $C(Q)$

Motivating Example



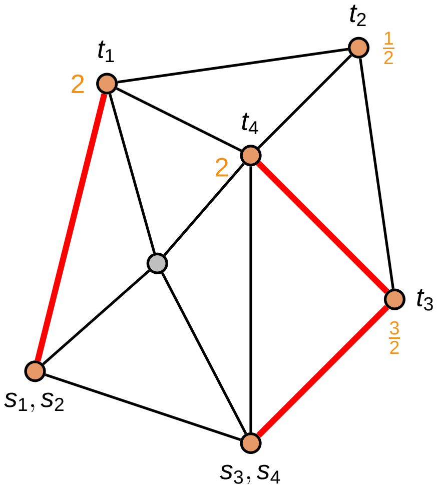
Player $i \in U$:

- valuation v_i (private!)
- bid b_i (public)
- **goal:** maximize $v_i - p_i$

Cost Sharing Mechanism:

- selects a set Q of players whose requests are satisfied
- determines a payment p_i for every $i \in Q$ to distribute the cost $C(Q)$

Motivating Example



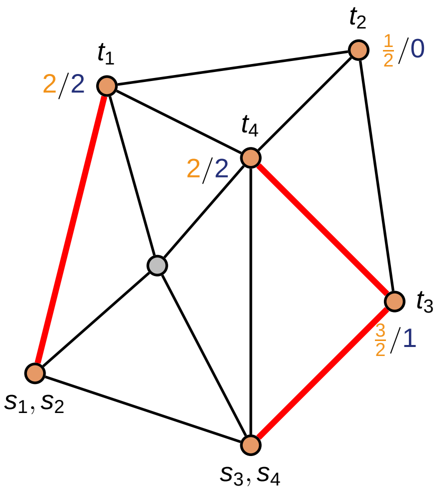
Player $i \in U$:

- valuation v_i (private!)
- bid b_i (public)
- **goal:** maximize $v_i - p_i$

Cost Sharing Mechanism:

- selects a set Q of players whose requests are satisfied
- determines a payment p_i for every $i \in Q$ to distribute the cost $C(Q)$

Motivating Example



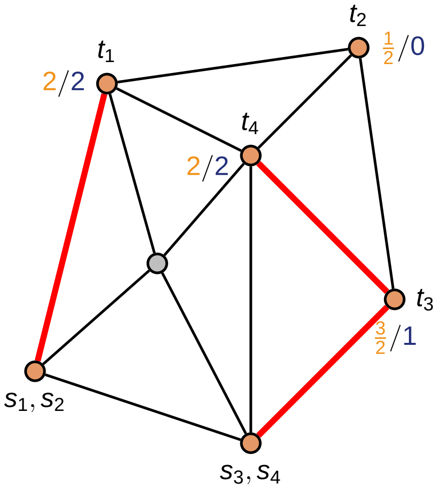
Player $i \in U$:

- valuation v_i (private!)
- bid b_i (public)
- **goal:** maximize $v_i - p_i$

Cost Sharing Mechanism:

- selects a set Q of players whose requests are satisfied
- determines a payment p_i for every $i \in Q$ to distribute the cost $C(Q)$

Motivating Example



Objectives:

1. **Truthfulness:** bidding truthfully is a dominant strategy for every player
2. **Budget Balance:** payments recover solution cost
3. **Efficiency:** selected player set realizes "social efficiency" objective

Cost Sharing Model

Given:

- set U of players (interested in some service)
- every player $i \in U$:
  - valuation v_i: value (private!) of the service
  - bid b_i: maximum amount he is willing to pay
- player-set dependent cost function $C : 2^U \rightarrow \mathbb{R}^+$
  - defined implicitly: cost function of combinatorial optimization problem $\mathcal{P}$ (e.g., Steiner forest, scheduling, etc.)
  - $C(S) =$ optimal solution cost for player set $S \subseteq U$

Cost Sharing Mechanism

Cost Sharing Mechanism M: collects bids $(b_i)_{i \in U}$ from players and computes

- set $Q \subseteq U$ of players that receive service (selection scheme)
  Notation: $q_i = 1$ if $i \in Q$ and $q_i = 0$ otherwise
- payment p_i for every player $i \in U$ to distribute the cost $C(Q)$ (payment scheme)

1. **No Positive Transfer:** $p_i \geq 0$ for all $i \in Q$
2. **Voluntary Participation:** $p_i = 0$ for all $i \notin Q$ and $p_i \leq b_i$ for all $i \in Q$
3. **Consumer Sovereignty:** for every $i \in U$ there exists a bid b_i^* for which i is guaranteed to receive service

Truthfulness

Strategic Behavior: every player $i \in U$ acts selfishly and attempts to maximize his quasi-linear utility function:

$$u_i(q, p) := q_i(v_i - p_i)$$

$\Rightarrow$ player i will misreport his valuation ($b_i \neq v_i$) if this leads to larger utility

Strategyproofness: utility of every player $i \in U$ is maximized if he bids truthfully $b_i = v_i$ (independently of other players' bids)

Group-Strategyproofness: same holds true even if players form coalitions to coordinate their bids

Truthfulness

Strategic Behavior: every player $i \in U$ acts selfishly and attempts to maximize his quasi-linear utility function:

$$u_i(q,p) := q_i(v_i - p_i)$$

$\Rightarrow$ player i will misreport his valuation ($b_i \neq v_i$) if this leads to larger utility

Strategyproofness: utility of every player $i \in U$ is maximized if he bids truthfully $b_i = v_i$ (independently of other players' bids)

Group-Strategyproofness: same holds true even if players form coalitions to coordinate their bids

Truthfulness

Strategic Behavior: every player $i \in U$ acts selfishly and attempts to maximize his quasi-linear utility function:

$$u_i(q, p) := q_i(v_i - p_i)$$

$\Rightarrow$ player i will misreport his valuation ($b_i \neq v_i$) if this leads to larger utility

Strategyproofness: utility of every player $i \in U$ is maximized if he bids truthfully $b_i = v_i$ (independently of other players' bids)

Group-Strategyproofness: same holds true even if players form coalitions to coordinate their bids

Illustration: Group-Strategyproofness

Definition

A cost sharing mechanism M is group-strategyproof iff for all $S \subseteq U$

$$u_i(\tilde{q},\tilde{p}) \geq u_i(q,p) \;\; \forall i \in S \quad \Rightarrow \quad u_i(\tilde{q},\tilde{p}) = u_i(q,p) \;\; \forall i \in S$$

(q,p): outcome if $b_i = v_i$ for every $i \in S$
$(\tilde{q},\tilde{p})$: outcome if $b_i = \cdot$ for every $i \in S$

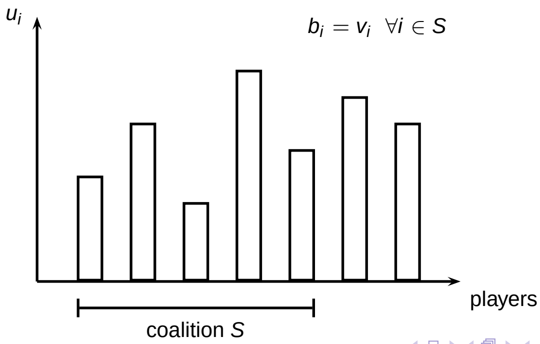


Illustration: Group-Strategyproofness

Definition

A cost sharing mechanism M is group-strategyproof iff for all $S \subseteq U$

$$\exists i \in S: \quad u_i(\tilde{q}, \tilde{p}) > u_i(q, p) \quad \Rightarrow \quad \exists j \in S: \quad u_j(\tilde{q}, \tilde{p}) < u_j(q, p)$$

(q, p): outcome if $b_i = v_i$ for every $i \in S$
$(\tilde{q}, \tilde{p})$: outcome if $b_i = \cdot$ for every $i \in S$

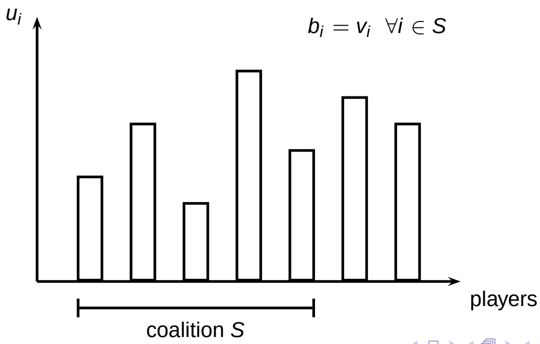


Illustration: Group-Strategyproofness

Definition

A cost sharing mechanism M is group-strategyproof iff for all $S \subseteq U$

$$\exists i \in S: \quad u_i(\tilde{q},\tilde{p}) > u_i(q,p) \quad \Rightarrow \quad \exists j \in S: \quad u_j(\tilde{q},\tilde{p}) < u_j(q,p)$$

(q,p): outcome if $b_i = v_i$ for every $i \in S$
$(\tilde{q},\tilde{p})$: outcome if $b_i = \cdot$ for every $i \in S$

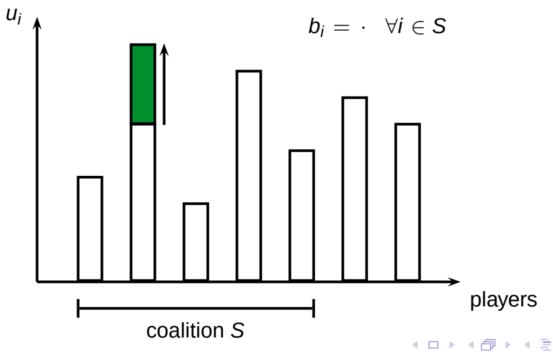


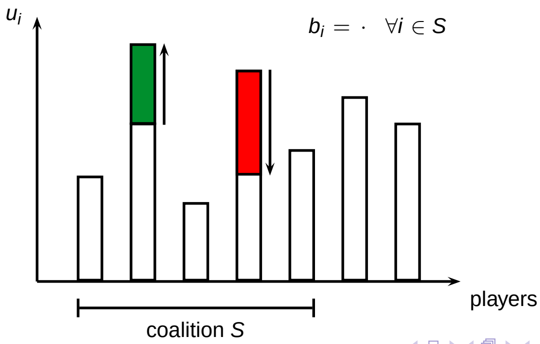
Illustration: Group-Strategyproofness

Definition

A cost sharing mechanism M is group-strategyproof iff for all $S \subseteq U$

$$\exists i \in S: \quad u_i(\tilde{q}, \tilde{p}) > u_i(q, p) \quad \Rightarrow \quad \exists j \in S: \quad u_j(\tilde{q}, \tilde{p}) < u_j(q, p)$$

(q, p): outcome if $b_i = v_i$ for every $i \in S$
$(\tilde{q}, \tilde{p})$: outcome if $b_i = \cdot$ for every $i \in S$



"Classical" Objectives

1. **Budget Balance:** payments equal servicing cost

$$\sum_{i \in Q} p_i = C(Q)$$

2. **Group-Strategyproofness**

3. **Efficiency:** assuming truthful bidding, selected player set maximizes social welfare

$$\sum_{i \in Q} v_i - C(Q) = \max_{S \subseteq U} \sum_{i \in S} v_i - C(S)$$

Computational Issues

Want to design mechanisms that are computationally efficient

Problems:

1. underlying optimization problem $\mathcal{P}$ is often computationally hard
2. truthfulness, budget balance and efficiency cannot be achieved simultaneously

[Green et al. '76] [Roberts '79]
[Feigenbaum et al., TCS '03]

Solutions:

1. use approximation algorithm to compute an approximate solution of cost $\bar{C}(Q) \leq \beta \cdot C(Q)$ where $\beta \geq 1$
2. consider different (but equivalent) social efficiency objective

Approximate Budget Balance and Efficiency

Approximate Budget Balance: cost sharing mechanism M is β-budget balanced if

$$\bar{C}(Q) \leq \sum_{i \in Q} p_i \leq \beta \cdot C(Q) \qquad (\beta \geq 1)$$

Define the social cost of a set $S \subseteq U$ as

$$\Pi(S) := \sum_{i \notin S} v_i + C(S) = \sum_{i \in U} v_i - \Big(\sum_{i \in S} v_i - C(S) \Big)$$

Approximate Efficiency: cost sharing mechanism M is α-approximate if, assuming truthful bidding,

$$\sum_{i \notin Q} v_i + \bar{C}(Q) \leq \alpha \cdot \min_{S \subseteq U} \Pi(S) \qquad (\alpha \geq 1)$$

[Roughgarden and Sundararajan, JACM '09]

Approximate Budget Balance and Efficiency

Approximate Budget Balance: cost sharing mechanism M is β-budget balanced if

$$\bar{C}(Q) \leq \sum_{i \in Q} p_i \leq \beta \cdot C(Q) \qquad (\beta \geq 1)$$

Define the social cost of a set $S \subseteq U$ as

$$\Pi(S) := \sum_{i \notin S} v_i + C(S) = \sum_{i \in U} v_i - \Big(\sum_{i \in S} v_i - C(S) \Big)$$

Approximate Efficiency: cost sharing mechanism M is α-approximate if, assuming truthful bidding,

$$\sum_{i \notin Q} v_i + \bar{C}(Q) \leq \alpha \cdot \min_{S \subseteq U} \Pi(S) \qquad (\alpha \geq 1)$$

[Roughgarden and Sundararajan, JACM '09]

Approximate Budget Balance and Efficiency

Approximate Budget Balance: cost sharing mechanism M is β-budget balanced if

$$\bar{C}(Q) \leq \sum_{i \in Q} p_i \leq \beta \cdot C(Q) \qquad (\beta \geq 1)$$

Define the social cost of a set $S \subseteq U$ as

$$\Pi(S) := \sum_{i \notin S} v_i + C(S) = \sum_{i \in U} v_i - \Big(\sum_{i \in S} v_i - C(S) \Big)$$

Approximate Efficiency: cost sharing mechanism M is α-approximate if, assuming truthful bidding,

$$\sum_{i \notin Q} v_i + \bar{C}(Q) \leq \alpha \cdot \min_{S \subseteq U} \Pi(S) \qquad (\alpha \geq 1)$$

[Roughgarden and Sundararajan, JACM '09]

Objectives at a Glance

1. **Computational Efficiency**

2. **Approximate Budget Balance:**

$$\bar{C}(Q) \leq \sum_{i \in Q} p_i \leq \beta \cdot C(Q) \qquad (\beta \geq 1)$$

3. **Group-Strategyproofness**

4. **Approximate Efficiency:**

$$\sum_{i \notin Q} v_i + \bar{C}(Q) \leq \alpha \cdot \min_{S \subseteq U} \left\{ \sum_{i \notin S} v_i + C(S) \right\} \qquad (\alpha \geq 1)$$

How to achieve

β-budget balance?

$$\left(\bar{C}(Q) \leq \sum_{i \in Q} p_i \leq \beta \cdot C(Q)\right)$$

How to achieve

group-strategyproofness?

(Not everybody in the coalition is better off by misreporting his valuation.)

Moulin's Framework

Cost Sharing Function: $\xi : U \times 2^U \to \mathbb{R}^+$

$\xi_i(S) =$ cost share of player i with respect to set $S \subseteq U$

β-Budget Balance:

$$\bar{C}(S) \leq \sum_{i \in S} \xi_i(S) \leq \beta \cdot C(S) \quad \forall S \subseteq U$$

Cross-Monotonicity: cost share of player i does not decrease if other players leave the game:

$$\forall S \subseteq T, \forall i \in S : \quad \xi_i(S) \geq \xi_i(T)$$

Moulin's Framework

Cost Sharing Function: $\xi : U \times 2^U \to \mathbb{R}^+$
$\xi_i(S) =$ cost share of player i with respect to set $S \subseteq U$

β-Budget Balance:

$$\bar{C}(S) \leq \sum_{i \in S} \xi_i(S) \leq \beta \cdot C(S) \quad \forall S \subseteq U$$

Cross-Monotonicity: cost share of player i does not decrease if other players leave the game:

$$\forall S \subseteq T, \forall i \in S : \quad \xi_i(S) \geq \xi_i(T)$$

Moulin's Framework

Moulin Mechanism $M(\xi)$:

1: Initialize: $Q \leftarrow U$
2: If for each player $i \in Q$: $\xi_i(Q) \leq b_i$ then STOP
3: Otherwise, remove from Q all players with $\xi_i(Q) > b_i$ and repeat

Theorem

If ξ is cross-monotonic and β-budget balanced, then the Moulin mechanism $M(\xi)$ is group-strategyproof and β-budget balanced.

[Moulin, SCW '99]

How to achieve

α-approximability?

$$\left(\sum_{i \notin Q} v_i + \bar{C}(Q) \leq \alpha \cdot \min_{S \subseteq U} \sum_{i \notin S} v_i + C(S) \right)$$

Summability

Suppose we are given an arbitrary order σ on the players in U.

Order each subset $S \subseteq U$ according to σ:

$$S := \{i_1, \ldots, i_{|S|}\} \text{ with } i_j \prec_\sigma i_k \text{ for all } 1 \leq j < k \leq |S|.$$

Let S_j refer to the first j players of S.

A cost sharing function ξ is α-summable if for every order σ of the players in U

$$\forall S \subseteq U: \quad \sum_{j=1}^{|S|} \xi_{i_j}(S_j) \leq \alpha \cdot C(S)$$

Summability

Suppose we are given an arbitrary order σ on the players in U. Order each subset $S \subseteq U$ according to σ:

$$S := \{i_1, \dots, i_{|S|}\} \text{ with } i_j \prec_\sigma i_k \text{ for all } 1 \leq j < k \leq |S|.$$

Let S_j refer to the first j players of S.

A cost sharing function ξ is α-summable if for every order σ of the players in U

$$\forall S \subseteq U : \quad \sum_{j=1}^{|S|} \xi_{i_j}(S_j) \leq \alpha \cdot C(S)$$

Summability

Suppose we are given an arbitrary order σ on the players in U. Order each subset $S \subseteq U$ according to σ:

$$S := \{i_1, \dots, i_{|S|}\} \text{ with } i_j \prec_\sigma i_k \text{ for all } 1 \le j < k \le |S|.$$

Let S_j refer to the first j players of S.

A cost sharing function ξ is α-summable if for every order σ of the players in U

$$\forall S \subseteq U : \quad \sum_{j=1}^{|S|} \xi_{i_j}(S_j) \le \alpha \cdot C(S)$$

Summability

Suppose we are given an arbitrary order σ on the players in U. Order each subset $S \subseteq U$ according to σ:

$$S := \{i_1, \dots, i_{|S|}\} \text{ with } i_j \prec_\sigma i_k \text{ for all } 1 \le j < k \le |S|.$$

Let S_j refer to the first j players of S.

A cost sharing function ξ is α-summable if for every order σ of the players in U

$$\forall S \subseteq U : \quad \sum_{j=1}^{|S|} \xi_{i_j}(S_j) \le \alpha \cdot C(S)$$

Summability and Approximability

Theorem

Let ξ be a cross-monotonic cost sharing function and let α, β be the smallest numbers such that ξ is α-summable and β-budget balanced. Then the Moulin mechanism $M(\xi)$ is $(\alpha + \beta)$-approximate and no better than $\max\{\alpha, \beta\}$-approximate.

[Roughgarden, Sundararajan, JACM '09]

Moulin Mechanisms: Known Results I

Upper bounds		β
[Moulin, Shenker, ET '01]	submodular cost	1
[Jain, Vazirani, STOC '01]	minimum spanning tree	1
	Steiner tree and TSP	2
[Pál, Tardos, FOCS '03]	facility location	3
	single-sink rent-or-buy	15
[Leonardi, Schäfer, EC '03], [Gupta et al., APPROX '04]	single-sink rent-or-buy	4
[Leonardi, Schäfer, EC '03]	connected facility location	30
[Könemann, Leonardi, Schäfer, SODA '05]	Steiner forest	2
[Gupta et al., SODA '07]	price-collecting Steiner forest	3
[Bleischwitz, Monien, CIAC '07]	makespan scheduling	2

Lower bounds		β
[Immorlica et al., SODA '05]	set cover, vertex cover	n^c
	facility location	3
[Könemann et al., SODA '05]	Steiner tree	2
[Bleischwitz, Monien, CIAC '07]	makespan scheduling	2
[Brenner, Schäfer, STACS '07]	completion time scheduling, etc.	n/c

Moulin Mechanisms: Known Results I

Upper bounds		β
[Moulin, Shenker, ET '01]	submodular cost	1
[Jain, Vazirani, STOC '01]	minimum spanning tree	1
	Steiner tree and TSP	2
[Pál, Tardos, FOCS '03]	facility location	3
	single-sink rent-or-buy	15
[Leonardi, Schäfer, EC '03], [Gupta et al., APPROX '04]	single-sink rent-or-buy	4
[Leonardi, Schäfer, EC '03]	connected facility location	30
[Könemann, Leonardi, Schäfer, SODA '05]	Steiner forest	2
[Gupta et al., SODA '07]	price-collecting Steiner forest	3
[Bleischwitz, Monien, CIAC '07]	makespan scheduling	2

Lower bounds		β
[Immorlica et al., SODA '05]	set cover, vertex cover	n^c
	facility location	3
[Könemann et al., SODA '05]	Steiner tree	2
[Bleischwitz, Monien, CIAC '07]	makespan scheduling	2
[Brenner, Schäfer, STACS '07]	completion time scheduling, etc.	n/c

Moulin Mechanisms: Known Results II

		β	α
[Roughgarden, Sundararajan, STOC '06]	submodular cost	1	$\Theta(\log n)$
	Steiner tree	2	$\Theta(\log^2 n)$
[Chawla et al., WINE '06]	Steiner forest	2	$\Theta(\log^2 n)$
[Roughgarden, Sundararajan, IPCO '07]	facility location	3	$\Theta(\log n)$
	SROB	4	$\Theta(\log^2 n)$
[Gupta et al., SODA '07]	price-collecting SF	3	$\Theta(\log^2 n)$
[Brenner, Schäfer, STACS '07]	makespan scheduling	2	$\Theta(\log n)$
	cost-stable problems		$\Omega(\log n)$

Cross-Monotonic Cost Shares for Steiner Forest

Steiner Forest Game

Goal: design a cost sharing mechanism for the Steiner forest game

- graph $G = (V, E)$ with edge costs $c : E \to \mathbb{R}^+$
- player i requests connection between terminals $s_i, t_i \in V$
  identify players with terminal pairs: $U = \{(s_1, t_1), \dots, (s_n, t_n)\}$
- $C(S)$ = cost of a minimum cost Steiner forest connecting all terminal pairs in $S \subseteq U$

Theorem

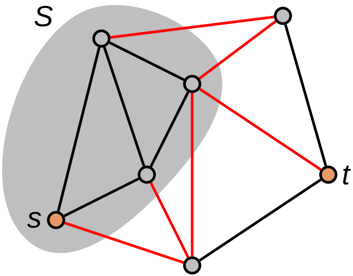
There is a cross-monotonic and 2-budget balanced cost sharing function for the Steiner forest game.

[Könemann, Leonardi, Schäfer, van Zwam, SICOMP '08]

Primal-Dual Steiner Forest Algorithm

Fix a set $Q \subseteq U$ of terminal pairs. We sketch the primal-dual algorithm AKR(Q) of [Agrawal, Klein, Ravi, SICOMP '95] for the Steiner forest problem with terminal pair set Q.

A subset $S \subseteq V$ of nodes is a Steiner cut if it separates at least one terminal pair $(s, t) \in Q$. Let $\mathcal{S}$ be the set of all such cuts.



Observation: for every Steiner cut $S \in \mathcal{S}$, any feasible Steiner forest must contain at least one of the red edges

$$\delta(S) = \{uv \in E \ : \ u \in S, \, v \notin S\}$$

Undirected Cut Formulation

Integer Program:

$$\min \sum_{e \in E} c_e \cdot x_e$$

$$\text{s.t.} \quad \sum_{e \in \delta(S)} x_e \geq 1 \quad \forall S \in \mathcal{S}$$

$$x_e \in \{0, 1\} \quad \forall e \in E$$

Undirected Cut Formulation

Primal LP:

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e \cdot x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(S)} x_e \geq 1 \quad \forall S \in \mathcal{S} \\ & x_e \geq 0 \quad \forall e \in E \end{aligned}$$

Undirected Cut Formulation

Primal LP:

$$\min \quad \sum_{e \in E} c_e \cdot x_e$$

$$\text{s.t.} \quad \sum_{e \in \delta(S)} x_e \geq 1 \quad \forall S \in \mathcal{S}$$

$$x_e \geq 0 \quad \forall e \in E$$

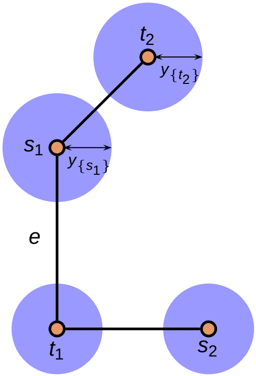
Dual LP:

$$\max \quad \sum_{S \in \mathcal{S}} y_S$$

$$\text{s.t.} \quad \sum_{S : e \in \delta(S)} y_S \leq c_e \quad \forall e \in E$$

$$y_S \geq 0 \quad \forall S \in \mathcal{S}$$

Visualizing the Dual

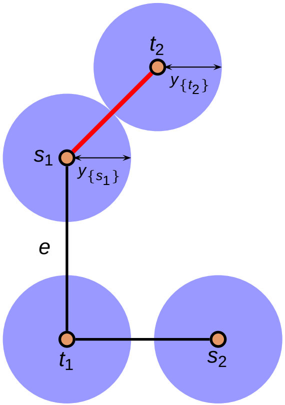


The dual y_S of Steiner cut S is visualized as moat around S of radius y_S

An edge e is said to be tight if its corresponding dual constraint is tight:

$$\sum_{S:e\in\delta(S)} y_S = c_e$$

Visualizing the Dual

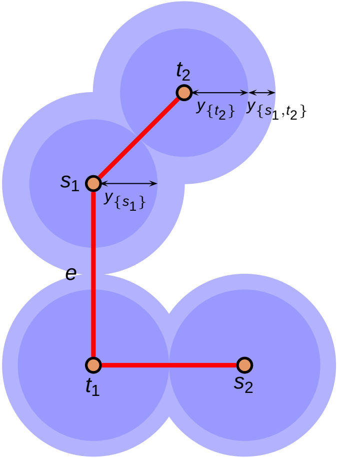


The dual y_S of Steiner cut S is visualized as moat around S of radius y_S

An edge e is said to be tight if its corresponding dual constraint is tight:

$$\sum_{S: e \in \delta(S)} y_S = c_e$$

Visualizing the Dual



The dual y_S of Steiner cut S is visualized as moat around S of radius y_S

An edge e is said to be tight if its corresponding dual constraint is tight:

$$\sum_{S:e\in\delta(S)} y_S = c_e$$

High-Level Description

Execution of AKR can be seen as a process over time τ:

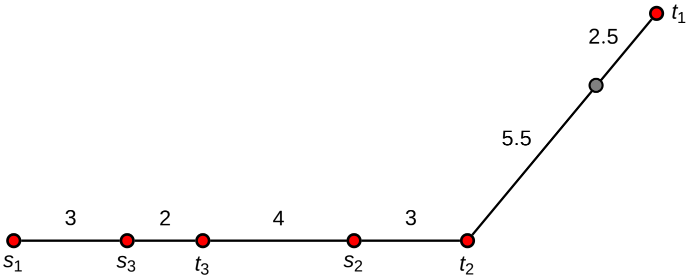
- $(F^\tau, y^\tau) =$ forest and dual packing
- terminal v is active if it is separated from its mate in F^τ
- $\bar{F}^\tau =$ subgraph induced by tight edges with respect to y^τ
- connected components of $\bar{F}^\tau$ are called moats
- moat is active if it contains an active terminal

Algorithm AKR:

1: $F^0 = \emptyset$, $y^0 = 0$
2: **repeat**
3: simultaneously increase duals of all active moats until some path P between two active terminals becomes tight
4: add tight segments of P to the current forest F^τ
5: **until** all terminals are inactive

Illustration: AKR

$\tau = 0.0$



$\tau = 0.3$

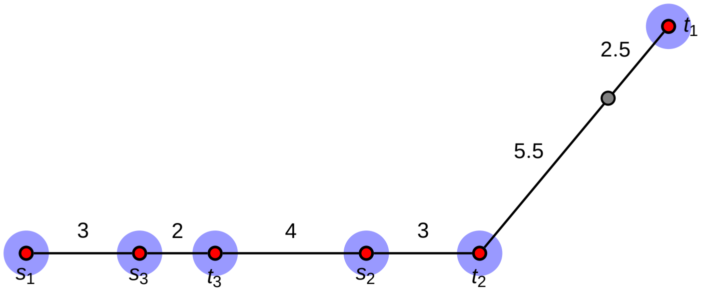


Illustration: AKR

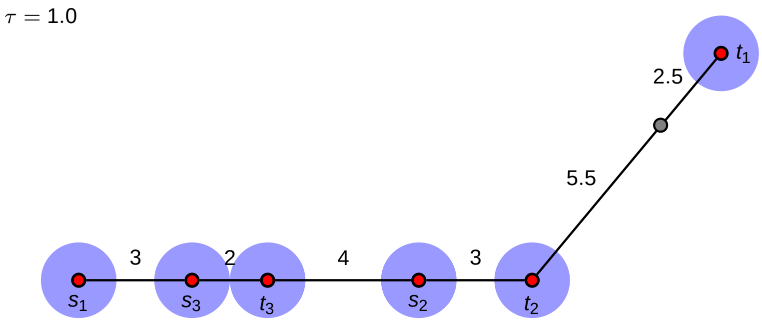


Illustration: AKR

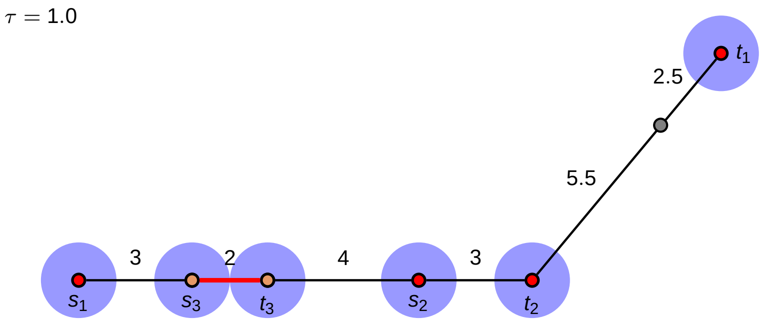


Illustration: AKR

$\tau = 1.5$

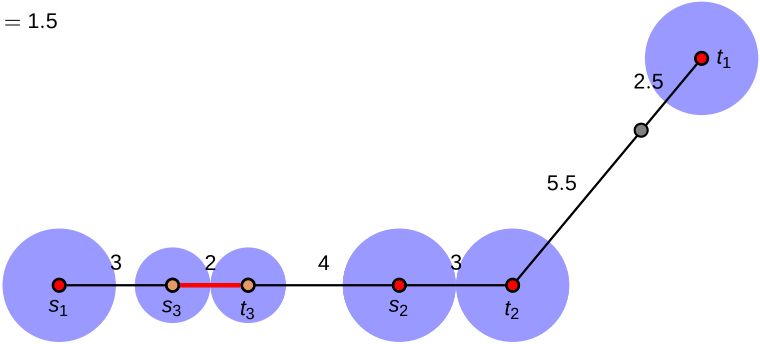


Illustration: AKR

$\tau = 1.5$

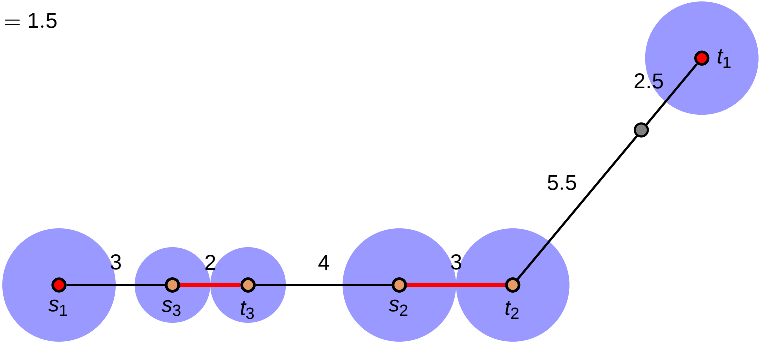


Illustration: AKR

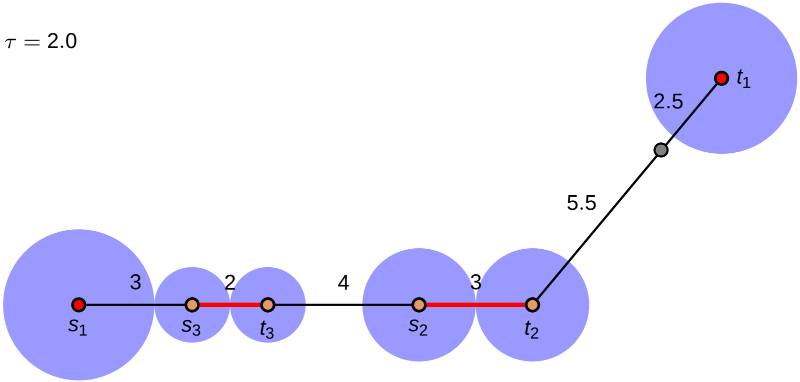


Illustration: AKR

$\tau = 2.5$

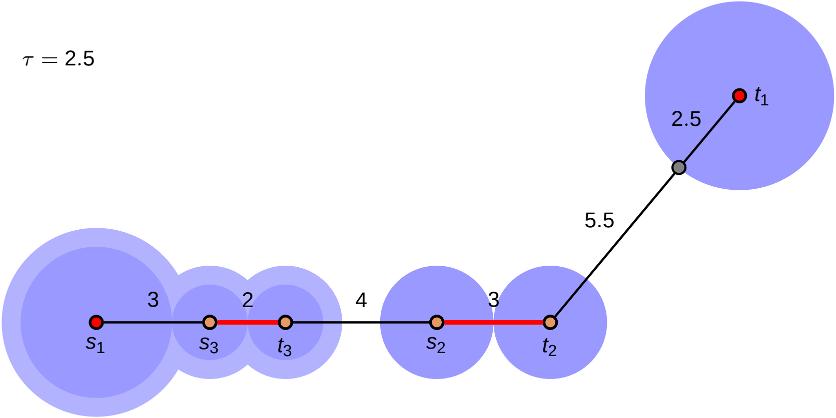


Illustration: AKR

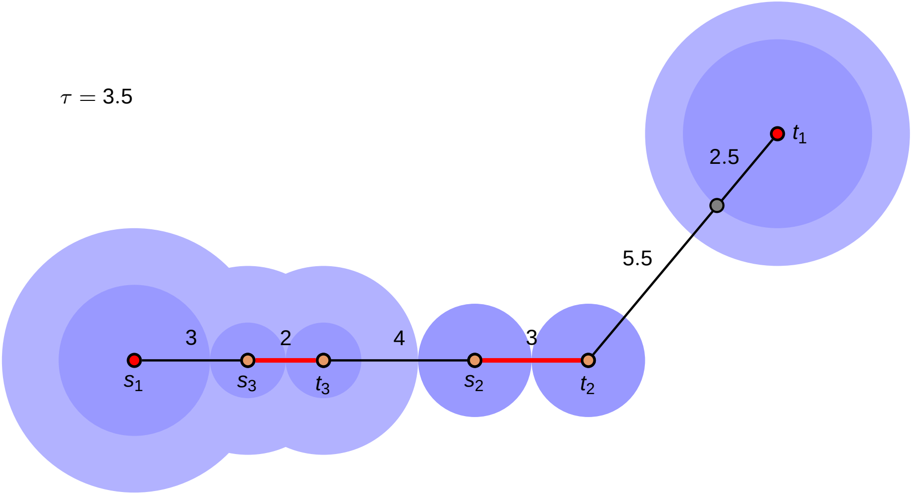


Illustration: AKR

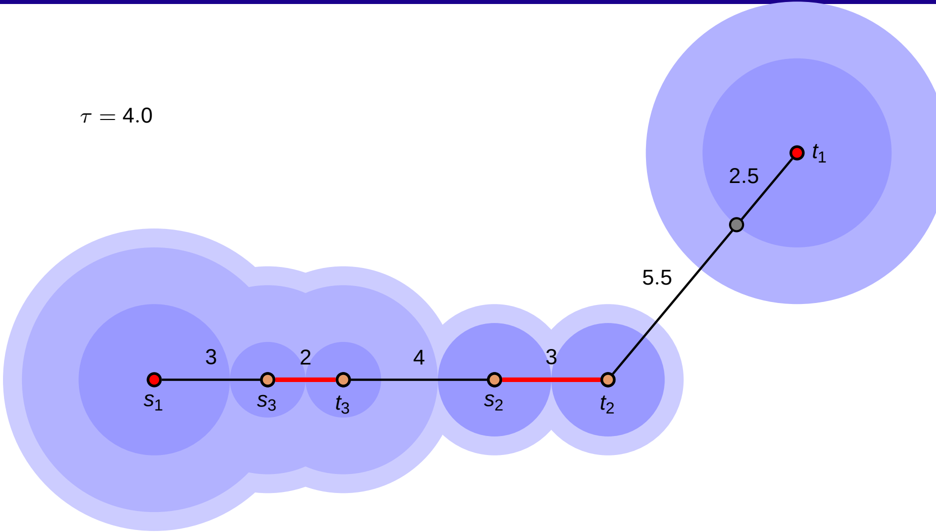


Illustration: AKR

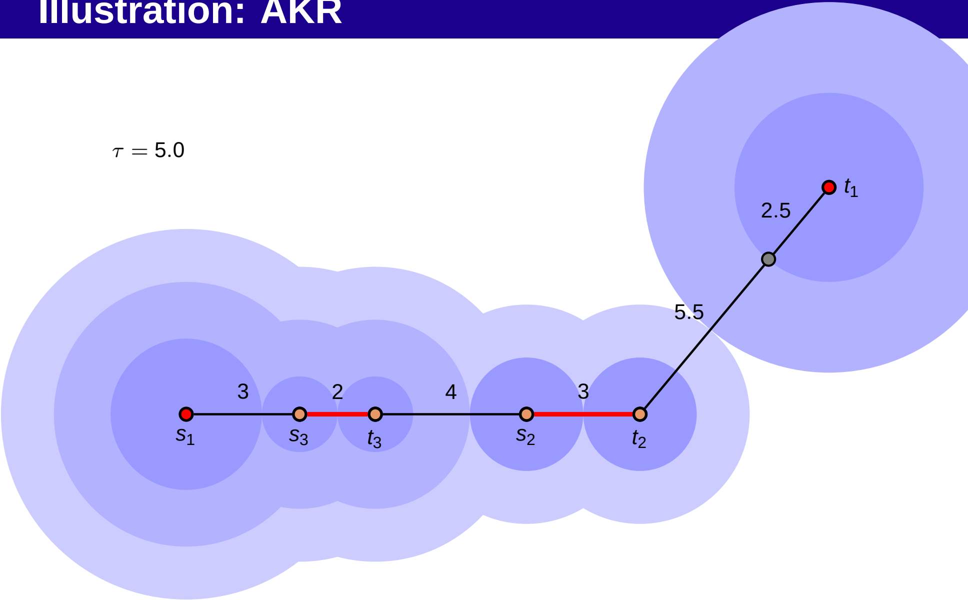
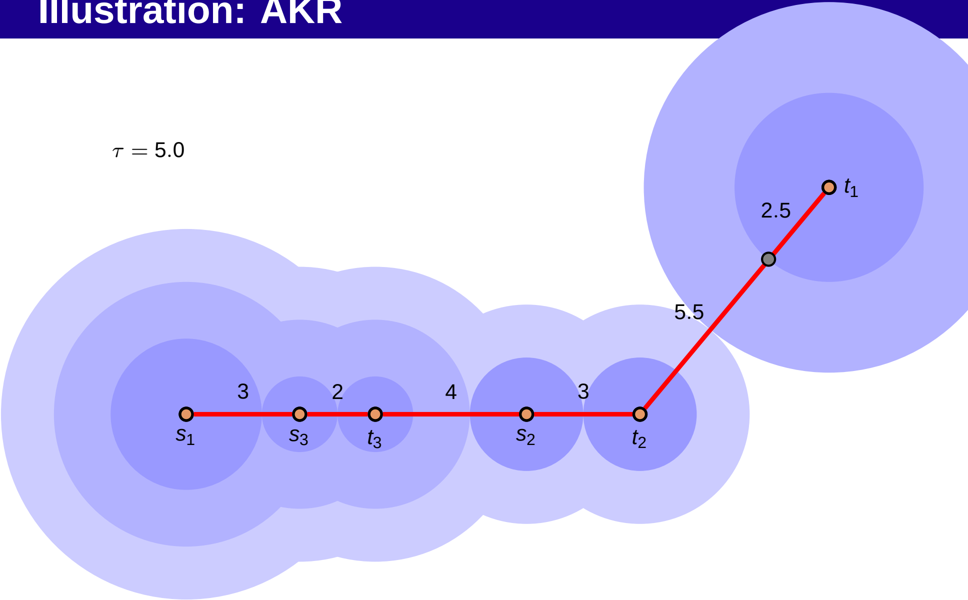


Illustration: AKR



Approximation Guarantee

Theorem

The algorithm $AKR(Q)$ computes a feasible forest F for terminal pair set Q and a feasible dual $(y_S)_{S \in \mathcal{S}}$ such that

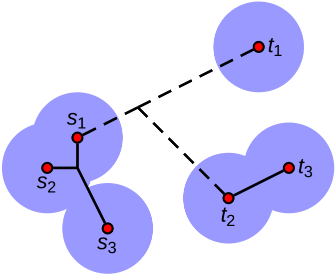
$$c(F) \leq \left(2 - \frac{1}{k}\right) \sum_{S \in \mathcal{S}} y_S \leq \left(2 - \frac{1}{k}\right) OPT(Q),$$

where k is the number of terminal pairs in Q.

[Agrawal, Klein, Ravi, SICOMP '95]

Idea: run AKR and distribute (twice) the total dual among the terminals

Sharing the Dual Growth



Example:

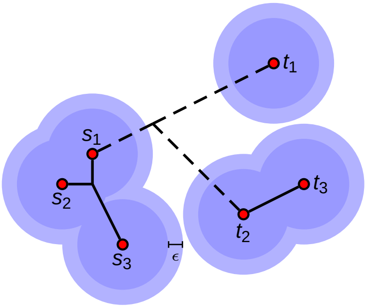
- all terminals are active
- grow active moats by ϵ
- growth of each moat is shared evenly among active terminals:

$s_1 : \epsilon/3$

$t_2 : \epsilon/2$

$t_1 : \epsilon$

Sharing the Dual Growth



Example:

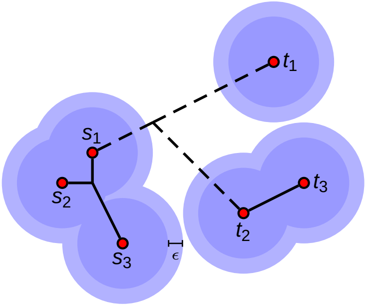
- all terminals are active
- grow active moats by ϵ
- growth of each moat is shared evenly among active terminals:

$s_1 : \epsilon/3$

$t_2 : \epsilon/2$

$t_1 : \epsilon$

Sharing the Dual Growth

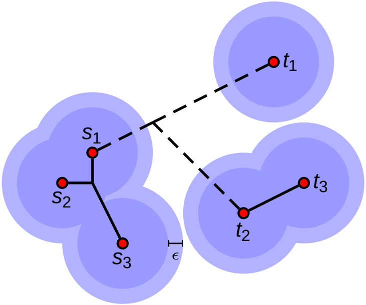


Example:

- all terminals are active
- grow active moats by ϵ
- growth of each moat is shared evenly among active terminals:

$$s_1 : \epsilon/3$$
$$t_2 : \epsilon/2$$
$$t_1 : \epsilon$$

Sharing the Dual Growth



a_v^τ = number of active terminals in the moat containing v at time τ

Suppose terminal $v \in Q$ becomes inactive at time T. Define the cost share of v as

$$\xi_v(Q) = \int_0^T \frac{1}{a_v^\tau} \, d\tau$$

For terminal pair $(s, t) \in Q$:

$$\xi_{st}(Q) = 2 \cdot (\xi_s(Q) + \xi_t(Q))$$

Sharing the Dual Growth

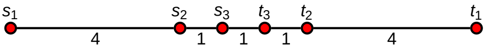
Problem: Activity time of terminal may depend on presence of other terminal pairs. Impossible to achieve cross-monotonicity.

Example: $Q = \{(s_1,t_1),(s_2,t_2),(s_3,t_3)\}$, $Q_0 = Q \setminus \{(s_3,t_3)\}$

AKR(Q)

ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
0.0	0.0	0.0	0.0	0.0	0.0

$\tau = 0.0$



Sharing the Dual Growth

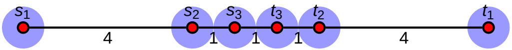
Problem: Activity time of terminal may depend on presence of other terminal pairs. Impossible to achieve cross-monotonicity.

Example: $Q = \{(s_1, t_1), (s_2, t_2), (s_3, t_3)\}$, $Q_0 = Q \setminus \{(s_3, t_3)\}$

AKR(Q)

ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
0.5	0.5	0.5	0.5	0.5	0.5

$\tau = 0.5$



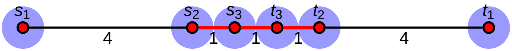
Sharing the Dual Growth

Problem: Activity time of terminal may depend on presence of other terminal pairs. Impossible to achieve cross-monotonicity.

Example: $Q = \{(s_1, t_1), (s_2, t_2), (s_3, t_3)\}$, $Q_0 = Q \setminus \{(s_3, t_3)\}$

AKR(Q)	ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
	0.5	0.5	0.5	0.5	0.5	0.5

$\tau = 0.5$



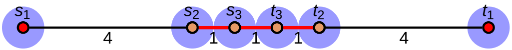
Sharing the Dual Growth

Problem: Activity time of terminal may depend on presence of other terminal pairs. Impossible to achieve cross-monotonicity.

Example: $Q = \{(s_1, t_1), (s_2, t_2), (s_3, t_3)\}$, $Q_0 = Q \setminus \{(s_3, t_3)\}$

AKR(Q)	ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
	0.5	0.5	0.5	0.5	0.5	0.5

$\tau = 0.5$



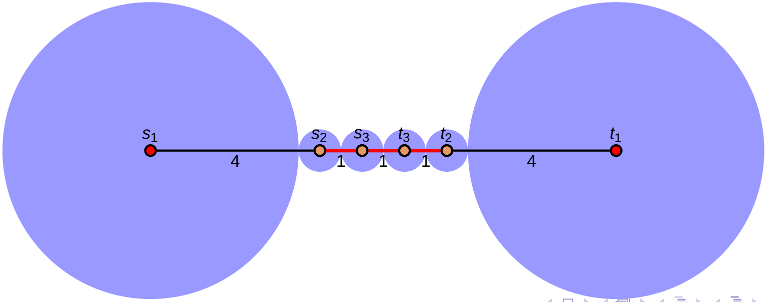
Sharing the Dual Growth

Problem: Activity time of terminal may depend on presence of other terminal pairs. Impossible to achieve cross-monotonicity.

Example: $Q = \{(s_1, t_1), (s_2, t_2), (s_3, t_3)\}$, $Q_0 = Q \setminus \{(s_3, t_3)\}$

AKR(Q)	ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
	3.5	0.5	0.5	0.5	0.5	3.5

$\tau = 3.5$



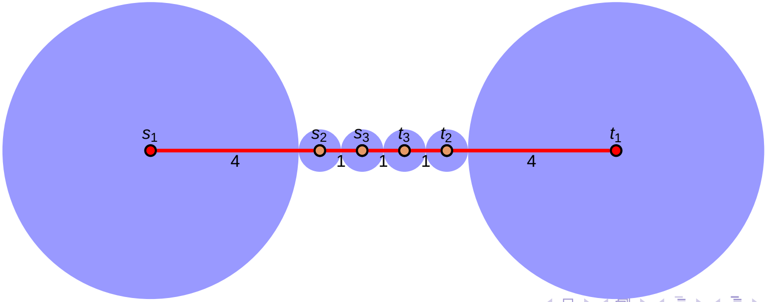
Sharing the Dual Growth

Problem: Activity time of terminal may depend on presence of other terminal pairs. Impossible to achieve cross-monotonicity.

Example: $Q = \{(s_1, t_1), (s_2, t_2), (s_3, t_3)\}$, $Q_0 = Q \setminus \{(s_3, t_3)\}$

AKR(Q)	ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
	3.5	0.5	0.5	0.5	0.5	3.5

$\tau = 3.5$



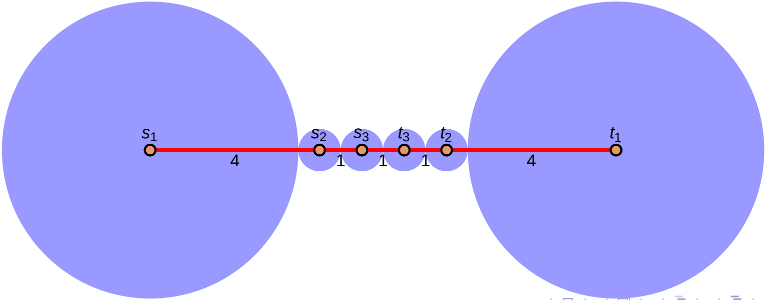
Sharing the Dual Growth

Problem: Activity time of terminal may depend on presence of other terminal pairs. Impossible to achieve cross-monotonicity.

Example: $Q = \{(s_1, t_1), (s_2, t_2), (s_3, t_3)\}$, $Q_0 = Q \setminus \{(s_3, t_3)\}$

AKR(Q)	ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
	3.5	0.5	0.5	0.5	0.5	3.5

$\tau = 3.5$



Sharing the Dual Growth

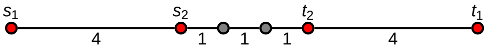
Problem: Activity time of terminal may depend on presence of other terminal pairs. Impossible to achieve cross-monotonicity.

Example: $Q = \{(s_1, t_1), (s_2, t_2), (s_3, t_3)\}$, $Q_0 = Q \setminus \{(s_3, t_3)\}$

	ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
AKR(Q)	3.5	0.5	0.5	0.5	0.5	3.5

	ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
AKR(Q_0)	0.0	0.0	–	–	0.0	0.0

$\tau = 0.0$



Sharing the Dual Growth

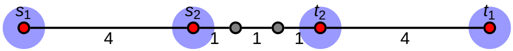
Problem: Activity time of terminal may depend on presence of other terminal pairs. Impossible to achieve cross-monotonicity.

Example: $Q = \{(s_1, t_1), (s_2, t_2), (s_3, t_3)\}$, $Q_0 = Q \setminus \{(s_3, t_3)\}$

	ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
AKR(Q)	3.5	0.5	0.5	0.5	0.5	3.5

	ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
AKR(Q_0)	0.5	0.5	–	–	0.5	0.5

$\tau = 0.5$



Sharing the Dual Growth

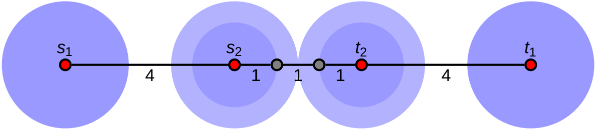
Problem: Activity time of terminal may depend on presence of other terminal pairs. Impossible to achieve cross-monotonicity.

Example: $Q = \{(s_1, t_1), (s_2, t_2), (s_3, t_3)\}$, $Q_0 = Q \setminus \{(s_3, t_3)\}$

	ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
AKR(Q)	3.5	0.5	0.5	0.5	0.5	3.5

	ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
AKR(Q_0)	1.5	1.5	–	–	1.5	1.5

$\tau = 1.5$



Sharing the Dual Growth

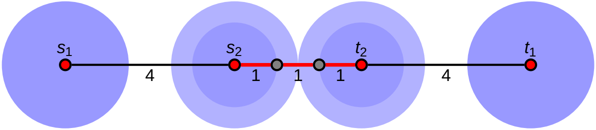
Problem: Activity time of terminal may depend on presence of other terminal pairs. Impossible to achieve cross-monotonicity.

Example: $Q = \{(s_1, t_1), (s_2, t_2), (s_3, t_3)\}$, $Q_0 = Q \setminus \{(s_3, t_3)\}$

AKR(Q)	ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
	3.5	0.5	0.5	0.5	0.5	3.5

AKR(Q_0)	ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
	1.5	1.5	–	–	1.5	1.5

$\tau = 1.5$



Sharing the Dual Growth

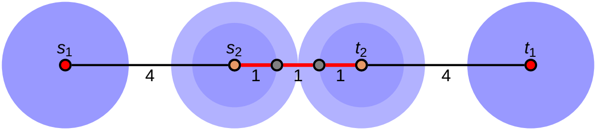
Problem: Activity time of terminal may depend on presence of other terminal pairs. Impossible to achieve cross-monotonicity.

Example: $Q = \{(s_1, t_1), (s_2, t_2), (s_3, t_3)\}$, $Q_0 = Q \setminus \{(s_3, t_3)\}$

	ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
AKR(Q)	3.5	0.5	0.5	0.5	0.5	3.5

	ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
AKR(Q_0)	1.5	1.5	–	–	1.5	1.5

$\tau = 1.5$



Sharing the Dual Growth

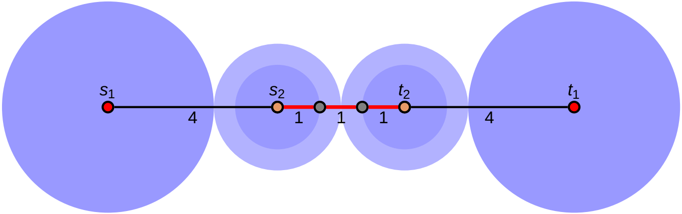
Problem: Activity time of terminal may depend on presence of other terminal pairs. Impossible to achieve cross-monotonicity.

Example: $Q = \{(s_1, t_1), (s_2, t_2), (s_3, t_3)\}$, $Q_0 = Q \setminus \{(s_3, t_3)\}$

	ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
AKR(Q)	3.5	0.5	0.5	0.5	0.5	3.5

	ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
AKR(Q_0)	2.5	1.5	–	–	1.5	2.5

$\tau = 2.5$



Sharing the Dual Growth

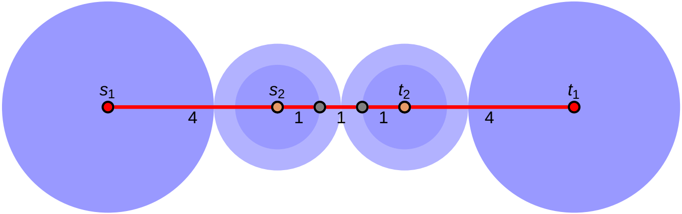
Problem: Activity time of terminal may depend on presence of other terminal pairs. Impossible to achieve cross-monotonicity.

Example: $Q = \{(s_1, t_1), (s_2, t_2), (s_3, t_3)\}$, $Q_0 = Q \setminus \{(s_3, t_3)\}$

AKR(Q)	ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
	3.5	0.5	0.5	0.5	0.5	3.5

AKR(Q_0)	ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
	2.5	1.5	–	–	1.5	2.5

$\tau = 2.5$



Sharing the Dual Growth

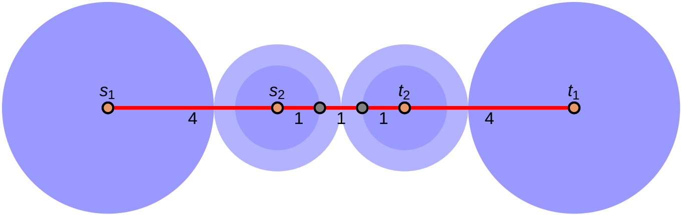
Problem: Activity time of terminal may depend on presence of other terminal pairs. Impossible to achieve cross-monotonicity.

Example: $Q = \{(s_1,t_1),(s_2,t_2),(s_3,t_3)\}$, $Q_0 = Q \setminus \{(s_3,t_3)\}$

	ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
AKR(Q)	3.5	0.5	0.5	0.5	0.5	3.5

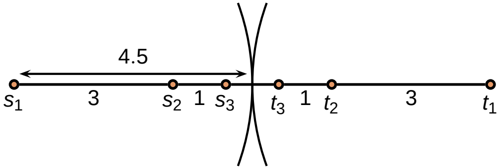
	ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
AKR(Q_0)	2.5	1.5	–	–	1.5	2.5

$\tau = 2.5$



Independent Activity Time

Question: How long would a terminal pair need to connect if all other terminal pairs were absent?



Death time: for each terminal pair $(s, t) \in U$ define

$$d(s) = d(t) = d(s, t) := \frac{1}{2}c(s, t),$$

where $c(s, t)$ is cost of minimum-cost s, t-path.

Cross-Monotonic Primal-Dual Algorithm

New Activity Notion: terminals s, t are active until time $\mathrm{d}(s, t)$

Primal-Dual Algorithm KLS: as before, but with modified activity notion

Cost Shares: define cost share of terminal $v \in Q$ as:

$$\xi_v(Q) = \int_0^{\mathrm{d}(v)} \frac{1}{a_v^\tau} \, d\tau$$

Theorem

The cost shares ξ computed by KLS are cross-monotonic and 2-budget balanced.

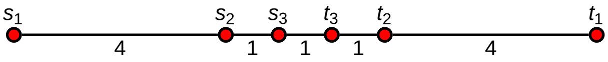
[Könemann, Leonardi, Schäfer, van Zwam, SICOMP '08]

Example

KLS(Q)

ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
0.0	0.0	0.0	0.0	0.0	0.0

$\tau = 0.0$

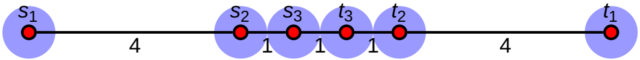


Example

KLS(Q)

ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
0.5	0.5	0.5	0.5	0.5	0.5

$\tau = 0.5$

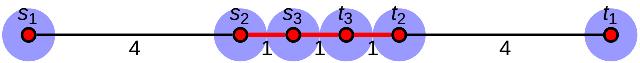


Example

KLS(Q)

ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
0.5	0.5	0.5	0.5	0.5	0.5

$\tau = 0.5$

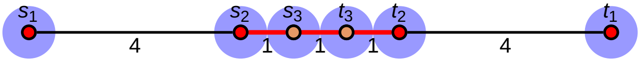


Example

KLS(Q)

ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
0.5	0.5	0.5	0.5	0.5	0.5

$\tau = 0.5$

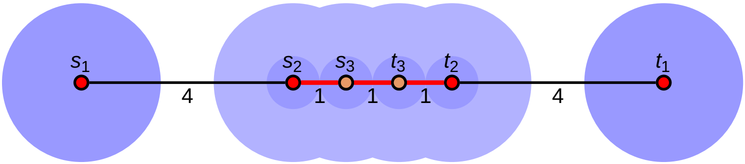


Example

KLS(Q)

ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
1.5	1.0	0.5	0.5	1.0	1.5

$\tau = 1.5$

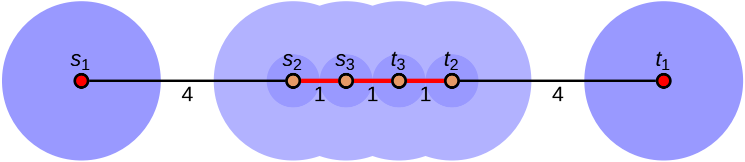


Example

KLS(Q)

ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
1.5	1.0	0.5	0.5	1.0	1.5

$\tau = 1.5$

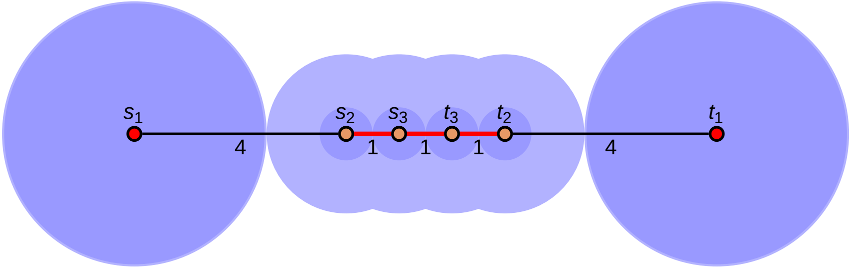


Example

KLS(Q)

ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
2.5	1.0	0.5	0.5	1.0	2.5

$\tau = 2.5$

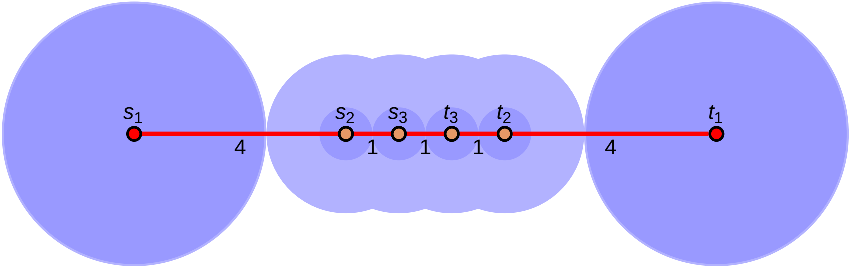


Example

KLS(Q)

ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
2.5	1.0	0.5	0.5	1.0	2.5

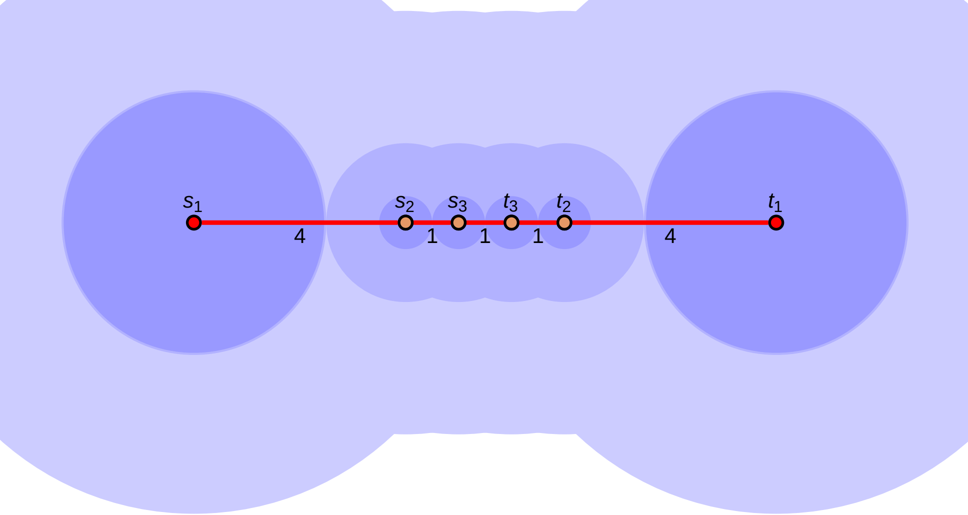
$\tau = 2.5$



Example

KLS(Q)	ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
	4.0	1.0	0.5	0.5	1.0	4.0

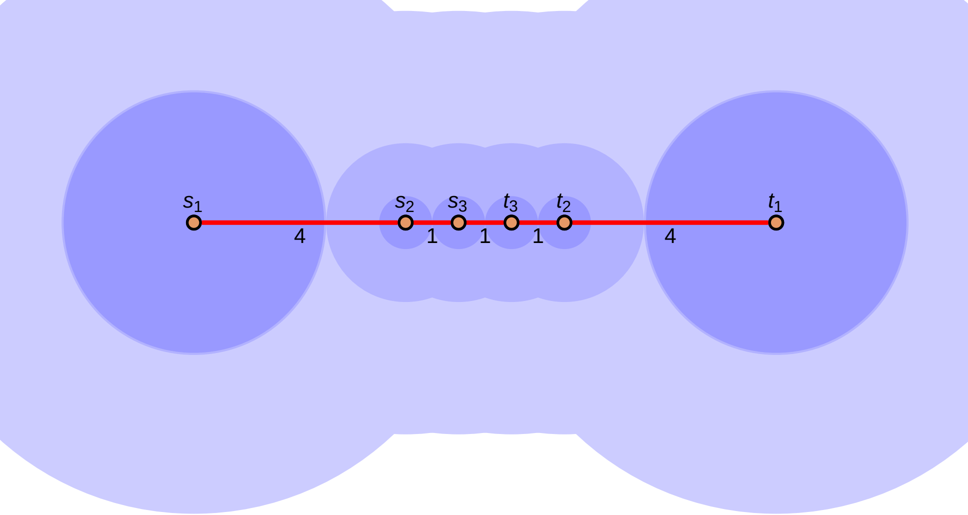
$\tau = 5.5$



Example

KLS(Q)	ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
	4.0	1.0	0.5	0.5	1.0	4.0

$\tau = 5.5$



Example

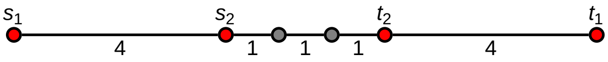
KLS(Q)

ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
4.0	1.0	0.5	0.5	1.0	4.0

KLS(Q_0)

ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
0.0	0.0	–	–	0.0	0.0

$\tau = 0.0$



Example

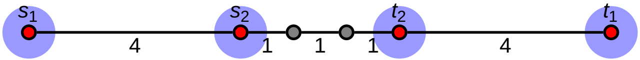
KLS(Q)

ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
4.0	1.0	0.5	0.5	1.0	4.0

KLS(Q_0)

ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
0.5	0.5	–	–	0.5	0.5

$\tau = 0.5$



Example

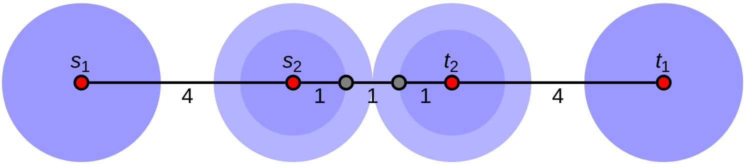
KLS(Q)

ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
4.0	1.0	0.5	0.5	1.0	4.0

KLS(Q_0)

ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
1.5	1.5	–	–	1.5	1.5

$\tau = 1.5$

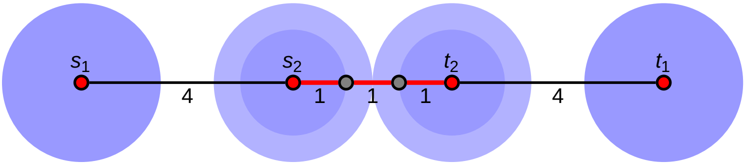


Example

KLS(Q)	ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
	4.0	1.0	0.5	0.5	1.0	4.0

$\tau = 1.5$

KLS(Q_0)	ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
	1.5	1.5	–	–	1.5	1.5



Example

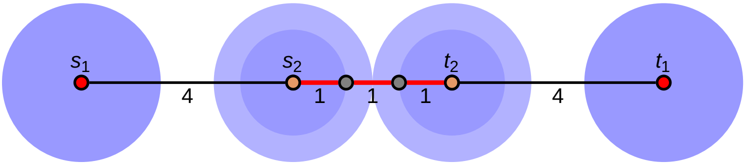
KLS(Q)

ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
4.0	1.0	0.5	0.5	1.0	4.0

KLS(Q_0)

ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
1.5	1.5	–	–	1.5	1.5

$\tau = 1.5$



Example

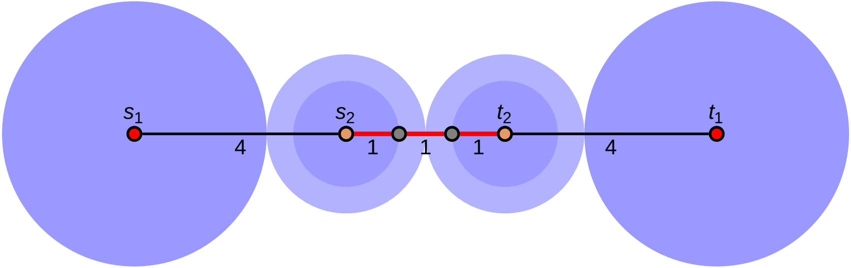
KLS(Q)

ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
4.0	1.0	0.5	0.5	1.0	4.0

KLS(Q_0)

ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
2.5	1.5	–	–	1.5	2.5

$\tau = 2.5$



Example

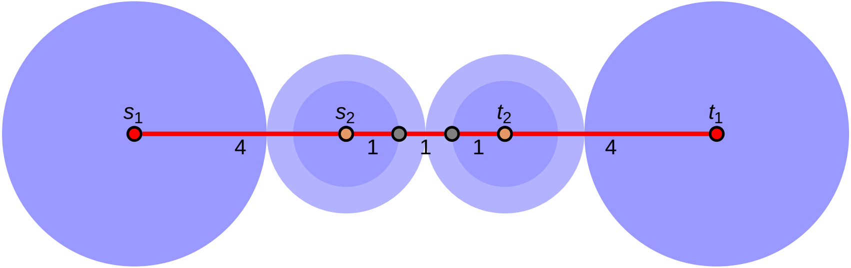
KLS(Q)

ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
4.0	1.0	0.5	0.5	1.0	4.0

$\tau = 2.5$

KLS(Q_0)

ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
2.5	1.5	–	–	1.5	2.5



Example

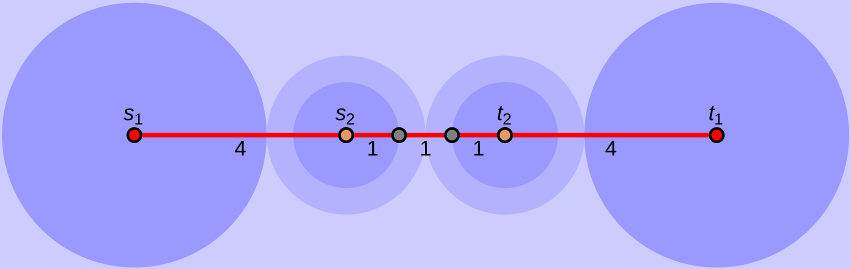
KLS(Q)

ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
4.0	1.0	0.5	0.5	1.0	4.0

KLS(Q_0)

ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
4.0	1.5	–	–	1.5	4.0

$\tau = 5.5$



Example

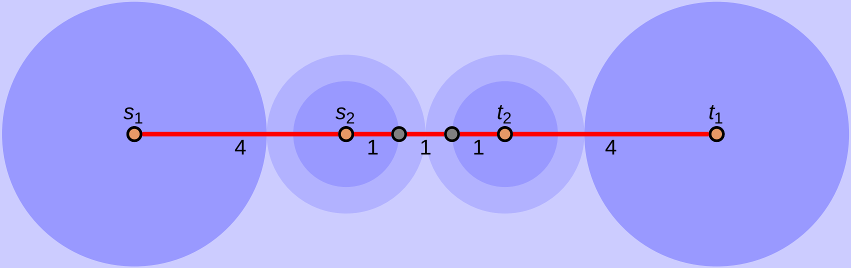
KLS(Q)

ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
4.0	1.0	0.5	0.5	1.0	4.0

$\tau = 5.5$

KLS(Q_0)

ξ_{s_1}	ξ_{s_2}	ξ_{s_3}	ξ_{t_3}	ξ_{t_2}	ξ_{t_1}
4.0	1.5	–	–	1.5	4.0



Proving Cross-Monotonicity

Lemma

The cost shares ξ computed by KLS are cross-monotonic.

Proof (sketch):
$\mathcal{M}^\tau(v)$ = moat of v at time τ in KLS(Q), $Q \subseteq U$
$\mathcal{M}_0^\tau(v)$ = moat of v at time τ in KLS(Q_0), $Q_0 = Q \setminus \{(s,t)\}$

Obs.: death-times of terminals are instance independent!

$$\begin{aligned} \mathcal{M}_0^\tau(v) \text{ active } &\Rightarrow \mathcal{M}^\tau(v) \text{ active} \\ \Rightarrow &\quad \mathcal{M}_0^\tau(v) \subseteq \mathcal{M}^\tau(v) \\ \Rightarrow &\quad a_0^\tau(v) \leq a^\tau(v) \end{aligned}$$

$$\xi_v(Q) = \int_0^{\mathrm{d}(v)} \frac{1}{a^\tau(v)}\, d\tau \leq \int_0^{\mathrm{d}(v)} \frac{1}{a_0^\tau(v)}\, d\tau = \xi_v(Q_0)$$

□

Proving Cross-Monotonicity

Lemma

The cost shares ξ computed by KLS are cross-monotonic.

Proof (sketch):

$\mathcal{M}^\tau(v)$ = moat of v at time τ in KLS(Q), $Q \subseteq U$

$\mathcal{M}_0^\tau(v)$ = moat of v at time τ in KLS(Q_0), $Q_0 = Q \setminus \{(s,t)\}$

Obs.: death-times of terminals are instance independent!

$$\mathcal{M}_0^\tau(v) \text{ active} \;\Rightarrow\; \mathcal{M}^\tau(v) \text{ active}$$

$$\Rightarrow \quad \mathcal{M}_0^\tau(v) \subseteq \mathcal{M}^\tau(v)$$

$$\Rightarrow \quad a_0^\tau(v) \le a^\tau(v)$$

$$\xi_v(Q) = \int_0^{d(v)} \frac{1}{a^\tau(v)}\, d\tau \le \int_0^{d(v)} \frac{1}{a_0^\tau(v)}\, d\tau = \xi_v(Q_0)$$

□

Proving Cross-Monotonicity

Lemma

The cost shares ξ computed by KLS are cross-monotonic.

Proof (sketch):

$\mathcal{M}^\tau(v) =$ moat of v at time τ in KLS(Q), $Q \subseteq U$

$\mathcal{M}_0^\tau(v) =$ moat of v at time τ in KLS(Q_0), $Q_0 = Q \setminus \{(s,t)\}$

Obs.: death-times of terminals are instance independent!

$$\mathcal{M}_0^\tau(v) \text{ active} \Rightarrow \mathcal{M}^\tau(v) \text{ active}$$

$$\Rightarrow \quad \mathcal{M}_0^\tau(v) \subseteq \mathcal{M}^\tau(v)$$

$$\Rightarrow \quad a_0^\tau(v) \leq a^\tau(v)$$

$$\xi_v(Q) = \int_0^{\mathrm{d}(v)} \frac{1}{a^\tau(v)}\, d\tau \leq \int_0^{\mathrm{d}(v)} \frac{1}{a_0^\tau(v)}\, d\tau = \xi_v(Q_0)$$

□

Proving Cross-Monotonicity

Lemma

The cost shares ξ computed by KLS are cross-monotonic.

Proof (sketch):

$\mathcal{M}^\tau(v)$ = moat of v at time τ in KLS(Q), $Q \subseteq U$

$\mathcal{M}_0^\tau(v)$ = moat of v at time τ in KLS(Q_0), $Q_0 = Q \setminus \{(s,t)\}$

Obs.: death-times of terminals are instance independent!

$$\mathcal{M}_0^\tau(v) \text{ active } \Rightarrow \mathcal{M}^\tau(v) \text{ active}$$

$$\Rightarrow \quad \mathcal{M}_0^\tau(v) \subseteq \mathcal{M}^\tau(v)$$

$$\Rightarrow \quad a_0^\tau(v) \le a^\tau(v)$$

$$\xi_v(Q) = \int_0^{\mathrm{d}(v)} \frac{1}{a^\tau(v)}\, d\tau \le \int_0^{\mathrm{d}(v)} \frac{1}{a_0^\tau(v)}\, d\tau = \xi_v(Q_0)$$

□

Proving Cross-Monotonicity

Lemma

The cost shares ξ computed by KLS are cross-monotonic.

Proof (sketch):

$\mathcal{M}^\tau(v)$ = moat of v at time τ in KLS(Q), $Q \subseteq U$

$\mathcal{M}_0^\tau(v)$ = moat of v at time τ in KLS(Q_0), $Q_0 = Q \setminus \{(s,t)\}$

Obs.: death-times of terminals are instance independent!

$$\mathcal{M}_0^\tau(v) \text{ active} \Rightarrow \mathcal{M}^\tau(v) \text{ active}$$

$$\Rightarrow \quad \mathcal{M}_0^\tau(v) \subseteq \mathcal{M}^\tau(v)$$

$$\Rightarrow \quad a_0^\tau(v) \leq a^\tau(v)$$

$$\xi_v(Q) = \int_0^{d(v)} \frac{1}{a^\tau(v)}\, d\tau \leq \int_0^{d(v)} \frac{1}{a_0^\tau(v)}\, d\tau = \xi_v(Q_0)$$

□

Proving Cross-Monotonicity

Lemma

The cost shares ξ computed by KLS are cross-monotonic.

Proof (sketch):

$\mathcal{M}^\tau(v)$ = moat of v at time τ in KLS(Q), $Q \subseteq U$

$\mathcal{M}_0^\tau(v)$ = moat of v at time τ in KLS(Q_0), $Q_0 = Q \setminus \{(s,t)\}$

Obs.: death-times of terminals are instance independent!

$$\mathcal{M}_0^\tau(v) \text{ active} \;\Rightarrow\; \mathcal{M}^\tau(v) \text{ active}$$

$$\Rightarrow \quad \mathcal{M}_0^\tau(v) \subseteq \mathcal{M}^\tau(v)$$

$$\Rightarrow \quad a_0^\tau(v) \leq a^\tau(v)$$

$$\xi_v(Q) = \int_0^{d(v)} \frac{1}{a^\tau(v)}\, d\tau \leq \int_0^{d(v)} \frac{1}{a_0^\tau(v)}\, d\tau = \xi_v(Q_0)$$

□

Proving Cross-Monotonicity

Lemma

The cost shares ξ computed by KLS are cross-monotonic.

Proof (sketch):

$\mathcal{M}^\tau(v) =$ moat of v at time τ in KLS(Q), $Q \subseteq U$

$\mathcal{M}_0^\tau(v) =$ moat of v at time τ in KLS(Q_0), $Q_0 = Q \setminus \{(s,t)\}$

Obs.: death-times of terminals are instance independent!

$$\begin{aligned} & \mathcal{M}_0^\tau(v) \text{ active} \;\Rightarrow\; \mathcal{M}^\tau(v) \text{ active} \\ \Rightarrow \quad & \mathcal{M}_0^\tau(v) \subseteq \mathcal{M}^\tau(v) \\ \Rightarrow \quad & a_0^\tau(v) \leq a^\tau(v) \end{aligned}$$

$$\xi_v(Q) = \int_0^{\mathrm{d}(v)} \frac{1}{a^\tau(v)}\, d\tau \leq \int_0^{\mathrm{d}(v)} \frac{1}{a_0^\tau(v)}\, d\tau = \xi_v(Q_0)$$

□

Proving Budget Balance

Lemma

The cost shares ξ computed by KLS are 2-budget balanced.

Proof (sketch):
(F, y) = forest and dual computed by KLS(Q), $Q \subseteq U$. Then

$$c(F) \leq 2 \sum_{S} y_S = \sum_{i \in Q} \xi_{s_i t_i}$$

But: y is not dual feasible since some active moats do not correspond to Steiner cuts. Can still show that $\sum y_S \leq OPT(Q)$!

Idea: charge dual growth against an optimal forest F^* for Q.

Proving Budget Balance

Lemma

The cost shares ξ computed by KLS are 2-budget balanced.

Proof (sketch):
$(F, y) =$ forest and dual computed by KLS(Q), $Q \subseteq U$. Then

$$c(F) \leq 2 \sum_{S} y_S = \sum_{i \in Q} \xi_{s_i t_i}$$

But: y is not dual feasible since some active moats do not correspond to Steiner cuts. Can still show that $\sum y_S \leq OPT(Q)$!

Idea: charge dual growth against an optimal forest F^* for Q.

Proving Budget Balance

Lemma

The cost shares ξ computed by KLS are 2-budget balanced.

Proof (sketch):
$(F, y) =$ forest and dual computed by KLS(Q), $Q \subseteq U$. Then

$$c(F) \leq 2 \sum_{S} y_S = \sum_{i \in Q} \xi_{s_i t_i}$$

But: y is not dual feasible since some active moats do not correspond to Steiner cuts. Can still show that $\sum y_S \leq OPT(Q)$!

Idea: charge dual growth against an optimal forest F^* for Q.

Proving Budget Balance

Lemma

The cost shares ξ computed by KLS are 2-budget balanced.

Proof (sketch):
$(F, y) =$ forest and dual computed by KLS(Q), $Q \subseteq U$. Then

$$c(F) \leq 2 \sum_S y_S = \sum_{i \in Q} \xi_{s_i t_i}$$

But: y is not dual feasible since some active moats do not correspond to Steiner cuts. Can still show that $\sum y_S \leq OPT(Q)$!

Idea: charge dual growth against an optimal forest F^* for Q.

Proving Budget Balance

Lemma

The cost shares ξ computed by KLS are 2-budget balanced.

Proof (sketch):
$(F, y) =$ forest and dual computed by KLS(Q), $Q \subseteq U$. Then

$$c(F) \leq 2 \sum_S y_S = \sum_{i \in Q} \xi_{s_i t_i}$$

But: y is not dual feasible since some active moats do not correspond to Steiner cuts. Can still show that $\sum y_S \leq OPT(Q)$!

Idea: charge dual growth against an optimal forest F^* for Q.

Proving Budget Balance

Lemma

The cost shares ξ computed by KLS are 2-budget balanced.

Proof (sketch):
$(F, y) =$ forest and dual computed by KLS(Q), $Q \subseteq U$. Then

$$c(F) \leq 2 \sum_{S} y_S = \sum_{i \in Q} \xi_{s_i t_i}$$

But: y is not dual feasible since some active moats do not correspond to Steiner cuts. Can still show that $\sum y_S \leq OPT(Q)$!

Idea: charge dual growth against an optimal forest F^* for Q.

Proving Budget Balance

Let $Q = \{(s_1, t_1), \ldots, (s_k, t_k)\}$ such that

$$d(s_1, t_1) \leq \cdots \leq d(s_k, t_k)$$

Define precedence order on terminals:

$$s_1 \prec t_1 \prec s_2 \prec t_2 \prec \cdots \prec s_k \prec t_k$$

Terminal v is responsible at time τ if $u \prec v$ for all $u \in \mathcal{M}^\tau(v)$. Define $r^\tau(v) = 1$ if v is responsible at time τ and $r^\tau(v) = 0$ otherwise. Let the responsibility time of v be

$$r(v) = \int_0^{d(v)} r^\tau(v)\, d\tau$$

Intuition: No sharing of dual growth; the responsible terminal gets it all! Suffices to bound total responsibility time by $OPT(Q)$.

Proving Budget Balance

Let $Q = \{(s_1, t_1), \ldots, (s_k, t_k)\}$ such that

$$d(s_1, t_1) \leq \cdots \leq d(s_k, t_k)$$

Define precedence order on terminals:

$$s_1 \prec t_1 \prec s_2 \prec t_2 \prec \cdots \prec s_k \prec t_k$$

Terminal v is responsible at time τ if $u \prec v$ for all $u \in \mathcal{M}^\tau(v)$. Define $r^\tau(v) = 1$ if v is responsible at time τ and $r^\tau(v) = 0$ otherwise. Let the responsibility time of v be

$$r(v) = \int_0^{d(v)} r^\tau(v)\, d\tau$$

Intuition: No sharing of dual growth; the responsible terminal gets it all! Suffices to bound total responsibility time by $OPT(Q)$.

Proving Budget Balance

Let $Q = \{(s_1, t_1), \dots, (s_k, t_k)\}$ such that

$$d(s_1, t_1) \leq \dots \leq d(s_k, t_k)$$

Define precedence order on terminals:

$$s_1 \prec t_1 \prec s_2 \prec t_2 \prec \dots \prec s_k \prec t_k$$

Terminal v is responsible at time τ if $u \prec v$ for all $u \in \mathcal{M}^\tau(v)$.
Define $r^\tau(v) = 1$ if v is responsible at time τ and $r^\tau(v) = 0$ otherwise. Let the responsibility time of v be

$$r(v) = \int_0^{d(v)} r^\tau(v)\, d\tau$$

Intuition: No sharing of dual growth; the responsible terminal gets it all! Suffices to bound total responsibility time by $OPT(Q)$.

Proving Budget Balance

Let $Q = \{(s_1, t_1), \ldots, (s_k, t_k)\}$ such that

$$\mathrm{d}(s_1, t_1) \leq \cdots \leq \mathrm{d}(s_k, t_k)$$

Define precedence order on terminals:

$$s_1 \prec t_1 \prec s_2 \prec t_2 \prec \cdots \prec s_k \prec t_k$$

Terminal v is responsible at time τ if $u \prec v$ for all $u \in \mathcal{M}^\tau(v)$. Define $r^\tau(v) = 1$ if v is responsible at time τ and $r^\tau(v) = 0$ otherwise. Let the responsibility time of v be

$$r(v) = \int_0^{\mathrm{d}(v)} r^\tau(v)\, d\tau$$

Intuition: No sharing of dual growth; the responsible terminal gets it all! Suffices to bound total responsibility time by $OPT(Q)$.

Proving Budget Balance

Let $Q = \{(s_1, t_1), \dots, (s_k, t_k)\}$ such that

$$\mathrm{d}(s_1, t_1) \leq \cdots \leq \mathrm{d}(s_k, t_k)$$

Define precedence order on terminals:

$$s_1 \prec t_1 \prec s_2 \prec t_2 \prec \cdots \prec s_k \prec t_k$$

Terminal v is responsible at time τ if $u \prec v$ for all $u \in \mathcal{M}^\tau(v)$. Define $r^\tau(v) = 1$ if v is responsible at time τ and $r^\tau(v) = 0$ otherwise. Let the responsibility time of v be

$$r(v) = \int_0^{\mathrm{d}(v)} r^\tau(v)\, d\tau$$

Intuition: No sharing of dual growth; the responsible terminal gets it all! Suffices to bound total responsibility time by $OPT(Q)$.

Proving Budget Balance

Let $Q = \{(s_1, t_1), \ldots, (s_k, t_k)\}$ such that

$$\mathrm{d}(s_1, t_1) \leq \cdots \leq \mathrm{d}(s_k, t_k)$$

Define precedence order on terminals:

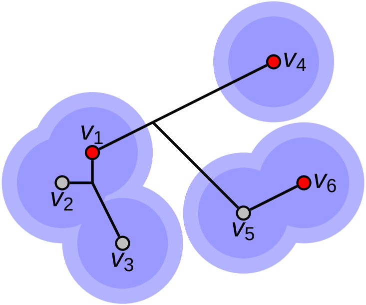
$$s_1 \prec t_1 \prec s_2 \prec t_2 \prec \cdots \prec s_k \prec t_k$$

Terminal v is responsible at time τ if $u \prec v$ for all $u \in \mathcal{M}^\tau(v)$. Define $r^\tau(v) = 1$ if v is responsible at time τ and $r^\tau(v) = 0$ otherwise. Let the responsibility time of v be

$$r(v) = \int_0^{\mathrm{d}(v)} r^\tau(v)\, d\tau$$

Intuition: No sharing of dual growth; the responsible terminal gets it all! Suffices to bound total responsibility time by $OPT(Q)$.

Proving Budget Balance



Consider a tree $T \in F^*$ and assume that T spans terminals $\{v_1, \dots, v_p\}$.

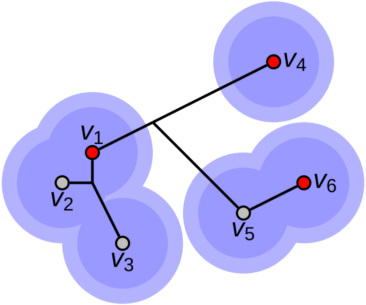
Every terminal v that is responsible at time τ loads a distinct part of T. **Note:** argument applies if there are at least two responsible terminals at time τ.

Let v_p be the terminal with highest responsibility time. Then

$$\sum_{i=1}^{p-1} r(v_i) \leq c(T).$$

Note: v_p's mate is in T as well.
$\Rightarrow r(v_p) \leq \mathrm{d}(v_p) \leq \frac{1}{2} c(T)$ □

Proving Budget Balance



Consider a tree $T \in F^*$ and assume that T spans terminals $\{v_1, \dots, v_p\}$.

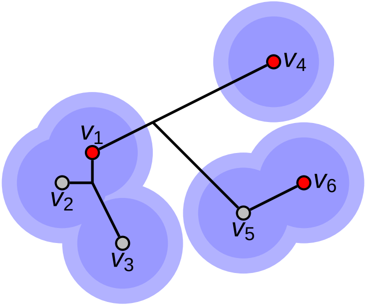
Every terminal v that is responsible at time τ loads a distinct part of T. **Note:** argument applies if there are at least two responsible terminals at time τ.

Let v_p be the terminal with highest responsibility time. Then

$$\sum_{i=1}^{p-1} r(v_i) \leq c(T).$$

Note: v_p's mate is in T as well.
$\Rightarrow r(v_p) \leq \mathrm{d}(v_p) \leq \frac{1}{2} c(T)$ □

Proving Budget Balance



Consider a tree $T \in F^*$ and assume that T spans terminals $\{v_1, \dots, v_p\}$.

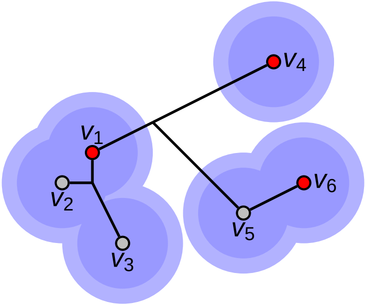
Every terminal v that is responsible at time τ loads a distinct part of T. **Note:** argument applies if there are at least two responsible terminals at time τ.

Let v_p be the terminal with highest responsibility time. Then

$$\sum_{i=1}^{p-1} r(v_i) \leq c(T).$$

Note: v_p's mate is in T as well.
$\Rightarrow r(v_p) \leq \text{d}(v_p) \leq \frac{1}{2}c(T)$ □

Proving Budget Balance



Consider a tree $T \in F^*$ and assume that T spans terminals $\{v_1, \ldots, v_p\}$.

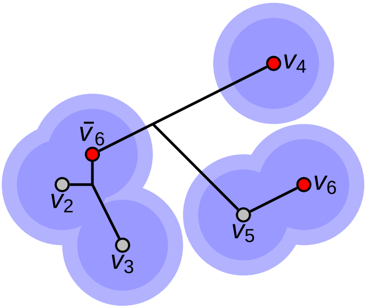
Every terminal v that is responsible at time τ loads a distinct part of T. **Note:** argument applies if there are at least two responsible terminals at time τ.

Let v_p be the terminal with highest responsibility time. Then

$$\sum_{i=1}^{p-1} r(v_i) \leq c(T).$$

Note: v_p's mate is in T as well.
$\Rightarrow r(v_p) \leq \mathrm{d}(v_p) \leq \frac{1}{2}c(T)$ □

Proving Budget Balance



Consider a tree $T \in F^*$ and assume that T spans terminals $\{v_1, \dots, v_p\}$.

Every terminal v that is responsible at time τ loads a distinct part of T. **Note:** argument applies if there are at least two responsible terminals at time τ.

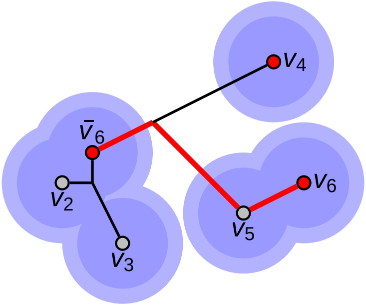
Let v_p be the terminal with highest responsibility time. Then

$$\sum_{i=1}^{p-1} r(v_i) \leq c(T).$$

Note: v_p's mate is in T as well.

$\Rightarrow r(v_p) \leq \mathrm{d}(v_p) \leq \frac{1}{2} c(T)$ □

Proving Budget Balance



Consider a tree $T \in F^*$ and assume that T spans terminals $\{v_1, \ldots, v_p\}$.

Every terminal v that is responsible at time τ loads a distinct part of T. **Note:** argument applies if there are at least two responsible terminals at time τ.

Let v_p be the terminal with highest responsibility time. Then

$$\sum_{i=1}^{p-1} r(v_i) \leq c(T).$$

Note: v_p's mate is in T as well.
$\Rightarrow r(v_p) \leq \mathrm{d}(v_p) \leq \frac{1}{2} c(T)$ □

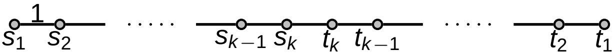
Further Consequences

Suppose our modified Steiner forest algorithm produces forest F and (infeasible) dual y for terminal pair set Q.

Surprisingly, can still show

$$c(F) \leq (2 - 1/k) \cdot OPT(Q)$$

Our dual is often much better than the AKR-dual!



$OPT(Q)$	$2k - 1$
Standard AKR-dual	k
Our dual	$2k - 1$

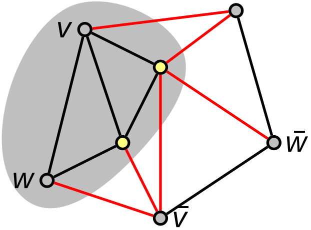
Recall: death-times induce precedence order $\prec$ on terminals

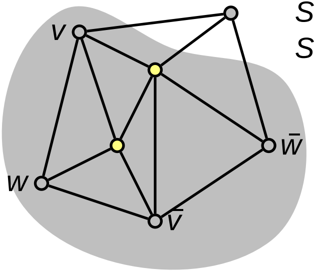
$$s_1 \prec t_1 \prec s_2 \prec t_2 \prec \cdots \prec s_k \prec t_k$$

Associate each cut $S \subseteq V$ with a terminal

Example: $v \prec \bar{v} \prec w \prec \bar{w}$

$S \in \mathcal{S}_w$





$S \in \mathcal{N}_w$
$S \in \mathcal{N}_{\bar{w}}$

Lifted-Cut Dual for Steiner Forests

$$
\begin{array}{rrcll}
OPT_{\mathrm{LC}} = \max & \sum_{S \subseteq V} y_S & & & \\
\text{s.t.} & \sum_{S \subseteq V : e \in \delta(S)} y_S & \leq & c_e & \forall e \in E \\
& \sum_{S \in \mathcal{S}_v} y_S + \sum_{S \in \mathcal{N}_v} y_S & \leq & \mathrm{d}(v) & \forall v \in R \\
& y_S & \geq & 0 & \forall S \subseteq V
\end{array}
$$

Theorem

1. $OPT_{UC} \leq OPT_{LC} \leq OPT$
2. *IP/LC gap is about 2*
3. *Additional strength of LC can be used to prove better approximation ratio of AKR for certain instances*

[Könemann, Leonardi, Schäfer, van Zwam, SICOMP ’08]

Related Results and Extensions

There is no $(2-\epsilon)$-budget balance cross-monotonic cost sharing scheme for the Steiner tree problem

[Könemann, Leonardi, Schäfer, van Zwam, SICOMP '08]

KLS is $\Theta(\log^2 n)$-approximate with respect to social cost

[Chawla, Roughgarden, Sundararajan, WINE '06]

Similar idea yields 3-budget balanced, $\Theta(\log^2 n)$-approximate, cross-monotonic cost sharing function for the price-collecting Steiner forest problem

[Gupta, Könemann, Leonardi, Ravi, Schäfer, SODA '07]

General Connectivity Problems

Idea:

- every player i has a cut-requirement function $f_i : 2^V \to \{0,1\}$
- model general connectivity game via the following LP

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e \cdot x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(S)} x_e \geq f_i(S) \quad \forall S \subseteq V,\ \forall i \in U \\ & x_e \in \{0,1\} \quad \forall e \in E \end{aligned}$$

- adapt approximation framework by Goemans and Williamson to obtain $O(1)$-budget balance, cross-monotonic cost sharing function [Könemann, Leonardi, Schäfer, Wheatley, manuscript]

Conclusions and Open Problems

Conclusions

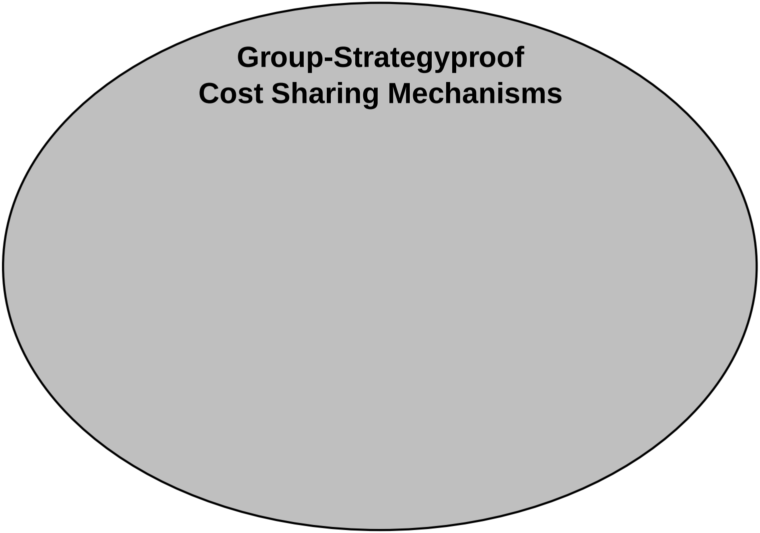
Moulin's framework enables to derive group-strategyproof cost sharing mechanisms through cross-monotonic cost sharing functions.

Have techniques at hand to bound social cost efficiency of Moulin mechanisms.

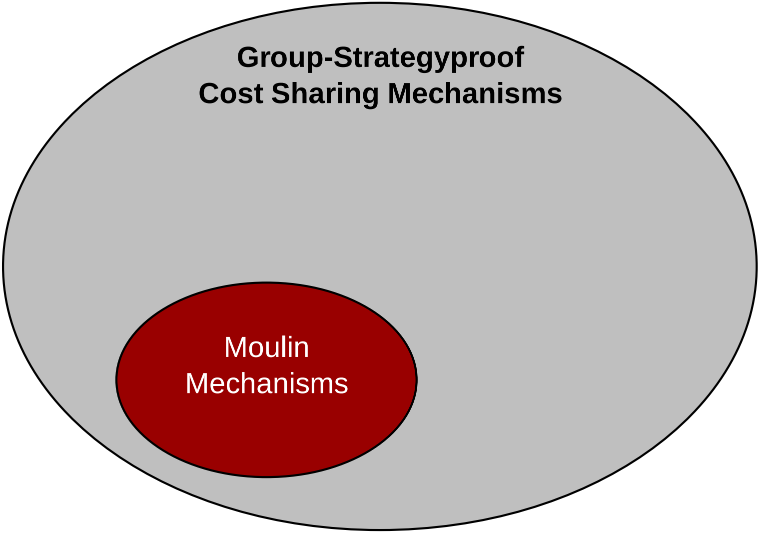
Trade-off between budget balance and social cost approximation guarantees of Moulin mechanisms are well-understood for several fundamental optimization problems.

Designing cross-monotonic cost sharing functions may lead to new insights that are useful in other contexts.

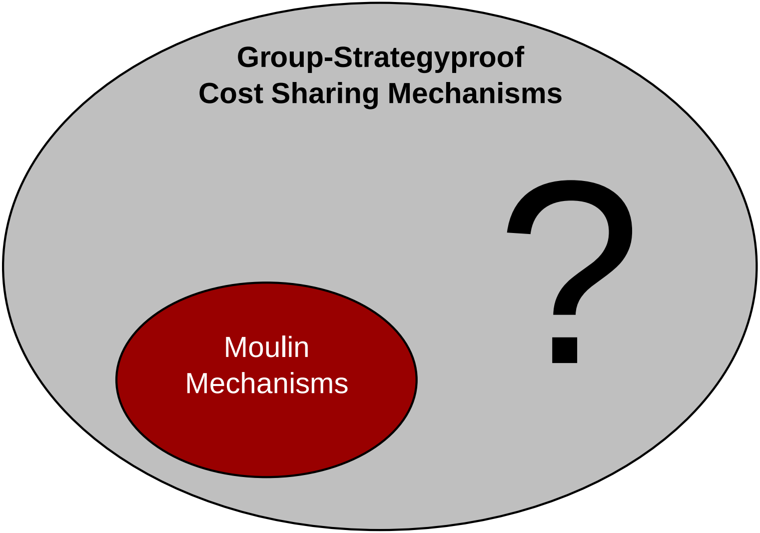
Characterization of GSP Mechanisms



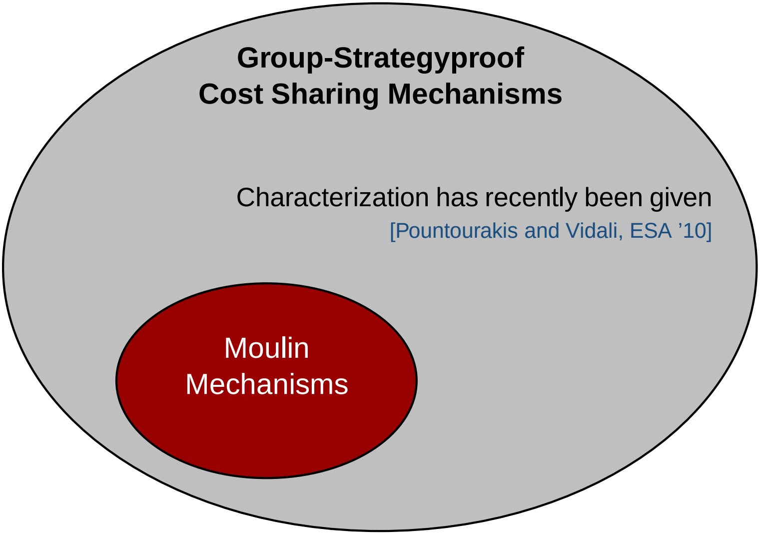
Characterization of GSP Mechanisms



Characterization of GSP Mechanisms



Characterization of GSP Mechanisms



Open Problems

Open Problem: Can we exploit the characterization of group-strategyproof cost sharing mechanisms algorithmically?

Open Problem: Are there other general techniques to derive group-strategyproof cost sharing mechanisms?

Open Problem: What are the trade-offs between group-strategyproofness and budget balance and social cost approximation guarantees?