ADFOCS 2010, Exercises

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The linear case of Fisher's market model consitst of a set B of n buyers and a set G of g divisible goods. Assume the buyers are numbered from 1 to n and the goods from 1 to g. Let u_{ij} be the utility derived by buyer i from 1 unit of good j. If x_{ij} is the amount of good j received by buyer i, then her total utility is given by $\sum_{j=1}^{g} u_{ij} \cdot x_{ij}$. Assume buyer i has m_i dollars and w.l.o.g. assume there is 1 unit of each good. The problem is to find *equilibrium prices and allocations*, i.e., prices s.t. if each buyer is given an optimal bundle of goods, the market clears.

1. Assume $|B| = 2$. Give a strongly polynomial algorithm for computing the equilibrium.
2. Assume that all u_{ij}'s are 0/1, and assume there are b_j units of good j. Give a polynomial time algorithm for computing the equilibrium.
   Extra credit: Extend to a strongly polynomial algorithm.
3. For Fisher's linear case, give a polynomial time algorithm for testing if given prices $p_1, \ldots, p_g$ are equilibrium prices.
4. Below is the Eisenberg-Gale program.

$$\begin{aligned} \text{maximize} \quad & \sum_{i \in B} m_i \log(u_i) \qquad (1) \\ \text{subject to} \quad & \forall i \in B: \; u_i = \sum_{j \in G} u_{ij} x_{ij} \\ & \forall j \in G: \; \sum_{i \in B} x_{ij} \le 1 \\ & \forall i \in B, \; \forall j \in G: \quad x_{ij} \ge 0 \end{aligned}$$

   The KKT conditions for this convex program are:

   (1) $\forall j \in G: \; p_j \ge 0$,
   (2) $\forall j \in G: \; p_j > 0 \Rightarrow \sum_{i \in B} x_{ij} = 1$.
   (3) $\forall i \in B, \; \forall j \in G \; p_j \ge \frac{m_i . u_{ij}}{u_i}$.
   (4) $\forall i \in B, \; \forall j \in G \; x_{ij} > 0 \Rightarrow p_j = \frac{m_i . u_{ij}}{u_i}$.

   Prove that (2) is a rational convex program and its optimal solution gives equilibrium allocations and utility.

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5. Use Lagrange relaxation to obtain the dual LP's for:

$$\begin{aligned} \text{maximize} \quad & c^T \cdot x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned} \tag{2}$$

$$\begin{aligned} \text{minimize} \quad & c^T \cdot x \\ \text{subject to} \quad & Ax \geq b \\ & x \geq 0 \end{aligned} \tag{3}$$