ADFOCS 2010, Exercises

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Fisher's market model consitst of a set B of n buyers and a set G of g divisible goods. Assume the buyers are numbered from 1 to n and the goods from 1 to g. Let f_i be the function specifying the utility derived by buyer i from a bundle of goods. If x_{ij} is the amount of good j received by buyer i, and $\boldsymbol{x}_i$ is the g-dimensional vector whose jth component is x_{ij}, then the utility she derives from this bundle is given by $f_i(\boldsymbol{x}_i)$. Assume buyer i has m_i dollars and w.l.o.g. assume there is 1 unit of each good. The problem is to find *equilibrium prices and allocations*, i.e., prices s.t. if each buyer is given an optimal bundle of goods, the market clears. Several different utility functions will be considered below.

1. **Additively-separable, piecewise-linear, concave utilities:** Assume buyers' utility functions are additively-separable, piecewise-linear, concave, and satisfying non-satiation. Give a polynomial time algorithm for testing if given prices $p_1, \ldots, p_g$ are equilibrium prices.

   Also, if all parameters are rational and equilibrium exists, show that there must be one that is rational.

2. **Leontief utilities:** A utility function is Leontief if, given parameters $a_{ij} \geq 0$,

$$f_i(\boldsymbol{x}_i) = \min_j \left\{ \frac{x_{ij}}{a_{ij}} \right\},$$

   (assume 0/0 is large enough to not affect the minimization; alternatively, the minimization is only over non-zero a_{ij}'s). Show that the following convex program captures equlibrium for this utility function. Is this a rational convex program?

$$\begin{aligned} \text{maximize} \quad & \sum_{i \in B} m_i \log(u_i) \qquad (1) \\ \text{subject to} \quad & \forall i \in B,\ \forall j \in G: \ u_i \leq \frac{x_{ij}}{a_{ij}} \\ & \forall j \in G: \ \sum_{i \in B} x_{ij} \leq 1 \\ & \forall i \in B,\ \forall j \in G: \ \ x_{ij} \geq 0 \end{aligned}$$

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3. **Linear utilities:** Show that the following convex program captures Fisher's linear case and that it is rational. Interpret b_{ij} as the amount of money spent by buyer i on good j and p_j as the price of good j.

$$
\begin{aligned}
\text{maximize} \quad & \sum_{i \in B,\ j \in G} b_{ij} \log \frac{u_{ij}}{p_j} \qquad (2)\\
\text{subject to} \quad & \forall j \in G: \ p_j = \sum_{i \in B} b_{ij} \\
& \forall i \in B: \ \sum_{j \in G} b_{ij} \le m_i \\
& \forall i \in B,\ \forall j \in G: \quad b_{ij} \ge 0
\end{aligned}
$$