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Cost Sharing and Approximation Algorithms
Exercise 2

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Problems 1 and 2 deal with the *Steiner tree problem*: We are given an undirected graph $G = (V,E)$ with non-negative edge costs $c : E \to \mathbb{R}^+$, a set of terminals $R = \{t_1, \ldots, t_k\} \subseteq V$ and a designated root node $r \in V$. For a given subset $S \subseteq R$ of terminals, a *Steiner tree* on S is a minimum cost tree in G that spans all nodes in $S \cup \{r\}$. We use $opt(S)$ to refer to its cost.

Problem 1. *In the cost sharing variant of the Steiner tree problem, the set of players corresponds to the set of terminal nodes, i.e.,* $U = R$. *Every player wants to connect her terminal* t_i *to the root node* r. *The cost* $C(S)$ *to connect all players in* $S \subseteq U$ *with* r *is defined as the cost* $\text{opt}(S)$ *of a Steiner tree on* S.

Develop a 2*-budget balanced and weakly group-strategyproof cost sharing mechanism for the Steiner tree cost sharing game.*

Problem 2. *Give a* 2*-approximate and* 2*-strict algorithm for the Steiner tree problem.*

Hints for Problems 1 and 2: Convince yourself about the following:

– We can assume without loss of generality that G is a complete graph and that the edge costs satisfy the triangle inequality.
– We obtain a 2-approximate Steiner tree for $S \subseteq R$ by computing a minimum spanning tree on $S \cup \{r\}$.

Problem 3. *Suppose we are given a set of players* U *and a cost function* $C : 2^U \to \mathbb{R}^+$. *A* cost allocation $(x_i)_{i \in N}$ *assigns a non-negative cost share* x_i *to every player* $i \in U$. *The cost allocation* $(x_i)_{i \in N}$ *is said to be in the* α-core *(*$\alpha \geq 1$*) if:*

1. $\frac{1}{\alpha} C(U) \leq \sum_{i \in U} x_i \leq C(U)$
2. $\sum_{i \in S} x_i \leq C(S)$ *for every* $S \subseteq U$.

(a) *Show that every cross-monotonic and* β*-budget balanced cost sharing function* ξ *gives rise to a cost allocation in the* α*-core with* $\alpha = \beta$.

(b) *Show that if the cost function* C *is such that* $C(U) \geq \beta \sum_{i \in U} C(\{i\})$ *for some* $\beta > 1$ *then there is no cost allocation in the* α*-core for* $\alpha < \beta$.