Algebraic approach to exact algorithms, Part IV: Matching connectivity matrix

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Matching connectivity matrix

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Let n be an even integer.

- $\mathcal{H}_n$ is a square matrix over the $\mathbb{Z}_2$ field with rows and columns labeled by all perfect matchings in K_n.
- $(\mathcal{H}_n)_{M_1,M_2} = [M_1 \cup M_2$ forms a Hamiltonian cycle in $K_t]$
- Dimension:
$$\frac{n!}{(n/2)!2^{n/2}} = \frac{(\frac{n}{e})^n}{(\frac{n}{2e})^{n/2}2^{n/2}} n^{O(1)} = (\tfrac{n}{e})^{n/2} n^{O(1)} = 2^{O(n\log n)}$$

Example: $\mathcal{H}_4$

			
	0	1	1
	1	0	1
	1	1	0

Matching connectivity matrix $\mathcal{H}_6$

Nr.		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1		0	0	0	0	1	1	0	1	1	1	1	0	1	1	0
2		0	0	0	1	0	1	1	1	0	0	1	1	1	0	1
3		0	0	0	1	1	0	1	0	1	1	0	1	0	1	1
4		0	1	1	0	0	0	0	1	1	1	0	1	1	0	1
5		1	0	1	0	0	0	1	0	1	0	1	1	1	1	0
6		1	1	0	0	0	0	1	1	0	1	1	0	0	1	1
7		0	1	1	0	1	1	0	0	0	0	1	1	0	1	1
8		1	1	0	1	0	1	0	0	0	1	0	1	1	1	0
9		1	0	1	1	1	0	0	0	0	1	1	0	1	0	1
10		1	0	1	1	0	1	0	1	1	0	0	0	0	1	1
11		1	1	0	0	1	1	1	0	1	0	0	0	1	0	1
12		0	1	1	1	1	0	1	1	0	0	0	0	1	1	0
13		1	1	0	1	1	0	0	1	1	0	1	1	0	0	0
14		1	0	1	0	1	1	1	1	0	1	0	1	0	0	0
15		0	1	1	1	0	1	1	0	1	1	1	0	0	0	0

Matching connectivity matrix $\mathcal{H}_n$

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- $\mathcal{H}_n$ is huge
- $\mathcal{H}_n$ has much redundancy
- What is the rank of $\mathcal{H}_n$?

Family of matchings $\mathbf{X}_n$

Partition the vertices $1, 2, \ldots, n$ into $n/2 + 1$ groups:

$$1 \mid 23 \mid 45 \mid \cdots \mid (n-2)(n-1) \mid n$$

Let $\mathrm{pm}(G)$ denote the set of all perfect matchings of G.

$$\mathbf{X}_n = \mathbf{X} =$$
$$\{M \in \mathrm{pm}(K_n) \ : \ M \text{ matches vertices from neighboring groups only}\}$$

Example: $n = 6$

Groups:

$$1 \mid 23 \mid 45 \mid 6$$

Matchings:

$$\mathbf{X}_6 = \{\{12, 34, 56\}, \{12, 35, 46\}, \{13, 24, 56\}, \{13, 25, 46\}\}$$

Indexing the matchings from $\mathbf{X}$

- $\mathbf{X}$ has $2^{n/2-1}$ matchings.
- The matchings are indexed by 0/1-strings of length $n/2-1$.
- Building a matching from the string $w_1 \ldots, w_{n/2-1}$:
  For $i = 1, \ldots, n/2-1$:
  - if $w_i = 1$ then the yet unmatched vertex of i-th group is matched with the first vertex of the $(i+1)$-th group,
  - if $w_i = 0$ then ... with the second ...

Example: $n = 6$

Groups:

$$1 \mid 23 \mid 45 \mid 6$$

Matchings:

$$\begin{array}{ll} \mathbf{X}(11) = \{12, 34, 56\} & \mathbf{X}(10) = \{12, 35, 46\} \\ \mathbf{X}(01) = \{13, 24, 56\} & \mathbf{X}(00) = \{13, 25, 46\} \end{array}$$

Properties of the $\mathcal{H}_{\mathbf{X},\mathbf{X}}$ submatrix

For $w \in \{0,1\}^\ell$ denote $\overline{w} = w \operatorname{xor} \underbrace{1 \cdots 1}_{\ell}$, e.g. $\overline{110} = 001$.

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$\mathbf{X}(w) \cup \mathbf{X}(u)$ is a Hamiltonian cycle iff $w = \overline{u}$.

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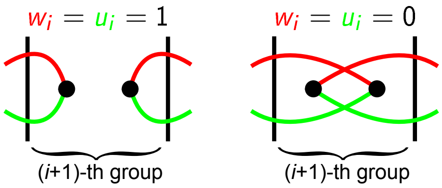

$\mathbf{X}(u) \cup \mathbf{X}(w)$ has at least two connected components.

Observation

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Proof:

Assume $w = \overline{u}$.

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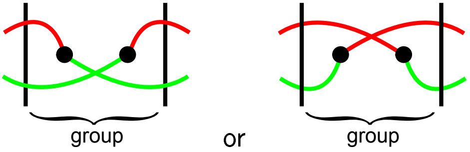

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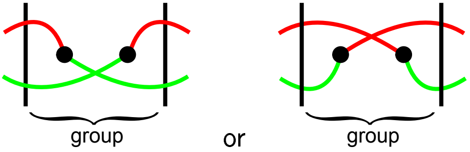

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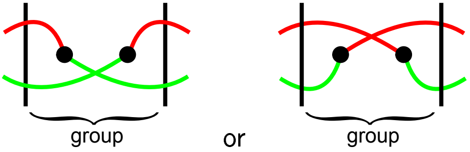

- Every vertex is adjacent to a vertex in the previous group.
- Hence, every vertex has a path to vertex 1.
- Hence there is only one connected component.
- Since all degrees are 2, this is a HC.

Properties of the $\mathcal{H}_{\mathbf{X},\mathbf{X}}$ submatrix

Order the rows/columns of $\mathcal{H}_{\mathbf{X},\mathbf{X}}$ in lexicographic order, i.e.:

$$\mathbf{X}(0\cdots000), \mathbf{X}(0\cdots001), \mathbf{X}(0\cdots010), \mathbf{X}(0\cdots011), \ldots, \mathbf{X}(1\cdots111).$$

Then, $\mathcal{H}_{\mathbf{X},\mathbf{X}} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \iddots & 0 & 0 \\ 1 & \cdots & 0 & 0 \end{bmatrix}$, so $\operatorname{rank} \mathcal{H}_{\mathbf{X},\mathbf{X}} = 2^{n/2-1}$.

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Corollary

$\operatorname{rank} \mathcal{H}_n \geq 2^{n/2-1}$.

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Corollary

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Rows $\mathbf{X}$ of $\mathcal{H}$ are linearly independent.

Question

Do they form a basis of the row space of $\mathcal{H}$?

Linear combination coefficients

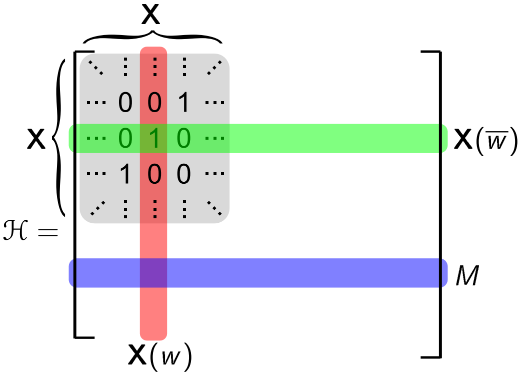
Assume that $\mathbf{X}$ is a basis of the row space of $\mathcal{H}$:
For any $M \in \mathrm{pm}(K_n)$, for some $c_{M,w} \in \{0,1\}$,

$$\mathcal{H}_M = \sum_{w \in \{0,1\}^{n/2-1}} c_{M,w} \mathcal{H}_{\mathbf{X}(w)}$$

- $\mathcal{H}_{M,\mathbf{X}(w)} = 0 \Rightarrow c_{M,\overline{w}} = 0$,
- $\mathcal{H}_{M,\mathbf{X}(w)} = 1 \Rightarrow c_{M,\overline{w}} = 1$.

Hence, $c_{M,w} = \mathcal{H}_{M,\mathbf{X}(\overline{w})}$ and

$$\mathcal{H}_M = \sum_{w \in \{0,1\}^{n/2-1}} \mathcal{H}_{M,\mathbf{X}(\overline{w})} \mathcal{H}_{\mathbf{X}(w)}$$



The representation formula

If $\mathbf{X}$ is a basis then $\mathcal{H}_M = \sum_{w \in \{0,1\}^{n/2-1}} \mathcal{H}_{M,\mathbf{X}(\overline{w})} \mathcal{H}_{\mathbf{X}(w)}$. Then,

The representation formula

$$\mathcal{H}_{M_1,M_2} = \sum_{w \in \{0,1\}^{n/2-1}} \mathcal{H}_{M_1,\mathbf{X}(\overline{w})} \mathcal{H}_{\mathbf{X}(w),M_2}.$$

Theorem (Cygan, Kratsch, Nederlof 2013)

The representation formula holds.

(technical inductive proof skipped here.)

Corollary (Cygan, Kratsch, Nederlof 2013)

$\operatorname{rank} H_n = 2^{n/2} - 1.$

Note: The representation formula holds in $GF(2^k)$ for every k.

Undirected Hamiltonicity in $O^*(1.888^n)$ time

- Let $G = (V, E)$ be an undirected graph.
- We want to test Hamiltonicity of G.
- W.l.o.g. $|V|$ is even.

Undirected Hamiltonicity in $O^*(1.888^n)$ time

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- We want to test Hamiltonicity of G.
- W.l.o.g. $|V|$ is even.
- Yet another hero (over $GF(2^{2n})$):

$$P(\mathbf{x}, \mathbf{y}) = \sum_{\substack{M_1, M_2 \in \mathrm{pm}(G) \\ M_1 \cup M_2 \text{ is a HC}}} \prod_{e \in M_1} x_e \prod_{e \in M_2} y_e$$

Polynomial P and Hamiltonicity

$$P(\mathbf{x},\mathbf{y}) = \sum_{\substack{M_1,M_2\in \mathrm{pm}(G) \\ M_1\cup M_2 \text{ is a HC}}} \prod_{e\in M_1} x_e \prod_{e\in M_2} y_e$$

Observation

$P \not\equiv 0$ iff G is Hamiltonian.

Proof:

($\Rightarrow$): Obvious.

($\Leftarrow$):

- Let H be a HC in G.
- Then $H = M_1 \cup M_2$ where M_1, M_2 are perfect matchings,
- The sum in the definition of P contains each of the monomials $\prod_{e\in M_1} x_e \prod_{e\in M_2} y_e$ and $\prod_{e\in M_2} x_e \prod_{e\in M_1} y_e$ exactly once.

Rewriting P

$$P(\{x_e\}_{e\in E},\{y_e\}_{e\in E}) = \sum_{\substack{M_1,M_2\in\mathrm{pm}(G)\\ M_1\cup M_2 \text{ is a HC}}} \prod_{e\in M_1} x_e \prod_{e\in M_2} y_e =$$

$$\sum_{M_1\in\mathrm{pm}(G)} \sum_{M_2\in\mathrm{pm}(G)} \mathcal{H}_{M_1,M_2} \prod_{e\in M_1} x_e \prod_{e\in M_2} y_e =^{(\mathrm{RF})}$$

$$\sum_{M_1\in\mathrm{pm}(G)} \sum_{M_2\in\mathrm{pm}(G)} \sum_{w\in\{0,1\}^{n/2-1}} \mathcal{H}_{M_1,\mathbf{X}(\overline{w})}\mathcal{H}_{\mathbf{X}(w),M_2} \prod_{e\in M_1} x_e \prod_{e\in M_2} y_e =$$

$$\sum_{w\in\{0,1\}^{n/2-1}} \underbrace{\left(\sum_{M_1\in\mathrm{pm}(G)} \mathcal{H}_{M_1,\mathbf{X}(\overline{w})} \prod_{e\in M_1} x_e\right)}_{\mathrm{ext}^G_{\mathbf{X}(\overline{w})}(\{x_e\}_{e\in E})} \cdot \underbrace{\left(\sum_{M_2\in\mathrm{pm}(G)} \mathcal{H}_{\mathbf{X}(w),M_2} \prod_{e\in M_2} y_e\right)}_{\mathrm{ext}^G_{\mathbf{X}(w)}(\{y_e\}_{e\in E})}$$

where for any $M \in \mathbf{X}$,

$$\mathrm{ext}^G_M(\{z_e\}_{e\in E}) = \sum_{\substack{M'\in\mathrm{pm}(G)\\ M\cup M' \text{ is a HC}}} \prod_{e\in M'} z_e$$

Evaluating P in $O^*(1.888^n)$ time

We got:

$$P(\{x_e\}_{e\in E}, \{y_e\}_{e\in E}) = \sum_{w\in\{0,1\}^{n/2-1}} \mathrm{ext}^G_{\mathbf{X}(\overline{w})}(\{x_e\}_{e\in E})\,\mathrm{ext}^G_{\mathbf{X}(w)}(\{y_e\}_{e\in E})$$

- Note that $|\mathbf{X}| = 2^{n/2-1} = O(1.42^n)$.
- Hence it suffices to precompute $\mathrm{ext}^G_M(\{x_e\}_{e\in E})$ and $\mathrm{ext}^G_M(\{y_e\}_{e\in E})$ for all $M \in \mathbf{X}$ in $O^*(1.888^n)$ time.

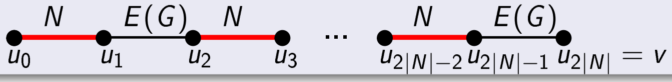
Evaluating $\mathrm{ext}_M^G(\{z_e\}_{e \in E}) = \sum_{\substack{M' \in \mathrm{pm}(G) \\ M \cup M' \text{ is a HC}}} \prod_{e \in M'} z_e$ in $O^*(1.888^n)$.

Fix any $u_0 \in V$.

N-alternating v-walk

Let N be a matching in K_n.
A walk $u_0, u_1, \ldots, u_t$ in K_n is called N-**alternating** v-**walk** if

- for every $i = 0, \ldots, n/2 - 1$, $u_{2i}u_{2i+1} \in N$ and $u_{2i+1}u_{2i+2} \in E(G)$.
- $t = 2|N|$,
- each edge of N is visited,
- $u_t = v$,



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N-alternating v-walk

For every matching N such that $N \subseteq M'$ for some $M' \in \mathbf{X}$, for every $v \in V$, compute

$$T[N, v] = \sum_{\substack{N\text{-alternating } v\text{-walk} \\ e_1, e_2, \ldots, e_{2|N|}}} \prod_{i=1}^{|N|} z_{e_{2i}}$$

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N-alternating v-walk

$$u_0 \overset{N}{\text{———}} u_1 \overset{E(G)}{\text{———}} u_2 \overset{N}{\text{———}} u_3 \;\cdots\; u_{2|N|-2} \overset{N}{\text{———}} u_{2|N|-1} \overset{E(G)}{\text{———}} u_{2|N|} = v$$

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Note that $\text{ext}_M^G(\{z_e\}_{e\in E}) = T[M, u_0]$.

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Dynamic programming formula

$$T[N,v] = \sum_{uv\in E} \sum_{u'u\in N} z_{uv}\, T[N\setminus\{u'u\}, u']$$

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Corollary

Let $\alpha(n) = |\{N \subseteq M \;:\; M \in \mathbf{X}_n\}|$.
All entries of $T[N,v]$ can be computed in $O^*(\alpha(n))$ time.

- Since $|\mathbf{X}_n| = 2^{n/2-1}$ and every $M \in \mathbf{X}_n$ has $2^{n/2}$ subsets, $\alpha(n) \le 2^{n-1}$.

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- ... but there are a lot of common subsets!

Bounding $\alpha(n)$

Let $\beta(n) = |\{N \subseteq M \;:\; M \in \mathbf{X}_n \text{ and } n \notin V(N)\}|$.
Then

$$\begin{cases} \alpha(n) = \overbrace{2\alpha(n-2)}^{\text{match vertex } n} + \overbrace{\beta(n)}^{\text{do not match vertex } n} \\ \beta(n) = \underbrace{4\alpha(n-4)}_{\text{match } n-2 \text{ or } n-1} + \underbrace{1 \cdot \beta(n-2)}_{\text{do not match them}} \end{cases}$$

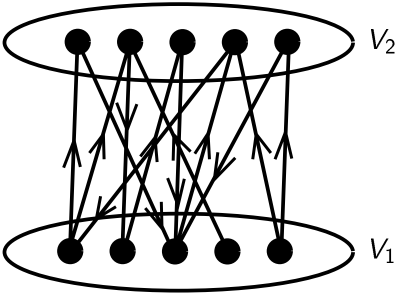
Solve it using your favorite method and get
$\alpha(n) = O((\frac{3+\sqrt{17}}{2})^{n/2}) = O(1.88721^n)$.

Theorem (Cygan, Kratsch, Nederlof 2013)

The Hamiltonian cycle problem in undirected graphs can be solved in $O^*(1.888^n)$ time.

Hamiltonicity in directed bipartite graphs in $O^*(1.888^n)$ time

$G = (V_1 \cup V_2, E)$ – a directed bipartite graph.



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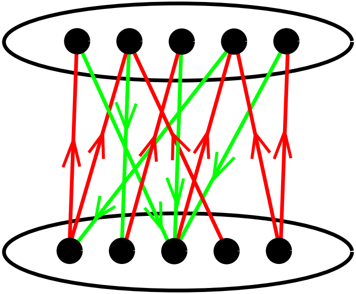

- $E_1 = \{(v_1, v_2) \in E \ : \ v_1 \in V_1 \text{ and } v_2 \in V_2\}$; $G_1 = (V, E_1)$
- $E_2 = \{(v_2, v_1) \in E \ : \ v_1 \in V_1 \text{ and } v_2 \in V_2\}$; $G_2 = (V, E_2)$

$$P = \sum_{\substack{M_1 \in \mathrm{pm}(G_1) \\ M_2 \in \mathrm{pm}(G_2) \\ M_1 \cup M_2 \text{ is a HC}}} \prod_{e \in M_1} x_e \prod_{e \in M_2} y_e = \sum_{w \in \{0,1\}^{n/2-1}} \mathrm{ext}^{G_1}_{\mathbf{X}(\overline{w})}(\{x_e\}_{e \in E_1})\, \mathrm{ext}^{G_2}_{\mathbf{X}(w)}(\{y_e\}_{e \in E_2})$$

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