Algebraic approaches to exact algorithms, part V: Systems of linear equations

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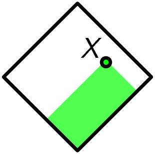
- There are t linear equations in variables $x_1, \ldots, x_t$ such that:
    - we can show the equations are linearly independent,
    - we can compute the coefficients and the constant terms of the equations **efficiently**.
- We solve the system using Gaussian Elimination in $O(t^3)$ time. (Or in $O(t^\omega)$ time if it matters.)

Recap: Fast Zeta transform ζ

Let $f : 2^U \to \mathbb{N}$.

Zeta transform

$(\zeta f)(X) = \sum_{Y \subseteq X} f(Y)$.



Trimmed zeta transform (Björklund, Husfeldt, Kaski, Koivisto)

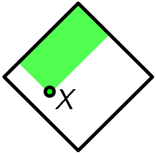
- For any set family $\mathcal{G} \subseteq 2^U$ we can compute **all values** of $\zeta f|_{\mathcal{G}}$ in $O^*(|\downarrow\mathcal{G}|)$ time.
- We can compute **all values** of ζf in $O^*(|\uparrow\mathrm{supp}(f)|)$ time.

Recap: Fast up-zeta transform $\zeta^{\uparrow}$

Let $f : 2^U \to \mathbb{N}$.

Up-zeta transform

$(\zeta^{\uparrow} f)(X) = \sum_{Y \supseteq X} f(Y)$.



Trimmed up-zeta transform (Björklund, Husfeldt, Kaski, Koivisto)

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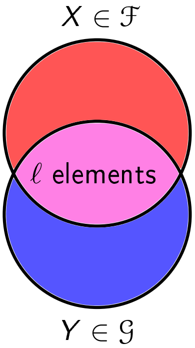
- We can compute **all values** of $\zeta^{\uparrow} f$ in $O^*(|\downarrow\mathrm{supp}(f)|)$ time.

Intersection Transform

- U is a given set, $|U| = n$.
- We are given $\mathcal{F}, \mathcal{G} \subseteq 2^U$.
- For every $Y \in \mathcal{G}$ and every $\ell \in \{0, \ldots, n\}$, compute

$$\iota_{\mathcal{F}}(\ell, Y) = |\{X \in \mathcal{F} \;:\; |X \cap Y| = \ell\}|$$

- $n + 1$ indeterminates $x_\ell^Y = \iota_{\mathcal{F}}(\ell, Y)$, for $\ell = 0, \ldots, n$
- $n + 1$ linear equations?



Intersection Transform

For every $Y \in \mathcal{G}$ and $\ell \in \{0, \ldots, n\}$, find $x_\ell^Y = |\{X \in \mathcal{F} \ : \ |X \cap Y| = \ell\}|$.

For every $Y \in G$ and $j \in \{0, \ldots, n\}$,

$$
\begin{aligned}
b_j^Y &= \sum_{\substack{Z \subseteq Y \\ |Z|=j}} |\{X \in \mathcal{F} \ : \ Z \subseteq X\}| = \\
&= \sum_{\substack{Z \subseteq Y \\ |Z|=j}} \sum_{\substack{X \in \mathcal{F} \\ Z \subseteq X}} 1 = \sum_{X \in \mathcal{F}} \sum_{\substack{Z \subseteq X \cap Y \\ |Z|=j}} 1 = \sum_{X \in \mathcal{F}} \binom{|X \cap Y|}{j} = \\
&= \sum_{\ell=0}^{n} \sum_{\substack{X \in \mathcal{F} \\ |X \cap Y|=\ell}} \binom{\ell}{j} = \sum_{\ell=0}^{n} \binom{\ell}{j} \sum_{\substack{X \in \mathcal{F} \\ |X \cap Y|=\ell}} 1 = \sum_{\ell=0}^{n} \binom{\ell}{j} x_\ell^Y
\end{aligned}
$$

Intersection Transform: linear equations

Intersection Transform

For every $Y \in \mathcal{G}$ and $\ell \in \{0,\ldots,n\}$, find $x_\ell^Y = |\{X \in \mathcal{F} \ : \ |X \cap Y| = \ell\}|$.

For every $Y \in G$ we got $(n+1)$ linear equations:

$$\sum_{\ell=0}^{n} \binom{\ell}{j} x_\ell^Y = b_j^Y, \qquad\qquad j = 0,\ldots,n$$

where $b_j^Y = \sum_{\substack{Z \subseteq Y \\ |Z|=j}} |\{X \in \mathcal{F} \ : \ Z \subseteq X\}|$

- Since for $\ell < j$, $\binom{\ell}{j} = 0$, *and* $\binom{\ell}{\ell} = 1$ and the coefficients matrix is non-singular.

Intersection Transform: linear equations

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- The coefficients can be evaluated fast.

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Intersection Transform

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For every $Y \in G$ we got $(n+1)$ linear equations:

$$\sum_{\ell=0}^{n} \binom{\ell}{j} x_\ell^Y = b_j^Y, \qquad\qquad j = 0, \ldots, n$$

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- Since for $\ell < j$, $\binom{\ell}{j} = 0$, *and* $\binom{\ell}{\ell} = 1$ and the coefficients matrix is non-singular.
- The coefficients can be evaluated fast.
- How fast can we evaluate b_j^Y?

Evaluating b_j^Y for every $Y \in \mathcal{G}$ and $j = 0, \ldots, n$

$$\begin{aligned} b_j^Y &= \sum_{\substack{Z \subseteq Y \\ |Z|=j}} |\{X \in \mathcal{F} \;:\; Z \subseteq X\}| = \\ &= \sum_{\substack{Z \subseteq Y \\ |Z|=j}} \sum_{X \supseteq Z} [X \in \mathcal{F}] = \sum_{\substack{Z \subseteq Y \\ |Z|=j}} (\zeta^{\uparrow} \mathbf{1}_{\mathcal{F}})(Z) = (\zeta f)(Y), \end{aligned}$$

where for every $Z \in \downarrow\mathcal{G}$,

$$f(Z) = (\zeta^{\uparrow} \mathbf{1}_{\mathcal{F}})(Z) \cdot [|Z| = j].$$

Algorithm for evaluating b_j^Y for every $Y \in \mathcal{G}$.

1. Compute $(\zeta^{\uparrow} \mathbf{1}_{\mathcal{F}})(Z)$ for all $Z \in \downarrow\mathcal{G}$ in $O(|\downarrow \mathrm{supp}(\mathbf{1}_{\mathcal{F}})|) = O^*(|\downarrow\mathcal{F}|)$ time; from this compute $f(Z)$ easily.
2. Compute $(\zeta f)(Y)$ for all $Y \in \mathcal{G}$ in $O^*(\downarrow\mathcal{G})$ time.

Total running time: $O^*(|\downarrow\mathcal{F}| + |\downarrow\mathcal{G}|)$

Intersection Transform Algorithm

Algorithm

1. Compute b_j^Y for every $Y \in \mathcal{G}$ and $j = 0, \ldots, n$ in $O^*(|{\downarrow}\mathcal{F}| + |{\downarrow}\mathcal{G}|)$ time,
2. For every $Y \in \mathcal{G}$, solve the system of linear equations with indeterminates x_ℓ^Y, $\ell = 0, \ldots, n$, using Gaussian Elimination in $O(n^3)$ time. (Actually one can derive an explicit formula, skipped here.)

Theorem (Björklund, Husfeldt, Kaski, Koivisto 2008)

Given $\mathcal{F}, \mathcal{G} \subseteq 2^U$, the values of

$$\iota_{\mathcal{F}}(Y, \ell) = |\{X \in \mathcal{F} \;:\; |X \cap Y| = \ell\}|$$

for **all** $Y \in \mathcal{G}$, $\ell = 0, \ldots, n$ can be found in $O^*(|{\downarrow}\mathcal{F}| + |{\downarrow}\mathcal{G}|)$ time.

With minor modifications to what we have just seen we can show:

Theorem (Björklund, Husfeldt, Kaski, Koivisto 2008)

Given $\mathcal{F}, \mathcal{G} \subseteq 2^U$, and a function $f : \mathcal{F} \to \mathbb{N}$, the values of

$$f\iota(Y, \ell) = \sum_{\substack{X \in \mathcal{F} \\ |X \cap Y| = \ell}} f(X)$$

for **all** $Y \in \mathcal{G}$, $\ell = 0, \ldots, n$ can be found in $O^*(|\downarrow\mathcal{F}| + |\downarrow\mathcal{G}|)$ time.

(By putting $f = \mathbf{1}_{\mathcal{F}}$ we get the previous version.)

Corollary

Given two functions $f, g : \binom{U}{q} \to \mathbb{N}$, we can compute the number

$$f \boxtimes_\ell g = \sum_{\substack{X, Y \in \binom{U}{q} \\ |X \cap Y| = \ell}} f(X) g(Y)$$

for all $\ell = 0, \ldots, n$ in $O^*(n^q)$ time.

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for all $\ell = 0, \ldots, n$ in $O^*(n^q)$ time.

Fast Intersection Trasform

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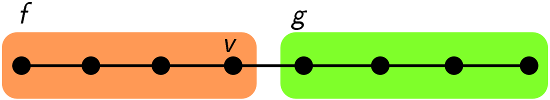
Application: counting k-paths in $O^*(n^{k/2})$ time

Problem

Given a directed graph $G = (V, E)$ count the number of k-vertex paths.

The problem is $\#W[1]$-hard when parameterized by k.



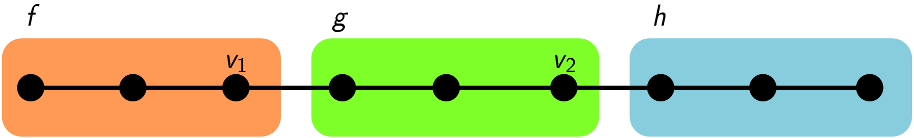
Algorithm (assume w.l.o.g. k is even)

For every $v \in V$ find the number of paths where v is the $k/2$-th vertex:
Define functions $f, g : \binom{V}{k/2} \to \mathbb{N}$

1. $f(S)$ is the number of paths P that end in v and $V(P) = S$;
2. $g(S)$ is the number of paths P that start in v and $V(P) = S \cup \{v\}$.

Compute $f \boxtimes_0 g = \sum_{\substack{X,Y \in \binom{V}{k/2} \\ |X \cap Y| = 0}} f(X)g(Y)$ in $O^*(n^{k/2})$ time.

Counting k-paths by disjoint triples



Algorithm (w.l.o.g. assume k is a multiple of 3)

For every $v_1, v_2 \in V(G)$ count paths where v_i is the $\frac{i}{3}k$-th vertex:

Define functions $f, g, h : \binom{V}{k/3} \to \mathbb{N}$

1. $f(S)$ is the number of paths P that end in v_1 and $V(P) = S$;
2. $g(S)$ is the number of paths P from v_1 to v_2 and $V(P) = S \cup \{v_1\}$.
3. $h(S)$ is the number of paths P that start in v_2 and $V(P) = S \cup \{v_2\}$.

Compute $$\Delta(f,g,h) = \sum_{\substack{A,B,C \in \binom{U}{q} \\ |A\cap B| = |A \cap C| = |B \cap C| = \emptyset}} f(A)g(B)h(C).$$

Counting disjoint triples

Problem (slightly simplified)

Let $q \le |U|/3$. Given $\mathcal{F} \subseteq \binom{U}{q}$, compute

$$x_{3q} = |\{(A, B, C) \in \mathcal{F}^3 \ : \ |A \cap B| = |A \cap C| = |B \cap C| = \emptyset\}|.$$

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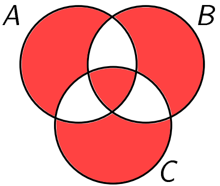
For a triple $(A, B, C) \in \mathcal{F}^3$ define

$$type(A, B, C) = |A \oplus B \oplus C|,$$

where $\oplus$ is the symmetric difference (xor).



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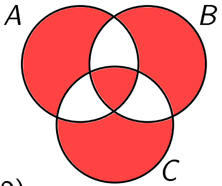

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Note: $|A \oplus B \oplus C| \equiv |A| + |B| + |C| = 3q \equiv q \pmod 2$.

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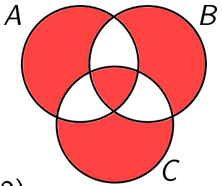

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Auxiliary indeterminates ($\lfloor \frac{3q}{2} \rfloor$ indeterminates in total)

For $j \equiv q \pmod 2$, $j \in \{0, \ldots, 3q\}$,

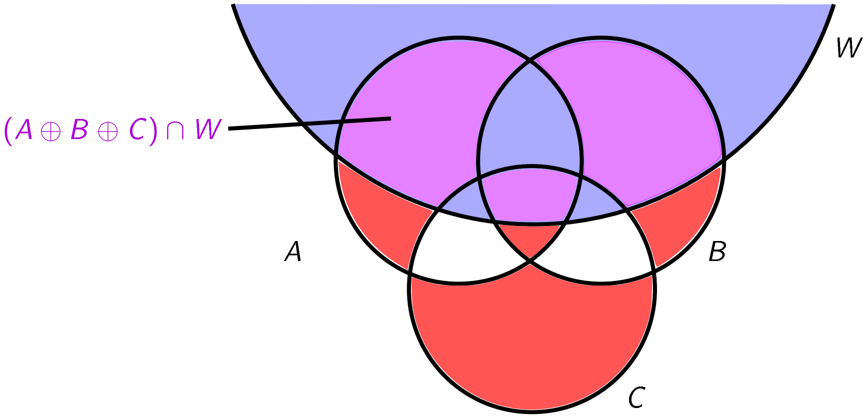
$$x_j = \{(A, B, C) \in \mathcal{F}^3 \ : \ |A \oplus B \oplus C| = j\}$$

First source of linear equations: Intersection Parity Counting

Intersection parity

For $W \in 2^U$ and $p = 0, 1$ let

$$T_p(W) = |\{(A, B, C) \in \mathcal{F}^3 : |(A \oplus B \oplus C) \cap W| \equiv p \pmod 2\}|$$



Linear equations

For $W \in \binom{U}{\leq i}$ and $p = 0, 1$ let

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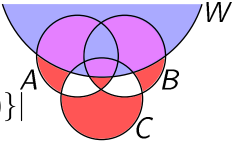


Linear equations

For every $i \geq 1$, $$\sum_{\substack{0 \leq j \leq 3q \\ j \equiv q \pmod 2}} (n-2j)^i x_j = \sum_{d_1, \ldots, d_i \in U} T_0(\oplus\{d_r\}_{r=1}^i) - T_1(\oplus\{d_r\}_{r=1}^i)$$

Proof: Let (A, B, C) be a triple of type j.
We show that (A, B, C) is counted $(n-2j)^i$ times in the RHS.
Define $v_i^p = |\{(d_1, \ldots, d_i) \in U^i : |(A \oplus B \oplus C) \cap \oplus\{d_r\}_{r=1}^i| \equiv p \pmod 2\}|$
Then (A, B, C) is counted $v_i^0 - v_i^1$ times in RHS.

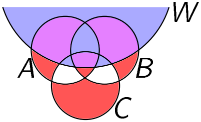
$$\begin{cases} v_i^0 = \overbrace{(n-j)v_{i-1}^0}^{d_i \notin A \oplus B \oplus C} + \overbrace{jv_{i-1}^1}^{d_i \in A \oplus B \oplus C} \\ v_i^1 = jv_{i-1}^0 + (n-j)v_{i-1}^1 \end{cases}$$

$$v_0^0 = 1, \quad v_0^1 = 0.$$

$$\begin{aligned} v_i^0 - v_i^1 &= (n-2j)(v_{i-1}^0 - v_{i-1}^1) \\ &= (n-2j)^i. \end{aligned}$$

Computing $b_i = \sum_{d_1,\dots,d_i \in U} T_0(\oplus\{d_r\}_{r=1}^{i}) - T_1(\oplus\{d_r\}_{r=1}^{i})$ in $O^*(n^i + n^q)$ time

$T_p(W) = |\{(A, B, C) \in \mathcal{F}^3 : |(A \oplus B \oplus C) \cap W| \equiv p \pmod 2\}|$



Note: It suffices to compute $T_p(W)$ for every $W \in \binom{U}{\leq i}$.

$$\begin{aligned}|(A \oplus B \oplus C) \cap W| &= |(A \cap W) \oplus (B \cap W) \oplus (C \cap W)| \\ &\equiv |A \cap W| + |B \cap W| + |C \cap W| \pmod 2\end{aligned}$$

Observation: $|(A \oplus B \oplus C) \cap W| \equiv 0$ iff

- **all** $|A \cap W|$, $|B \cap W|$, $|C \cap W|$ even or
- **exactly one** of $|A \cap W|$, $|B \cap W|$, $|C \cap W|$ even.

Let $n_p(W) = |\{S \in \mathcal{F} : |S \cap W| \equiv p \pmod 2\}|$, for $p = 0, 1$. Then,

$$T_0(W) = n_0(W)^3 + 3n_0(W)n_1(W)^2; \quad T_1(W) = |\mathcal{F}|^3 - T_0(W).$$

Computing $b_i = \sum_{d_1,\dots,d_i \in U} T_0(\oplus\{d_r\}_{r=1}^i) - T_1(\oplus\{d_r\}_{r=1}^i)$ in $O^*(n^i + n^q)$ time

- It suffices to compute $T_p(W)$ for every $W \in \binom{U}{\leq i}$.
- We showed

$$T_0(W) = n_0(W)^3 + 3n_0(W)n_1(W)^2; \quad T_1(W) = |\mathcal{F}|^3 - T_0(W),$$

  where Let $n_p(W) = |\{S \in \mathcal{F} : |S \cap W| \equiv p \pmod 2\}|$ for $p = 0, 1$.
- $$n_p(W) = \sum_{j \equiv p} |\{S \in \mathcal{F} : |S \cap W| = j\}|$$
- We find $\iota_{\mathcal{F}}(W, j) = |\{S \in \mathcal{F} : |S \cap W| = j\}|$ for every j and $W \in \binom{n}{\leq i}$ in $O^*(|\downarrow\mathcal{F}| + |\downarrow\binom{n}{\leq i}|) = O^*(n^q + n^i)$ time using Fast Intersection Transform.
- From the values of $\iota_{\mathcal{F}}(W, j)$ we can compute any value of $n_p(W)$ in $O^*(1)$ time, so b_i can be found in $O(n^i)$ time.

First source of linear equations: Summary

Corollary

The coefficients / constant term of the equation:

$$\sum_{\substack{0\le j\le 3q \\ j\equiv q \pmod 2}} (n-2j)^i x_j = \sum_{d_1,\ldots,d_i\in U} T_0(\oplus\{d_r\}_{r=1}^i) - T_1(\oplus\{d_r\}_{r=1}^i)$$

can be computed in $O^*(n^i + n^q)$ time, for any $i \geq 0$.

Observation

For the k-path application, $q = k/3$;
If we use only the first source we need $\lfloor \frac{3q}{2} \rfloor + 1 = \lfloor k/2 \rfloor + 1$ equations,
which results in total $O^*(n^{k/2})$ time.

Second source of linear equations: computing x_j for small j

Computing x_j for small j: summing over all possible $A \oplus B$

Consider a triple (A, B, C) of type j. Let $\ell = |A \oplus B|$.

Since $|A \oplus B| = |\overbrace{A \oplus B \oplus C}^{j} \oplus \overbrace{C}^{q}|$,

$$q - j \leq \ell \leq q + j$$

Note that $\ell = |A| + |B| - 2|A \cap B| = 2q - 2|A \cap B| \equiv 0 \pmod 2$.

Since $\overbrace{|A \oplus B \oplus C|}^{j} = \overbrace{|A \oplus B|}^{\ell} + \overbrace{|C|}^{q} - 2|(A \oplus B) \cap C|$,

$$|(A \oplus B) \cap C| = \tfrac{\ell+q-j}{2}$$

$$x_j = \sum_{\substack{q-j\leq\ell\leq q+j \\ \ell\equiv 0 \pmod 2}} \sum_{D\in\binom{U}{\ell}} |\oplus^{-1}(D)| \cdot |\{C \in \mathcal{F} \ :\ |D \cap C| = \tfrac{\ell+q-j}{2}\}|,$$

where $\oplus^{-1}(D) = \{(A, B) \in \mathcal{F}^2 \ :\ A \oplus B = D\}$

Computing x_j for small j: summing over all possible $A \oplus B$

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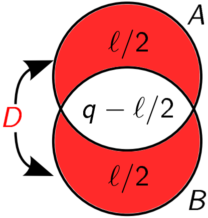
- $|\{C \in \mathcal{F} \ : \ |D \cap C| = \frac{\ell+q-j}{2}\}| = \iota_{\mathcal{F}}(D, \frac{\ell+q-j}{2})$ can be computed for all $D \in \binom{U}{\ell}$ in $O^*(|\downarrow\binom{U}{\ell}| + |\downarrow\mathcal{F}|) = O^*(n^{q+j})$ time using fast intersction transform.
- How fast can we compute $\oplus^{-1}$?

Computing $|\oplus^{-1}(D)| = |\{(A,B) \in \mathcal{F}^2 \;:\; A \oplus B = D\}|$



Let M be a matrix with

- rows indexed by sets $S \in \binom{U}{\ell/2}$,
- columns indexed by sets $X \in \binom{U}{q-\ell/2}$,
- $M_{SX} = [S \cup X \in \mathcal{F}]$.

Let $B = MM^T$. B is indexed by sets $S \in \binom{U}{\ell/2}$.

$$B_{RS} = \sum_{X \in \binom{U}{q-\ell/2}} [R \cup X \in \mathcal{F}] \cdot [S \cup X \in \mathcal{F}] = \left|\left\{X \in \binom{U}{q-\ell/2} \;:\; R \cup X, S \cup X \in \mathcal{F}\right\}\right|.$$

Then, $$|\oplus^{-1}(D)| = \sum_{R \cup S = D} B_{RS}$$

- B can be computed in $O(\max\{n^{(\omega-2)\ell/2+q}, n^{\omega\ell/2}\})$ time.
- Hence, within the same time we can find $|\oplus^{-1}(D)|$ for **all** $D \in \binom{U}{\ell}$.

Computing x_j for small j: summing over all possible $A \oplus B$

$$x_j = \sum_{\substack{q-j \le \ell \le q+j \\ \ell \equiv 0 \pmod 2}} \sum_{D \in \binom{U}{\ell}} |\oplus^{-1}(D)| \cdot |\{C \in \mathcal{F} \;:\; |D \cap C| = \tfrac{\ell+q-j}{2}\}|,$$

where $\oplus^{-1}(D) = \{(A,B) \in \mathcal{F}^2 \;:\; A \oplus B = D\}$

- $|\{C \in \mathcal{F} \;:\; |D \cap C| = \frac{\ell+q-j}{2}\}|$ can be computed for all $D \in \binom{U}{\ell}$ in $O^*(|\downarrow\binom{U}{\ell}| + |\downarrow\mathcal{F}|) = O^*(n^{q+j})$ time using fast intersction transform.
- $|\oplus^{-1}(D)|$ can be computed in

  $$O(\max\{n^{(\omega-2)\ell/2+q}, n^{\omega\ell/2}\}) = O(\max\{n^{(\omega-2)(q+j)/2+q}, n^{\omega(q+j)/2}\}) = O(\max\{n^{\omega(q+j)/2-j}, n^{\omega(q+j)/2}\}) = O(n^{\omega(q+j)/2})$$

  time for **all** $D \in \binom{U}{\ell}$.
- Overall, x_j can be computed in $O(n^{\omega(q+j)/2})$ time.

Second source of linear equations: Summary

Corollary

The constant term of the equation:

$$x_j = \sum_{\substack{q-j \le \ell \le q+j \\ \ell \equiv 0 \pmod 2}} \sum_{D \in \binom{U}{\ell}} |\oplus^{-1}(D)| \cdot |\{C \in \mathcal{F} \,:\, |D \cap C| = \tfrac{\ell+q-j}{2}\}|$$

can be computed in $O(n^{\omega(q+j)/2})$ time, for any $j = 0, \ldots, \lfloor \frac{3q}{2} \rfloor$, $j \equiv q$.

Setting up the system if linear equations

- Pick r equations from the first source :

$$\sum_{\substack{0\le j\le 3q \\ j\equiv q \pmod 2}} (n-2j)^i x_j = \sum_{d_1,\ldots,d_i\in U} T_0(\oplus\{d_r\}_{r=1}^i) - T_1(\oplus\{d_r\}_{r=1}^i); \quad i=0,\ldots,r-1$$

in $\sum_{i=0}^{r-1} O^*(n^i + n^q) = O^*(n^r + n^q)$ time;

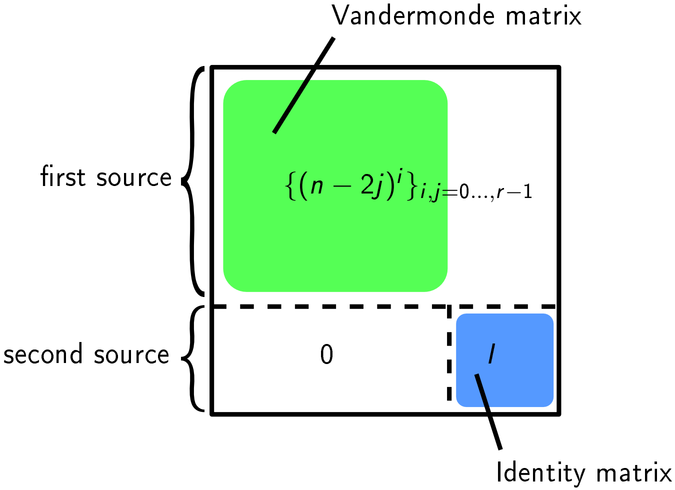
- Pick $\lfloor \frac{3q}{2} \rfloor + 1 - r$ equations from the second source:

$$x_j = \sum_{\substack{q-j\le \ell\le q+j \\ \ell\equiv 0 \pmod 2}} \sum_{D\in\binom{U}{\ell}} |\oplus^{-1}(D)| \cdot \iota_{\mathcal{F}}(D, \tfrac{l+q-j}{2}), \qquad j\equiv q$$

in $O^*(n^{\omega(q+2(\frac{3q}{2}-r))/2}) = O^*(n^{\omega(2q-r)})$ time;

- Both running times meet at $r = \frac{2\omega q}{1+\omega} \approx 1.408q$

The missing piece: linear independence



Conclusion

Corollary

One can count disjoint triples of a family of q-subsets of n-element universe in $O^*(n^{1.408q})$ time.

By essentialy the same arguments we can get...

Corollary

One can compute $\Delta(f,g,h) = \sum_{\substack{A,B,C \in \binom{U}{q} \\ |A \cap B| = |A \cap C| = |B \cap C| = \emptyset}} f(A)g(B)h(C)$ in $O^*(n^{1.408q})$ time.

Corollary

One can count the number of k-paths in an n-vertex graph in $O^*(n^{0.47k})$ time.

After some improvements...

Theorem (Björklund, Kaski, K. 2013)

- One can count disjoint triples of a family of q-subsets of n-element universe in $O^*(n^{1.364q})$ time.
- One can compute $\Delta(f,g,h) = \sum_{\substack{A,B,C \in \binom{U}{q} \\ |A\cap B|=|A\cap C|=|B\cap C|=\emptyset}} f(A)g(B)h(C)$ in $O^*(n^{1.364q})$ time.
- One can count the number of k-paths in an n-vertex graph in $O^*(n^{0.455k})$ time.
- One can count the number of occurences of a fixed k-vertex pathwidth p subgraph in an n-vertex graph in $O^*(n^{0.455k+2p})$ time.