ADFOCS 2013
Parameterized Algorithms using Matroids – Exercise I

August 6th 2013

Iterative Compression

1. In the Vertex Cover problem, we are given a graph $G = (V, E)$ and a positive integer k, and the problem is to test whether there exists a vertex subset $X \subseteq V(G)$ such that $|X| \leq k$ and $G \setminus X$ is an independent set. Obtain a $2^k n^{\mathcal{O}(1)}$-time algorithm for this problem using iterative compression.

2. In the Feedback Vertex Set problem, we are give a graph $G = (V, E)$ and a positive integer k, and the problem is to test whether there exists a vertex subset $X \subseteq V(G)$ such that $|X| \leq k$ and $G \setminus X$ is a forest.

   In the following steps, we design an algorithm for this problem with running time $5^k n^{\mathcal{O}(1)}$ using the method of iterative compression.

   (a) Consider an iterative *compression step*. Here, we are given a feedback vertex set, say F, of size $k + 1$, and the objective is to find another feedback vertex set $X \subseteq V$ such that $X \cap F = \emptyset$ and $|X| \leq k$.

   - Devise reduction rules such that every vertex in $V \setminus F$ either has degree at least 3 or has at least two neighbors in F.
   - Let $\mu = k + \gamma$, where γ is the number of connected components of $G[F]$. Using μ as a measure devise a branching algorithm to find the desired X (if exists) in time $4^k n^{\mathcal{O}(1)}$. Hint: Branch on a vertex of degree at most 1 of $G[V \setminus F]$.

   (b) Use the previous step to design a $5^k n^{\mathcal{O}(1)}$-time algorithm for Feedback Vertex Set.

3. Let $G = (V, E)$ be a graph and $Q \subseteq V$ be such that $G \setminus Q$ is bipartite with color classes A, B. Then, show that the size of the minimum odd cycle transversal is the minimum over all partitions $Q = L \cup R \cup C$ of the following value:

$$|C| + \underset{G' \setminus (C_A \cup C_B)}{\text{mincut}}((R_A \cup L_B), (L_A \cup R_B))$$

   Here, G' has been obtained from G as follows. Vertices in G' are $A \cup B \cup Q_A \cup Q_B$. Edges within $G'[A \cup B]$ are same as in G, while for $q \in Q$ a vertex q_a is connected to $N_G(q) \cap A$ and q_b to $N_G(q) \cap B$.

Matroid Basics

1. Show that the following families form matroid.

   (a) Let $G = (V, E)$ be a graph. Let $M = (U, \mathcal{I})$ be a matroid defined on G, where $U = E$ and $\mathcal{I}$ contains all *forests* of G. (**Graphic Matroid**)

   (b) Let $G = (V, E)$ be a connected graph. Let $M = (U, \mathcal{I})$ be a matroid defined on G, where $U = E$ and $\mathcal{I}$ contains all $E' \subseteq E$ such that $G' = (V, E \setminus E')$ is connected. (**Co-Graphic Matroid**)

2. Obtain a representation matrix for the following matroid.

   (a) Graphic Matroid.

   (b) Uniform Matroids – $M = (U, \mathcal{I})$ where $\mathcal{I}$ contains all subsets of U of size at most k for some fixed constant k.

   (c) Partition Matroids – It is defined by a ground set U being partitioned into (disjoint) sets $U_1, \ldots, U_\ell$ and by ℓ non-negative integers $k_1, \ldots, k_\ell$. A set $X \subseteq U$ is independent if and only if $|X \cap U_i| \leq k_i$ for all $i \in \{1, \ldots, \ell\}$. That is,

   $$\mathcal{I} = \Big\{X \subseteq U \mid |X \cap U_i| \leq k_i,\ i \in \{1, \ldots, \ell\}\Big\}.$$

   (d) Direct Sum of Matroids – Let $M_1 = (U_1, \mathcal{I}_1)$, $M_2 = (U_2, \mathcal{I}_2)$, $\cdots$, $M_t = (U_t, \mathcal{I}_t)$ be t matroids with $U_i \cap U_j = \emptyset$ for all $1 \leq i \neq j \leq t$. The direct sum $M_1 \oplus \cdots \oplus M_t$ is a matroid $M = (U, \mathcal{I})$ with $U := \bigcup_{i=1}^{t} U_i$ and $X \subseteq U$ is independent if and only if for all $i \leq t$, $X \cap U_i \in \mathcal{I}_i$.

3. Let $M_1 = (U_1, \mathcal{I}_1)$ and $M_2 = (U_2, \mathcal{I}_2)$ be two matroids such that $U = U_1 = U_2$. Define $M_1 \cap M_2$ as $M = (U, \mathcal{I})$ such that $X \in \mathcal{I}$ if and only if $X \in \mathcal{I}_1$ and $X \in \mathcal{I}_2$. Is M always a matroid? (**Matroid Intersection**)

4. Express the following as intersection of matroids (possibly more than two).

   (a) Finding a maximum matching in a bipartite graph $G = (A \cup B, E)$.

   (b) Testing whether a graph $G = (V, E)$ contains two edge disjoint spanning trees.

   (c) Finding a hamiltonian path in a directed graph $D = (V, A)$ between a pair of vertices s and t of D.