Parameterized Algorithms using Matroids

Lecture I: Matroid Basics and its use as data structure

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Introduction and Kernelization

Fixed Parameter Tractable (FPT) Algorithms

For decision problems with input size n, and a parameter k, (which typically is the solution size), the goal here is to design an algorithm with running time $f(k) \cdot n^{\mathcal{O}(1)}$, where f is a function of k alone.

Problems that have such an algorithm are said to be fixed parameter tractable (FPT).

A Few Examples

VERTEX COVER
Input: A graph $G = (V, E)$ and a positive integer k.
Parameter: k
Question: Does there exist a subset $V' \subseteq V$ of size at most k such that for every edge $(u, v) \in E$ either $u \in V'$ or $v \in V'$?

PATH
Input: A graph $G = (V, E)$ and a positive integer k.
Parameter: k
Question: Does there exist a path P in G of length at least k?

Kernelization: A Method for Everyone

INFORMALLY: A kernelization algorithm is a polynomial-time transformation that transforms any given parameterized instance to an equivalent instance of the same problem, with size and parameter bounded by a function of the parameter.

Kernel: Formally

FORMALLY: A kernelization algorithm, or in short, a kernel for a parameterized problem $L \subseteq \Sigma^* \times \mathbb{N}$ is an algorithm that given $(x, k) \in \Sigma^* \times \mathbb{N}$, outputs in $p(|x| + k)$ time a pair $(x', k') \in \Sigma^* \times \mathbb{N}$ such that

- $(x, k) \in L \iff (x', k') \in L$,
- $|x'|, k' \leq f(k)$,

where f is an arbitrary computable function, and p a polynomial. Any function f as above is referred to as the size of the kernel.

Polynomial kernel $\implies$ f is polynomial.

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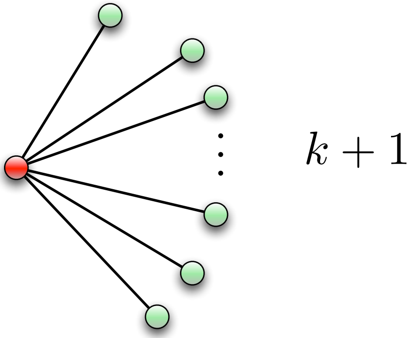

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Apply these rules until no longer possible.

What conclusions can we draw ?

Outcome 1: If G is not empty and k drops to 0 — the answer is NO.

Observation: Every vertex has degree at most k — number of edges they can cover is at most k^2.

Outcome 2: If $|E| > k^2$ — the answer is NO. Else $|E| \leq k^2$, $|V| \leq 2k^2$ and we have polynomial sized kernel of $O(k^2)$.

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Iterative Compression and Odd Cycle Transversal

Result from
Bruce A. Reed, Kaleigh Smith, Adrian Vetta: Finding odd cycle transversals. Operation Resarch Letters 32(4): 299-301 (2004)

Iterative compression

- A surprisingly small, but very powerful trick.
- Most useful for deletion problems: delete k things to achieve some property.
- Demonstration: Odd Cycle Transversal aka Bipartite Deletion aka Graph Bipartization: Given a graph G and an integer k, delete k vertices to make the graph bipartite.
- Forbidden induced subgraphs: odd cycles. There is no bound on the size of odd cycles.

Odd Cycle Transversal

ODD CYCLE TRANSVERSAL
Input: A graph $G = (V, E)$ and a positive integer k.
Parameter: k
Question: Does there exist a subset $V' \subseteq V$ of size at most k such that $G \setminus V'$ is bipartite?

Odd Cycle Transversal

Solution based on iterative compression:

- Step 1: Solve the **annotated problem** for bipartite graphs:

  Given a bipartite graph G, two sets $B, W \subseteq V(G)$, and an integer k, find a set S of at most k vertices such that $G \setminus S$ has a 2-coloring where $B \setminus S$ is black and $W \setminus S$ is white.

- Step 2: Solve the **compression problem** for general graphs:

  Given a graph G, an integer k, and a set Q of $k+1$ vertices such that $G \setminus Q$ is bipartite, find a set S of k vertices such that $G \setminus S$ is bipartite.

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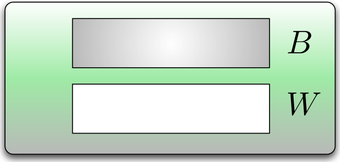

Find an arbitrary 2-coloring (B_0, W_0) of G.
$C := (B_0 \cap W) \cup (W_0 \cap B)$ should change color, while
$R := (B_0 \cap B) \cup (W_0 \cap W)$ should remain the same color.

Lemma: $G \setminus S$ has the required 2-coloring if and only if S separates C and R, i.e., no component of $G \setminus S$ contains vertices from both $C \setminus S$ and $R \setminus S$.

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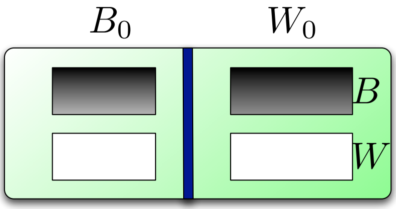

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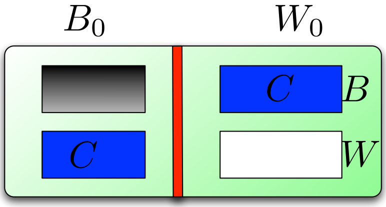

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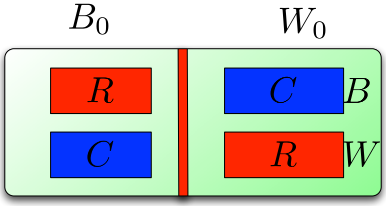

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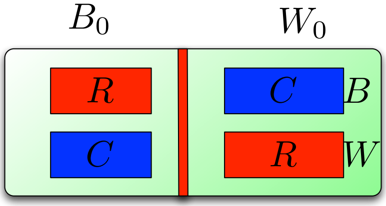

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Proof:

$\implies$ In a 2-coloring of $G \setminus S$, each vertex either remained the same color or changed color. Adjacent vertices do the same, thus every component either changed or remained.

$\Leftarrow$ Flip the coloring of those components of $G \setminus S$ that contain vertices from $C \setminus S$. No vertex of R is flipped.

Algorithm: Using max-flow min-cut techniques, we can check if there is a set S that separates C and R. It can be done in time $O(k|E(G)|)$ using k iterations of the Ford-Fulkerson algorithm.

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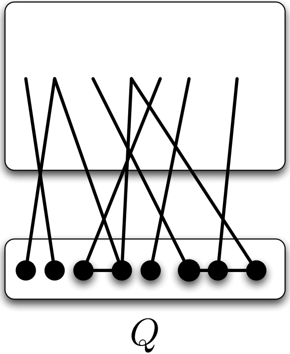

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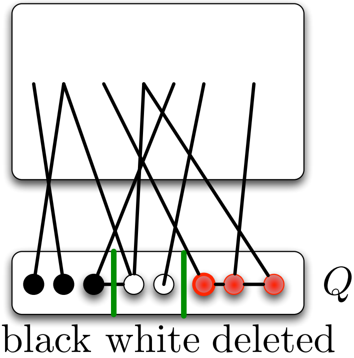

Branch into 3^{k+1} cases: each vertex of Q is either black, white, or deleted. Trivial check: no edge between two black or two white vertices.

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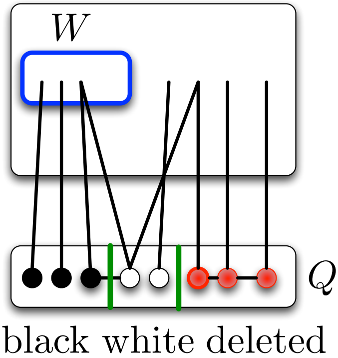

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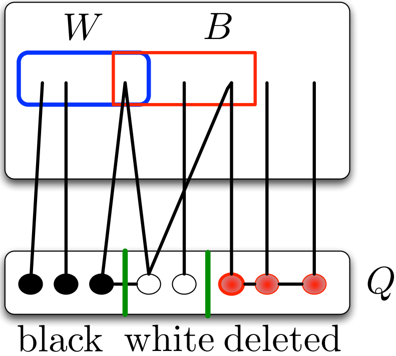

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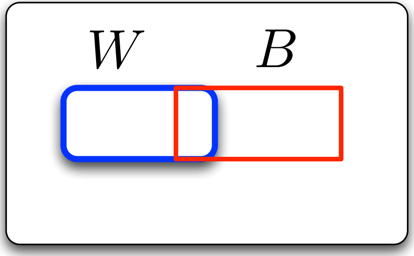

The vertices of Q can be disregarded. Thus we need to solve the annotated problem on the bipartite graph $G \setminus Q$.

Running time: $O(3^k \cdot k|E(G)|)$ time.

Step 3: Iterative compression

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Let $V(G) = \{v_1, \ldots, v_n\}$ and let G_i be the graph induced by $\{v_1, \ldots, v_i\}$.

For every i, we find a set S_i of size k such that $G_i \setminus S_i$ is bipartite.

- For G_k, the set $S_k = \{v_1, \ldots, v_k\}$ is a trivial solution.
- If S_{i-1} is known, then $S_{i-1} \cup \{v_i\}$ is a set of size $k+1$ whose deletion makes G_i bipartite $\Longrightarrow$ We can use the compression algorithm to find a suitable S_i in time $O(3^k \cdot k|E(G_i)|)$.

Step 3: Iterative Compression

Bipartite-Deletion(G, k)

1. $S_k = \{v_1, \ldots, v_k\}$
2. for $i := k + 1$ to n
3. Invariant: $G_{i-1} \setminus S_{i-1}$ is bipartite.
4. Call Compression($G_i, S_{i-1} \cup \{v_i\}$)
5. If the answer is "NO" $\implies$ return "NO"
6. If the answer is a set $X \implies S_i := X$
7. Return the set S_n

Running time: the compression algorithm is called n times and everything else can be done in linear time.
$\implies O(3^k \cdot k|V(G)| \cdot |E(G)|)$ time algorithm.

Useful Reformulation of the Algorithm

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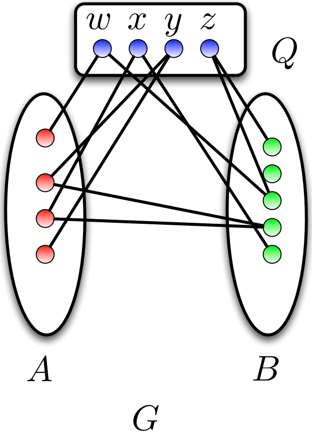

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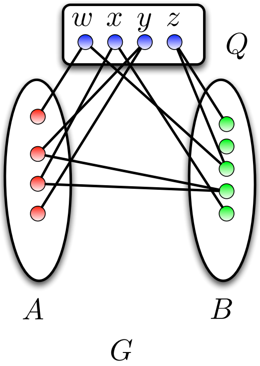
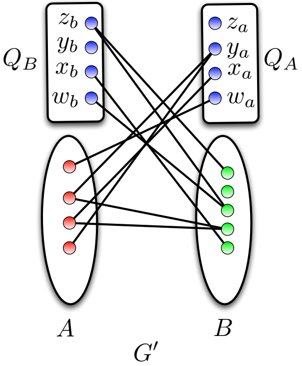
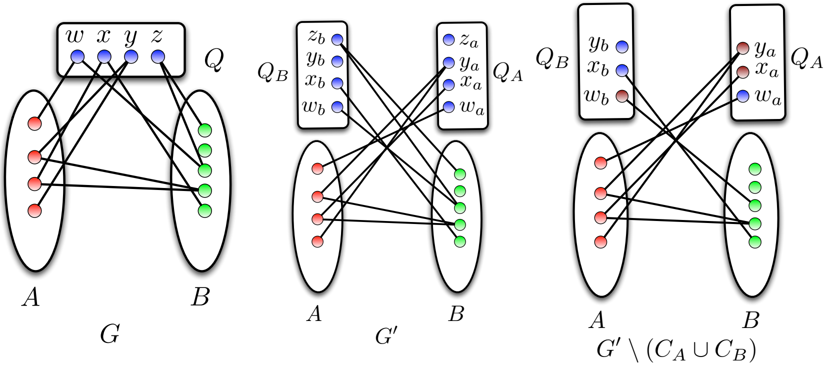




- Vertices in G' are $A \cup B \cup Q_A \cup Q_B$. Edges within $G'[A \cup B]$ are as in G, while for $q \in Q$ a vertex q_a is connected to $N_G(q) \cap A$ and q_b to $N_G(q) \cap B$.

For a partition $Q = L \cup R \cup C$ we are going to compute the minimum $(R_A \cup L_B)$, $(L_A \cup R_B)$-cut in $G' \setminus (C_A \cup C_B)$.

Example



For $L = \{w\}, R = \{x, y\}, C = \{z\} \implies L_A \cup R_B = \{w_a, x_b, y_b\}$ and $L_B \cup R_A = \{w_b, x_a, y_a\}$ and $C_A \cup C_B = \{z_a, z_b\}$

Want to compute cut between $L_A \cup R_B = \{w_a, x_b, y_b\}$ and $L_B \cup R_A = \{w_b, x_a, y_a\}$ in $G' \setminus (C_A \cup C_B)$.

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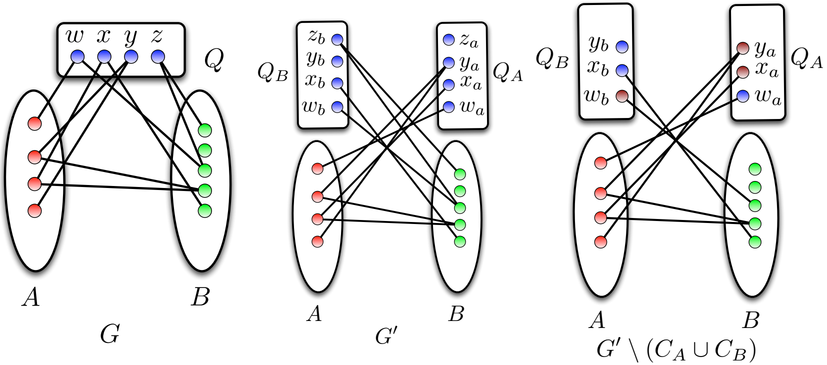

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Final Result

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Lemma: Let $G = (V, E)$ be a graph and $Q \subseteq V$ be such that $G \setminus Q$ is bipartite with color classes A, B. Then, the size of the minimum odd cycle transversal is the minimum over all partitions $Q = L \cup R \cup C$ of the following value:

$$|C| + \operatorname*{mincut}_{G' \setminus (C_A \cup C_B)} ((R_A \cup L_B), (L_A \cup R_B))$$

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Final Result: Restated

Let S, T and R be a partition of $Q_A \cup Q_B$. We say that (S, T, Z) is a *valid* partition if for all $x \in Q$ either

- $|\{x_1, x_2\} \cap S| = |\{x_1, x_2\} \cap T| = 1$; or
- $|\{x_1, x_2\} \cap Z| = 2$.

Lemma: Let $G = (V, E)$ be a graph and $Q \subseteq V$ be such that $G \setminus Q$ is bipartite with color classes A, B. Then, the size of the minimum odd cycle transversal is the minimum over all valid partitions of $Q_A \cup Q_B = S \cup T \cup Z$ of the following value:

$$\frac{|Z|}{2} + \underset{G' \setminus Z}{\text{mincut}}(S, T)$$

Final Result: Restated

Let S, T and R be a partition of $Q_A \cup Q_B$. We say that (S, T, Z) is a *valid* partition if for all $x \in Q$ either

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Matroids and its Representation

Definition

A pair $M = (E, \mathcal{I})$, where E is a ground set and $\mathcal{I}$ is a family of subsets (called independent sets) of E, is a *matroid* if it satisfies the following conditions:

(I1) $\emptyset \in \mathcal{I}$ or $\mathcal{I} \neq \emptyset$.
(I2) If $A' \subseteq A$ and $A \in \mathcal{I}$ then $A' \in \mathcal{I}$.
(I3) If $A, B \in \mathcal{I}$ and $|A| < |B|$, then $\exists\ e \in (B \setminus A)$ such that $A \cup \{e\} \in \mathcal{I}$.

The axiom (I2) is also called the *hereditary property* and a pair $M = (E, \mathcal{I})$ satisfying (I1) and (I2) is called *hereditary family* or *set-family*.

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Rank and Basis

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An inclusion wise maximal set of $\mathcal{I}$ is called a *basis* of the matroid. Using axiom (I3) it is easy to show that all the bases of a matroid have the same size. This size is called the *rank* of the matroid M, and is denoted by $\text{rank}(M)$.

Examples Of Matroids

Uniform Matroid

A pair $M = (E, \mathcal{I})$ over an n-element ground set E, is called a *uniform matroid* if the family of independent sets is given by

$$\mathcal{I} = \Big\{ A \subseteq E \mid |A| \leq k \Big\},$$

where k is some constant. This matroid is also denoted as $U_{n,k}$.
Eg: $E = \{1, 2, 3, 4, 5\}$ and $k = 2$ then

$$\begin{aligned} \mathcal{I} \;\; = \;\; & \Big\{ \{\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,2\}, \{1,3\}, \{1,4\}, \\ & \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\} \Big\} \end{aligned}$$

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Partition Matroid

A *partition matroid* $M = (E, \mathcal{I})$ is defined by a ground set E being partitioned into (disjoint) sets $E_1, \ldots, E_\ell$ and by ℓ non-negative integers $k_1, \ldots, k_\ell$. A set $X \subseteq E$ is independent if and only if $|X \cap E_i| \leq k_i$ for all $i \in \{1, \ldots, \ell\}$. That is,

$$\mathcal{I} = \left\{ X \subseteq E \mid |X \cap E_i| \leq k_i,\ i \in \{1, \ldots, \ell\} \right\}.$$

- If $X, Y \in \mathcal{I}$ and $|Y| > |X|$, there must exist i such that $|Y \cap E_i| > |X \cap E_i|$ and this means that adding any element e in $E_i \cap (Y \setminus X)$ to X will maintain independence.
- M in general would not be a matroid if E_i were not disjoint. Eg: $E_1 = \{1, 2\}$ and $E_2 = \{2, 3\}$ and $k_1 = 1$ and $k_2 = 1$ then both $Y = \{1, 3\}$ and $X = \{2\}$ have at most one element of each E_i but one can't find an element of Y to add to X.

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Graphic Matroid

Given a graph G, a *graphic matroid* is defined as $M = (E, \mathcal{I})$ where and

- $E = E(G)$ – edges of G are elements of the matroid
- $$\mathcal{I} = \Big\{ F \subseteq E(G) : F \text{ is a forest in the graph } G \Big\}$$

Co-Graphic Matroid

Given a graph G, a *co-graphic matroid* is defined as $M = (E, \mathcal{I})$ where and

- $E = E(G)$ – edges of G are elements of the matroid
- $$\mathcal{I} = \left\{ S \subseteq E(G) : G \setminus S \text{ is connected} \right\}$$

Direct Sum

Let $M_1 = (E_1, \mathcal{I}_1)$, $M_2 = (E_2, \mathcal{I}_2)$, $\cdots$, $M_t = (E_t, \mathcal{I}_t)$ be t matroids with $E_i \cap E_j = \emptyset$ for all $1 \leq i \neq j \leq t$.

The direct sum $M_1 \oplus \cdots \oplus M_t$ is a matroid $M = (E, \mathcal{I})$ with $E := \bigcup_{i=1}^{t} E_i$ and $X \subseteq E$ is independent if and only if for all $i \leq t$, $X \cap E_i \in \mathcal{I}_i$.

$$\mathcal{I} = \left\{ X \mid X \subseteq E,\ (X \cap E_i) \in \mathcal{I}_i,\ i \in \{1, \ldots, t\} \right\}$$

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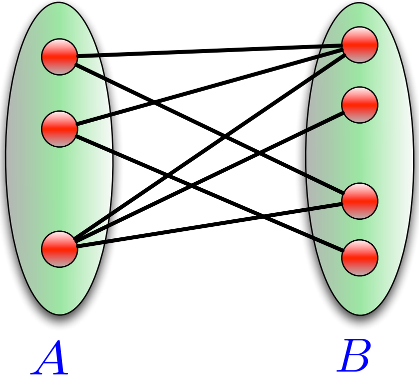
Transversal Matroid

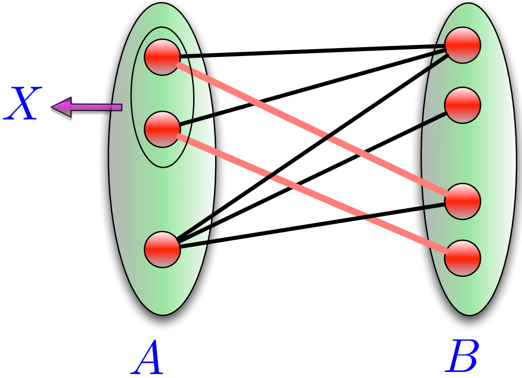
Let G be a bipartite graph with the vertex set $V(G)$ being partitioned as A and B.



Transversal Matroid

Let G be a bipartite graph with the vertex set $V(G)$ being partitioned as A and B. The *transversal matroid* $M = (E, \mathcal{I})$ of G has $E = A$ as its ground set,

$$\mathcal{I} = \Big\{ X \mid X \subseteq A, \text{ there is a matching that covers } X \Big\}$$



Gammoids

Let $D = (V, A)$ be a directed graph, and let $S \subseteq V$ be a subset of vertices. A subset $X \subseteq V$ is *said to be linked to* S if there are $|X|$ vertex disjoint paths going from S to X.

The paths are disjoint, not only internally disjoint. Furthermore, zero-length paths are also allowed if $X \cap S = \emptyset$.

Given a digraph $D = (V, A)$ and subsets $S \subseteq V$ and $T \subseteq V$, a *gammoid* is a matroid $M = (E, \mathcal{I})$ with $E = T$ and

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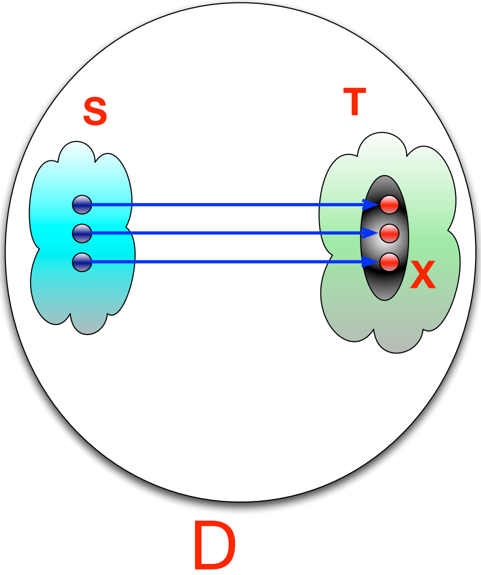
Gammoid: Example



Strict Gammoids

Given a digraph $D = (V, A)$ and subset $S \subseteq V$, a *strict gammoid* is a matroid $M = (E, \mathcal{I})$ with $E = V$ and

$$\mathcal{I} = \Big\{ X \mid X \subseteq V \text{ and } X \text{ is linked to } S \Big\}$$

Matroid Representation

Remark

- Need a compact representation for the family of independent sets.
- Also should be able to test easily, whether a set belongs to the family of independent sets.

Linear Matroid

Let A be a matrix over an arbitrary field $\mathbb{F}$ and let E be the set of columns of A. Given A we define the matroid $M = (E, \mathcal{I})$ as follows.
A set $X \subseteq E$ is independent (that is $X \in \mathcal{I}$) if the corresponding columns are *linearly independent* over $\mathbb{F}$.

$$A = \begin{bmatrix} * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * \end{bmatrix} \quad * \text{ are elements of } \mathbb{F}$$

The matroids that can be defined by such a construction are called *linear matroids*.

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Linear Matroids and Representable Matroids

If a matroid can be defined by a matrix A over a field $\mathbb{F}$, then we say that the matroid is *representable* over $\mathbb{F}$.

Linear Matroids and Representable Matroids

A matroid $M = (E, \mathcal{I})$ is representable over a field $\mathbb{F}$ if there exist vectors in $\mathbb{F}^\ell$ that correspond to the elements such that the linearly independent sets of vectors precisely correspond to independent sets of the matroid.
Let $E = \{e_1, \ldots, e_m\}$ and ℓ be a positive integer.

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ \ell \end{array}
\begin{array}{c}
\begin{array}{ccccc} e_1 & e_2 & e_3 & \cdots & e_m \end{array} \\
\left[\begin{array}{ccccc}
* & * & * & \cdots & * \\
* & * & * & \cdots & * \\
* & * & * & \cdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
* & * & * & \cdots & *
\end{array}\right]_{\ell \times m}
\end{array}
$$

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* & * & * & \cdots & * \\
* & * & * & \cdots & * \\
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* & * & * & \cdots & *
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Let $M = (E, \mathcal{I})$ be linear matroid and Let $E = \{e_1, \ldots, e_m\}$ and d=rank(M).

We can always assume (using Gaussian Elimination) that the matrix has following form:

$$\left[\begin{array}{c|c} I_{d\times d} & D \end{array}\right]_{d\times m}$$

Here $I_{d\times d}$ is a $d \times d$ identity matrix.

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$$a_{ij} = \begin{cases} z_{ij} & \text{if there is an edge between } a_i \text{ and } b_j, \\ 0 & \text{otherwise.} \end{cases}$$

where z_{ij} are in-determinants. Think of them as independent variables.

$$T = \begin{array}{c} \\ b_1 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{array} \begin{array}{c} \begin{array}{cccccc} a_1 & a_2 & \cdots & a_j & \cdots & a_\ell \end{array} \\ \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1j} & \cdots & z_{1\ell} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ z_{i1} & z_{i2} & \cdots & z_{ij} & \cdots & z_{i\ell} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{nj} & \cdots & z_{n\ell} \end{bmatrix} \end{array}$$

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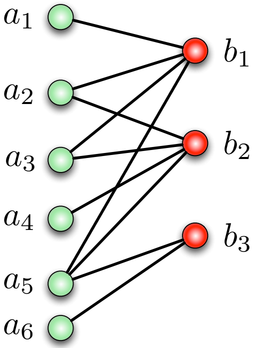
Example of the Construction



$$
\begin{array}{c c}
 & \begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{array} \\
\begin{array}{c} b_1 \\ b_2 \\ b_3 \end{array} &
\begin{bmatrix}
z_{11} & z_{12} & z_{13} & 0 & z_{15} & 0 \\
0 & z_{22} & z_{23} & z_{24} & z_{25} & 0 \\
0 & 0 & 0 & 0 & z_{35} & z_{36}
\end{bmatrix}
\end{array}
$$

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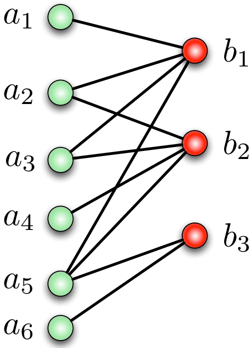

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$$

Permutation expansion of Determinants

THEOREM: Let

$$A = (a_{ij})_{n \times n}$$

be a $n \times n$ matrix with entries in $\mathbb{F}$. Then

$$\det(A) = \sum_{\pi \in S_n} \mathrm{sgn}(\pi) \prod_{i=1}^{n} a_{i\pi(i)}.$$

Proof that Transversal Matroid is Representable over $F[\vec{z}]$

Forward direction: (Board for Picture)

- Suppose some subset $X = \{a_1, \ldots, a_q\}$ is independent.
- Then there is a matching that saturates X. Let $Y = \{b_1, b_2, \ldots, b_q\}$ be the endpoints of this matching and $a_i b_i$ are the matching edges.

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- Then there is a matching that saturates X. Let $Y = \{b_1, b_2, \ldots, b_q\}$ be the endpoints of this matching and $a_i b_i$ are the matching edges.
- Consider $T[Y, X]$ – a submatrix with rows in Y and columns in X. Consider the determinant of $T[Y, X]$ then we have a term

$$\prod_{i=1}^{q} z_{ii}$$

which can not be cancelled by any other term! So these columns are linearly independent.

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Reverse direction: (Board for Picture)

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  and this gives us that there is a matching that saturates X in and hence X is independent.

- For this direction we do not use z_{ij}, the very fact that X forms independent set of column is enough to argue that there is a matching that saturates X.

Removing z_{ij}

To remove the z_{ij} we do the following.

Uniformly at random assign z_{ij} from values in finite field $\mathbb{F}$ of size P.

What should be the upper bound on P? What is the probability that the randomly obtained T is a representation matrix for the transversal matroid.

Removing z_{ij}

To remove the z_{ij} we do the following.

> *Uniformly at random assign z_{ij} from values in finite field $\mathbb{F}$ of size P.*

What should be the upper bound on P? What is the probability that the randomly obtained T is a representation matrix for the transversal matroid.

Using Zippel-Schwartz Lemma

THEOREM: Let $p(x_1, x_2, \ldots, x_n)$ be a non-zero polynomial of degree d over some field $\mathbb{F}$ and let S be an N element subset of $\mathbb{F}$. If each x_i is independently assigned a value from S with uniform probability, then $p(x_1, x_2, \ldots, x_n) = 0$ with probability at most $\frac{d}{N}$.

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- Thus probability that T is the representation is at least $1 - \frac{2^n n}{P}$. Take P to be some field with at least $2^n n 2^n$ elements :-).
- size of this representation with be like $n^{O(1)}$ bits!

Representation of Gammoids

- Let $D = (V, A)$ be a directed graph, $\varepsilon > 0$ be a given real number, and let S and T be possibly overlapping subsets of V.
- Let $M = (T, \mathcal{I})$, where $\mathcal{I} = \{Z \subseteq T : Z \text{ is linked to } S\}$, be the gammoid formed by (D, S) restricted to T.
- We can compute a representation of M as an $|S| \times |T|$ matrix over the rationals with entries of bit-length $O(\min\{|T|, |S| \log |T|\} + \log(1/\varepsilon) + \log |V|)$ in randomized polynomial time with one-sided error bounded by ε.

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Stefan Kratsch, Magnus Wahlström: Compression via matroids: A randomized polynomial kernel for odd cycle transversal. SODA 2012: 94-103

Kernelization for ODD CYCLE TRANSVERSAL

Result from

Stefan Kratsch, Magnus Wahlström: Compression via matroids: A randomized polynomial kernel for odd cycle transversal. SODA 2012: 94-103

Algorithm for OCT

Lemma: Let $G = (V, E)$ be a graph and $Q \subseteq V$ be such that $G \setminus Q$ is bipartite with color classes A, B. Then, the size of the minimum odd cycle transversal is the minimum over all valid partitions of $Q_A \cup Q_B = S \cup T \cup Z$ of the following value:

$$\frac{|Z|}{2} + \underset{G' \setminus Z}{\text{mincut}}(S, T)$$

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$$\frac{|Z|}{2} + \operatorname*{mincut}_{G' \setminus Z}(S, T)$$

- The idea is to encode the algorithm given by the above lemma using matroids.
- Note that if $M = (E, \mathcal{I})$ is representable then the corresponding matrix M succinctly represents all the sets in $\mathcal{I}$.
- The size of $\mathcal{I}$ could be huge, however the size of M is polynomial in the universe size and whether a set is in $\mathcal{I}$ or not can be tested by looking at the corresponding columns in M.

Algorithm for OCT

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- The idea is to encode the algorithm given by the above lemma using matroids.
- Want to exploit this tiny representation of matroids compared to $\left|\mathcal{I}\right|$.

Towards the kernel for OCT

Let $|Q| = q$.

- There are 3^q steps in the OCT algorithm. Want each step to be encoded by an *independent set* of a matroid whose representation matrix has size only $q^{\mathcal{O}(1)}$.
- Each step finds a minimum cut between a pair of subsets of $Q_A \cup Q_B$.

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Does this ring a bell about which matroid to use for our purpose?

Menger's Theorem

Let D be a (un)-directed graph and S and T (may not be disjoint) be vertex subsets.

max-dis-path$(S, T)(D)$ denotes the maximum number of vertex disjoint paths (even at ends).

mincut$(S, T)(D)$ denotes the minimum number of vertices required to disconnect S from T in D.

Mengers Theorem:

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So rather than remembering minimum cut we can remember maximum number of vertex disjoint paths.

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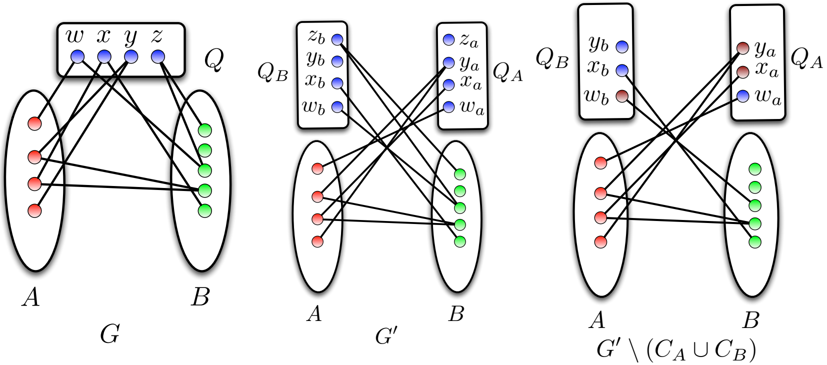
Gammoid!

Recall: Example



For $L = \{w\}, R = \{x, y\}, C = \{z\} \implies L_A \cup R_B = \{w_a, x_b, y_b\}$ and $L_B \cup R_A = \{w_b, x_a, y_a\}$ and $C_A \cup C_B = \{z_a, z_b\}$ Want to compute cut between $L_A \cup R_B = \{w_a, x_b, y_b\}$ and $L_B \cup R_A = \{w_b, x_a, y_a\}$ in $G' \setminus (C_A \cup C_B)$.

Gammoid for our purpose

Given $Q_A \cup Q_B$, we need a gammoid that does the following job:

- For every valid partition of $Q_A \cup Q_B = S \cup T \cup Z$, remembers the size of minimum cut/maximum number of vertex disjoint paths between S and T in $G' \setminus Z$.

We also need to encode deletion of vertices of Z.

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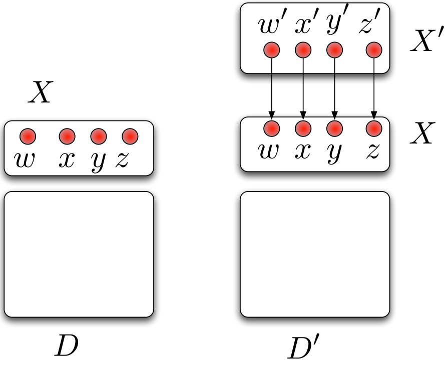
Gammoid for our purpose

Abstractly the problem we want to solve is the following:

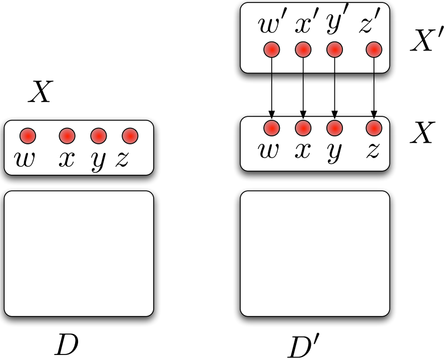
- Input: A directed graph D and a subset X of terminals.
- Output: A representation of a gammoid of size $|X|^{O(1)}$ which for every partition of X as $S \cup T \cup R \cup U$, has an independent set I from which we can *infer the maximum number of vertex disjoint paths between* S *and* T *in* $D \setminus R$.

Solving the Problem



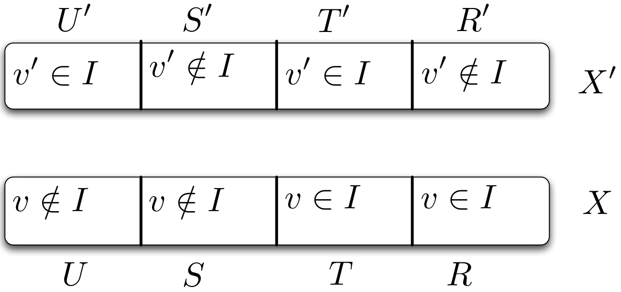
- Let $X' = \{x' \mid x \in X\}$ be a vertex set. The vertices x' and x are called conjugates of each other.
- Add X' to D and arcs (x', x) to D for every $x \in X$. Let the resulting digraph be D'.

Solving the Problem



- Obtain a gammoid with $\mathcal{S} = X'$ and $\mathcal{T} = X' \cup X$.
- Clearly, the size of the representation matrix is $|X| \times 2|X|$ (not the number of bits).

Correspondence between an Independent Set and a Partition



Let $I \subseteq X \cup X'$. Given I we define a partition of X, called P_I, as follows:

- S contains all vertices $v \in X$ with $v, v' \notin I$
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- $U = X \setminus (R \cup T \cup U)$

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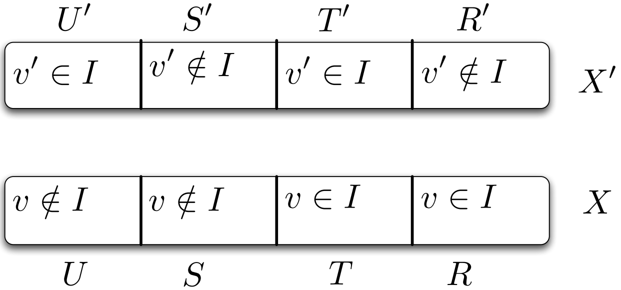

Given a partition $X = S \cup T \cup R \cup U$, the corresponding subset $I(S, T, R, U) \subseteq X \cup X'$ is $T \cup R \cup T' \cup U'$.

Proof

$I \subseteq X \cup X'$ is independent in the gammoid if and only if T is linked to S in $D \setminus R$.
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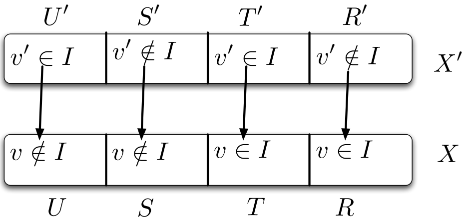

Proof: $\Rightarrow$

There exists $|I|$ vertex disjoint paths from X' to I. For every vertex in $X' \cap I = T' \cup U'$ the only path that is possible has the form v'. For every vertex w in R there is either a path of the form $w'w$ or $v'v \cdots w$ with $v' \in S'$. In later case we can replace the path $v'v \cdots w$ with $w'w$.

Proof

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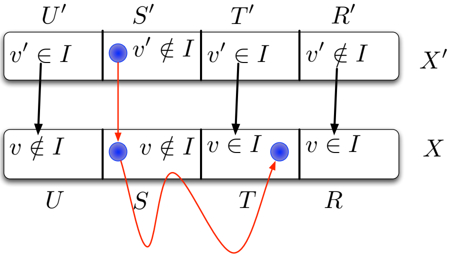

Proof: $\Rightarrow$ There exists $|I|$ vertex disjoint paths from X' to $X \cup X'$. For every vertex in T there exists a path of the form $v'v \cdots w$ with $v' \in S'$. All these paths do not contain any vertices of R and are vertex disjoint and in fact $v \cdots w$ is a path in $D \setminus R$. T is linked to S in $D \setminus R$.

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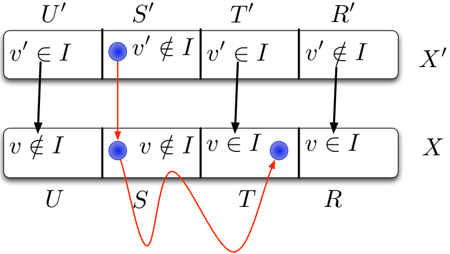


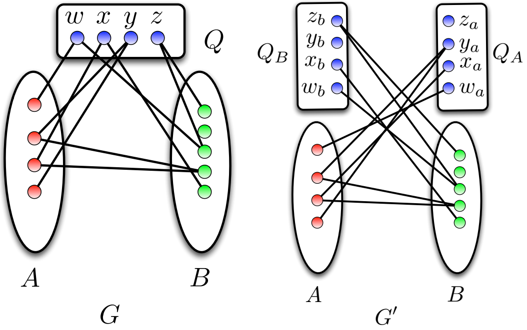
$\Leftarrow$ This obviously follows by taking paths v' and $w'w$ and appending paths from S to T by its conjugate in S'. So there exists vertex disjoint paths from X' to I. Thus I is independent.

Key Lemma

Given a partition $X = S \cup T \cup R \cup U$ let $I = I(S, T, R, U) \subseteq X \cup X'$ be the corresponding set. That is, $I = T \cup R \cup T' \cup U'$. Then

$$\text{mincut}(S, T)(D \setminus R) = r(I) - |X \setminus S|.$$

Compression Algorithm for OCT



Step 1: Create an auxiliary graph $(G', Q_A \cup Q_B)$ from (G, Q).

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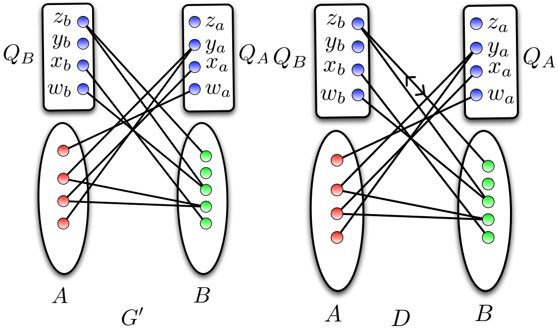

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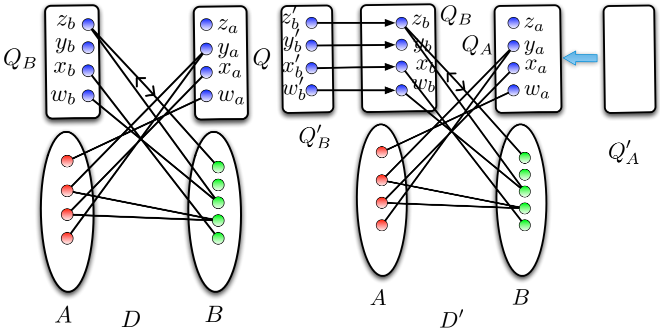

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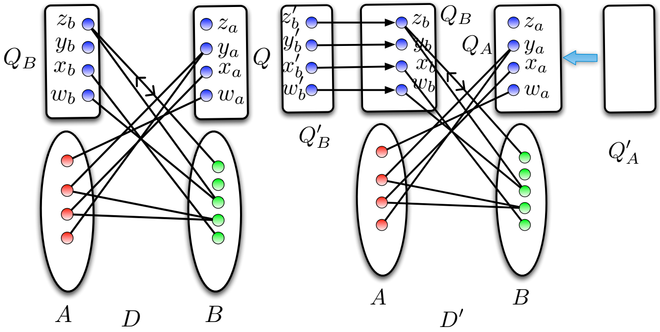

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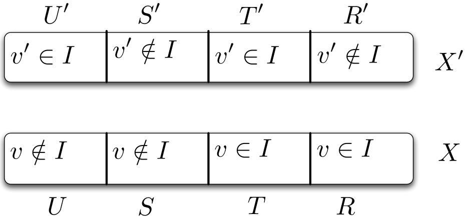
Step 4: Let A be the matrix representing the gammoid. Output A, k.

Compression Algorithm for OCT



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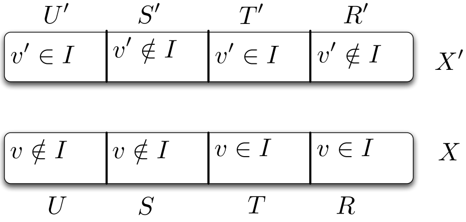
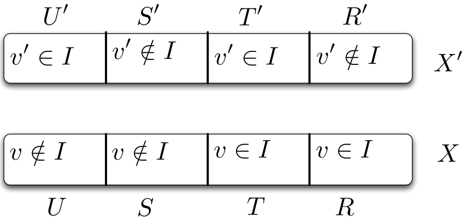


We call $I \subseteq X \cup X'$ an interesting set if $P_I = S \cup T \cup R (= Z)$ is a *valid partition* of $X = Q_A \cup Q_B$.

Compression Algorithm for OCT



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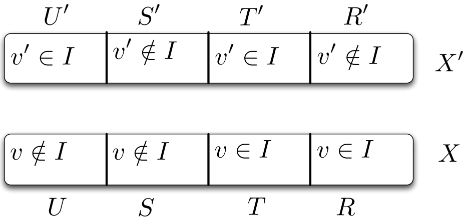
(G, k) has an odd cycle transversal of size k if and only if there exists an interesting set $I \subseteq X \cup X$ such that $\text{rank}(I) - |Z \setminus S| \leq k$.

Compression Algorithm for OCT



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For proof recall...

$$\text{mincut}(S, T)(D \setminus R) = r(I) - |X \setminus S|.$$

Size of A

- Let $D = (V, A)$ be a directed graph, $\varepsilon > 0$ a given real, and let $\mathcal{S}$ and $\mathcal{T}$ be possibly overlapping subsets of V.
- Let $M = (\mathcal{T}, \mathcal{I})$, where $\mathcal{I} = \{Z \subseteq \mathcal{T} : Z \text{ is linked to } \mathcal{S}\}$, be the gammoid formed by $(D, \mathcal{S})$ restricted to $\mathcal{T}$.
- We can compute a representation of M as an $|S| \times |T|$ matrix over the rationals with entries of bit-length $O(\min\{|\mathcal{T}|, |\mathcal{S}| \log |\mathcal{T}|\} + \log(1/\varepsilon) + \log |V|)$ in randomized polynomial time with one-sided error bounded by ε.

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Size of A in terms of bits $= \mathcal{O}(|Q|^3 \log |Q| + |Q|^2 \log(1/\varepsilon) + |Q|^2 \log |V|)$

How do we get Q and the final size.

- If $k \leq \log n$ then run the $\mathcal{O}(3^k mn)$ FPT algorithm and find solution in polynomial time.
- Apply the known $\alpha\sqrt{\log n}$ approximation algorithm for OCT and get a set Q. If the size of $|Q| > k\alpha\sqrt{\log n}$ output NO.
- Else $k > \log n$ and thus $|Q| \leq k\alpha\sqrt{\log n} \leq \mathcal{O}(k^{1.5})$
- So the size of A in terms of bits is at most $\mathcal{O}(k^{4.5} \log k)$.

Finally Kernel for OCT

Given (G, Q) and A checking whether a set I is interesting or not is within NP. And thus there exists a reduction from the compressed instance to an instance of Odd Cycle Transversal such that the size of the graph is $k^{\mathcal{O}(1)}$.

Final Slide

Thank You!
Any Questions?