Parameterized Algorithms using Matroids

Lecture II: Representative Sets

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Problems we would be interested in...

Vertex Cover
Input: A graph $G = (V, E)$ and a positive integer k.
Parameter: k
Question: Does there exist a subset $V' \subseteq V$ of size at most k such that for every edge $(u, v) \in E$ either $u \in V'$ or $v \in V'$?

Hamiltonian Path
Input: A graph $G = (V, E)$
Question: Does there exist a path P in G that spans all the vertices?

Path
Input: A graph $G = (V, E)$ and a positive integer k.
Parameter: k
Question: Does there exist a path P in G of length at least k?

REPRESENTATIVE SETS

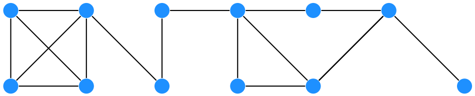
Why, What and How.

REPRESENTATIVE SETS

Why, What and How.

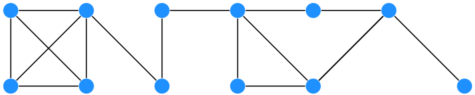
Dynamic Programming for Hamiltonian Path

Ham-Path



$1 \quad 2 \quad 3 \quad \cdots \quad i \quad \cdots \quad n-1 \quad n$

∘ Ham-Path



$1 \quad 2 \quad 3 \quad \cdots \quad i \quad \cdots \quad n-1 \quad n$

v_1

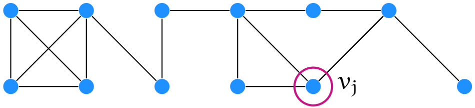
$\vdots$

v_j

$\vdots$

v_n

Ham-Path



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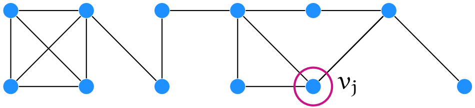

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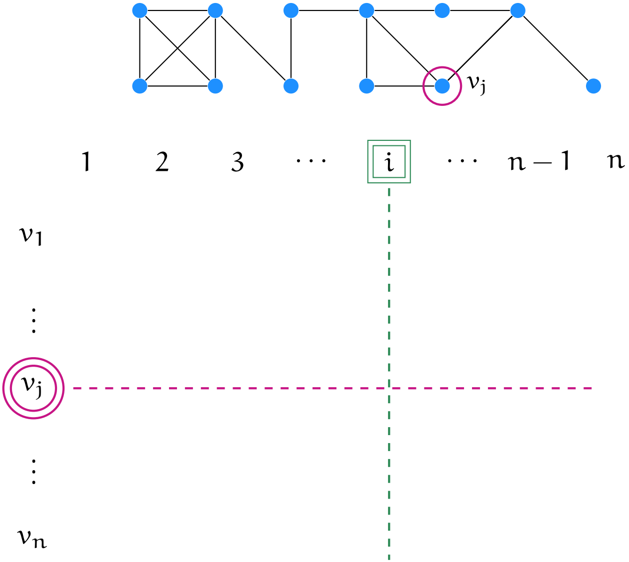
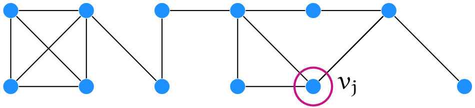


Ham-Path



$1 \quad 2 \quad 3 \quad \cdots \quad \boxed{i} \quad \cdots \quad n-1 \quad n$

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$\vdots$

v_j

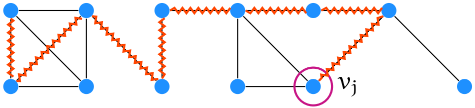
V[Paths of length i ending at v_j]

$\vdots$

v_n

◦ Ham-Path

Example:



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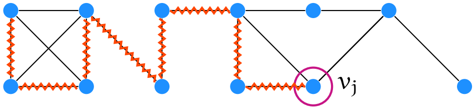
$\vdots$

v_j $\quad V$[Paths of length i ending at v_j]

$\vdots$

v_n

∘ HAM-PATH

Example: 

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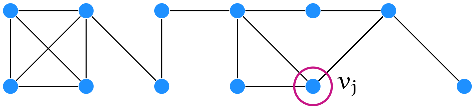

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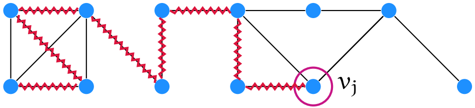
SETS, NOT SEQUENCES.

V[Paths of length i ending at v_j]

○ HAM-PATH

Example: 

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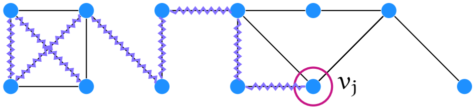
Example: 

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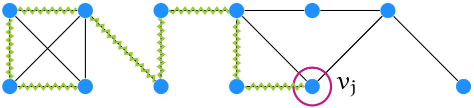
Example: 

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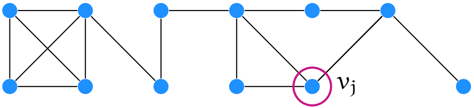

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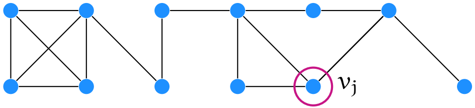
SETS, NOT SEQUENCES.

V[Paths of length i ending at v_j]

Two paths that use the same set of vertices but visit them in different orders are equivalent.

∘ HAM-PATH



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v_1

$\vdots$

v_j

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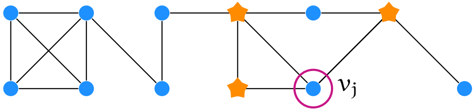
v_n

V[Paths of length i ending at v_j]

= V[Paths of length $(i-1)$ ending at u, avoiding v_j.]

∘ Ham-Path



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v_1

$\vdots$

v_j

$\vdots$

v_n

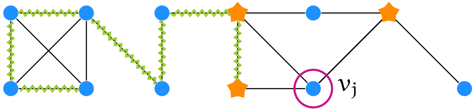
V[Paths of length i ending at v_j]

= V[Paths of length $(i-1)$ ending at u, avoiding v_j.]

$u \in N(v_j)$

∘ Ham-Path

Valid:



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v_n

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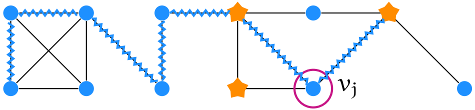
Invalid:



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v_1

$\vdots$

v_j

$\vdots$

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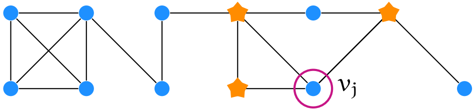


$1 \quad 2 \quad 3 \quad \cdots \quad \boxed{i} \quad \cdots \quad n-1 \quad n$

v_1

$\vdots$

Potentially storing $\binom{n}{i}$ sets.

v_j

V[Paths of length i ending at v_j]

$\vdots$

= V[Paths of length $(i-1)$ ending at u, avoiding v_j.]

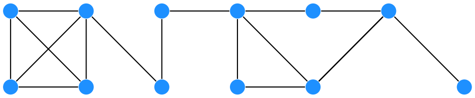
$u \in N(v_j)$

v_n

Let us now turn to k-Path.

To find paths of length at least k,
we may simply use the DP table for Hamiltonian Path
restricted to the first k columns.

K-Path



1 2 3 ⋯ i ⋯ $k-1$ k

v_1

⋮

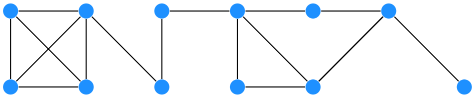
v_j

Worst case running time: $\mathcal{O}^\star\left(\binom{n}{k}\right)$

⋮

v_n

K-Path



$1 \quad 2 \quad 3 \quad \cdots \quad i \quad \cdots \quad k-1 \quad k$

v_1

$\vdots$

v_j

Worst case running time: $\mathcal{O}^\star(n^k)$

$\vdots$

v_n

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In the i^{th} column, we are storing paths of length i.

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Let P be a path of length k.

There may be several paths of length i that “latch on” to the last $(k - i)$ vertices of P.

Do we really need to store all these sets?

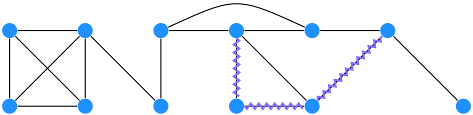
In the i^{th} column, we are storing paths of length i.

Let P be a path of length k.

There may be several paths of length i that "latch on" to the last $(k-i)$ vertices of P.

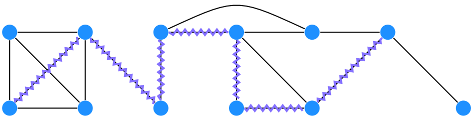
We need to store just one of them.

Example.



Example.

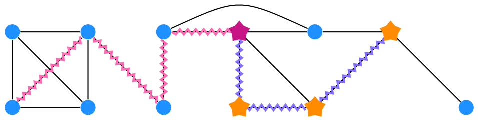
Suppose we have a path P on seven edges.

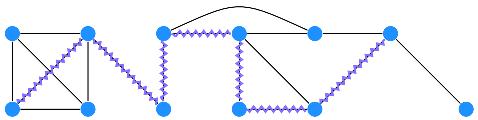


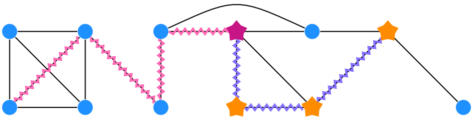
Example.

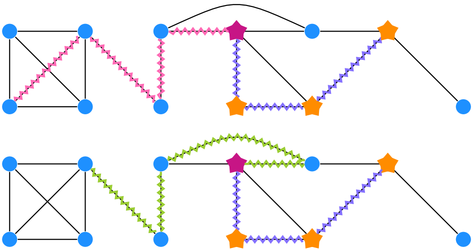
Suppose we have a path P on seven edges.

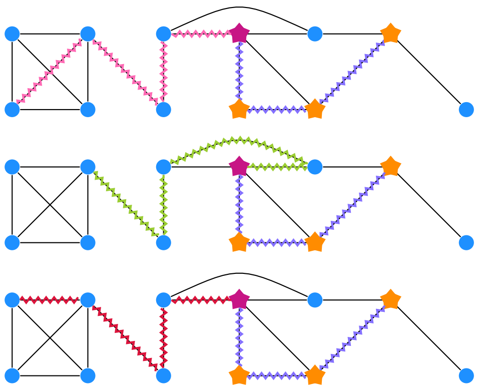
Consider it broken up into the first four and the last three edges.

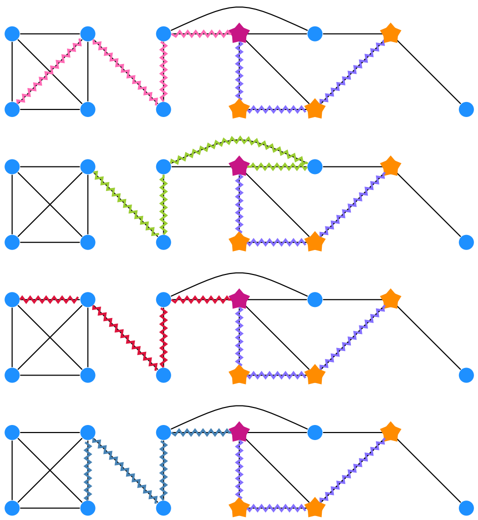


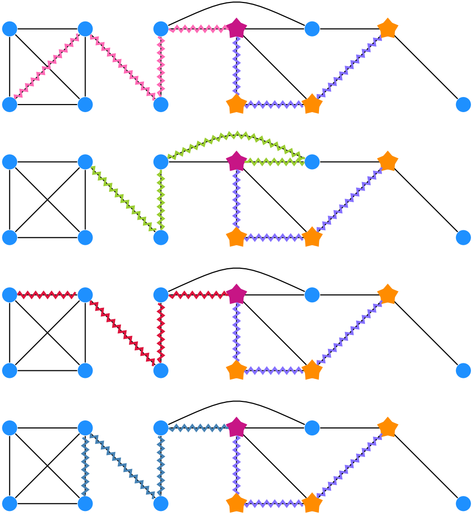






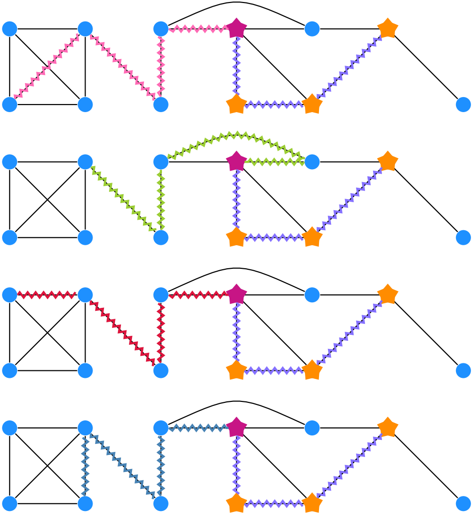






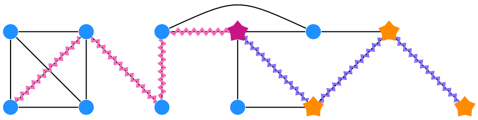
A Fixed Future ($v_{i+1} - \cdots - v_k$).

The Possibilities for Partial Solutions Compatible with $v_{i+1} - \cdots - v_k$.



A Fixed Future ($v_{i+1} - \cdots - v_k$).

Let's try a different example.



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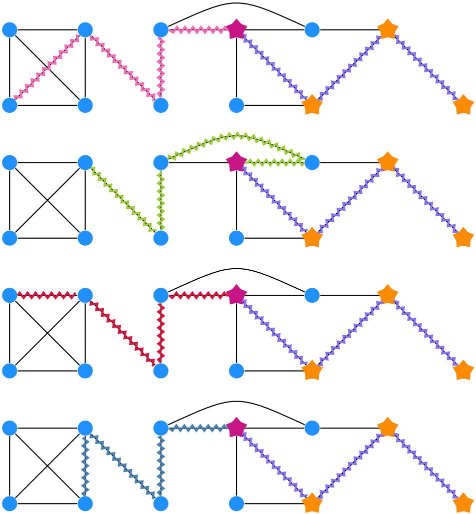
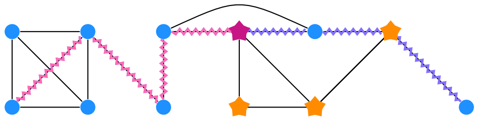


A Fixed Future ($v_{i+1} - \cdots - v_k$).

Here's one more example:



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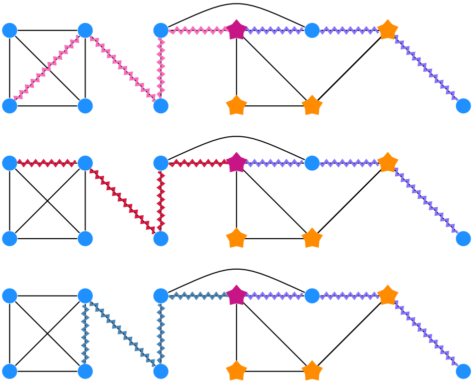

A Fixed Future ($v_{i+1} - \cdots - v_k$).

For any possible ending of length $(k - i)$, we want to be sure that we store at least one among the possibly many “prefixes”.

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This could also be $\binom{n}{k-i}$.

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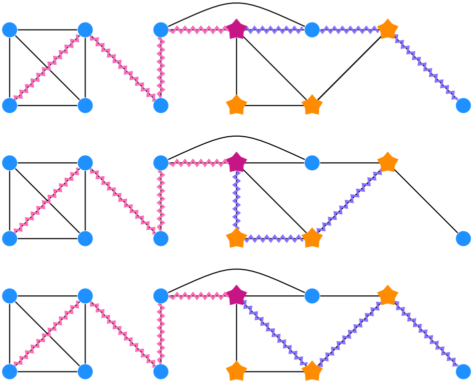
This could also be $\binom{n}{k-i}$.

The hope for "saving" comes from the fact that a single path of length i is potentially capable of being a prefix to several distinct endings.

For example...



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Partial solutions: paths of length j ending at v_i

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A "small" representative family.

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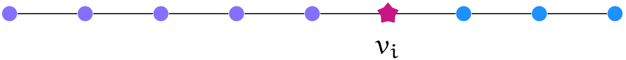

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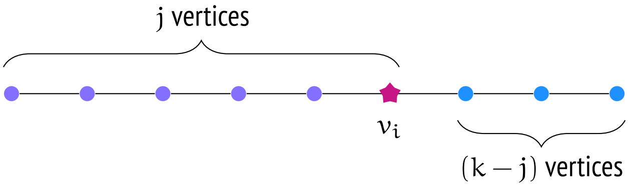

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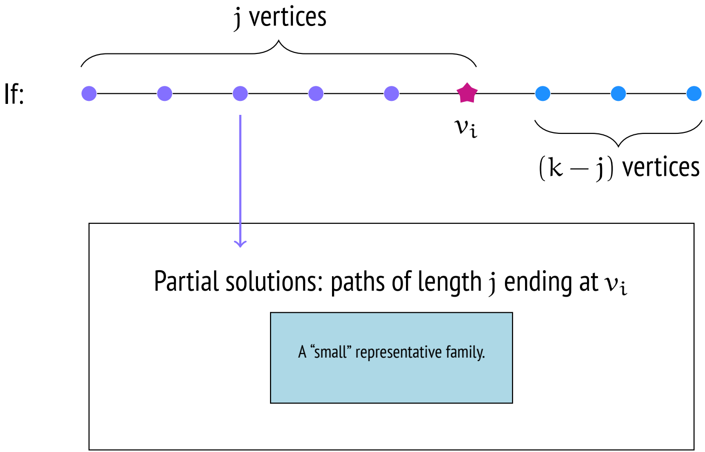
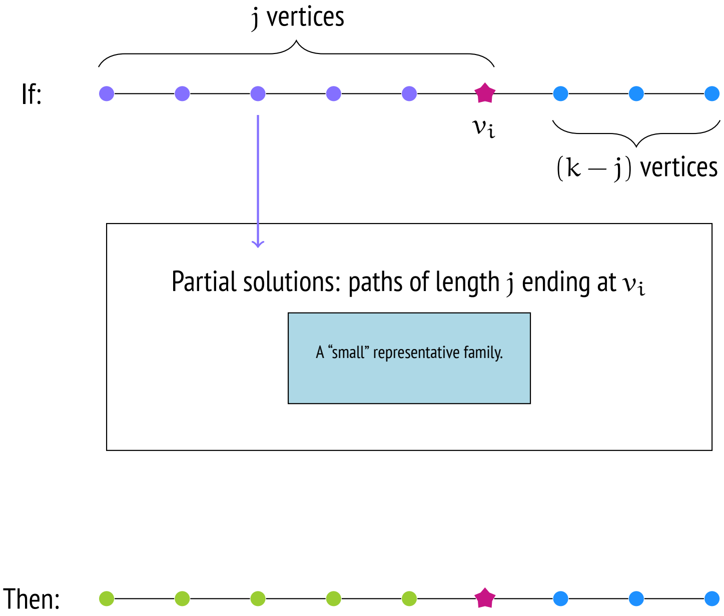


j vertices
If:
v_i
$(k-j)$ vertices
Partial solutions: paths of length j ending at v_i
A “small” representative family.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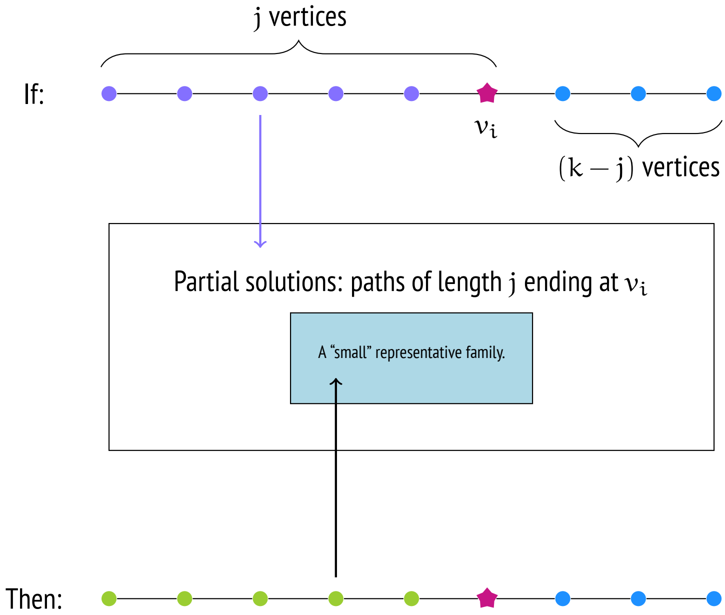
Then: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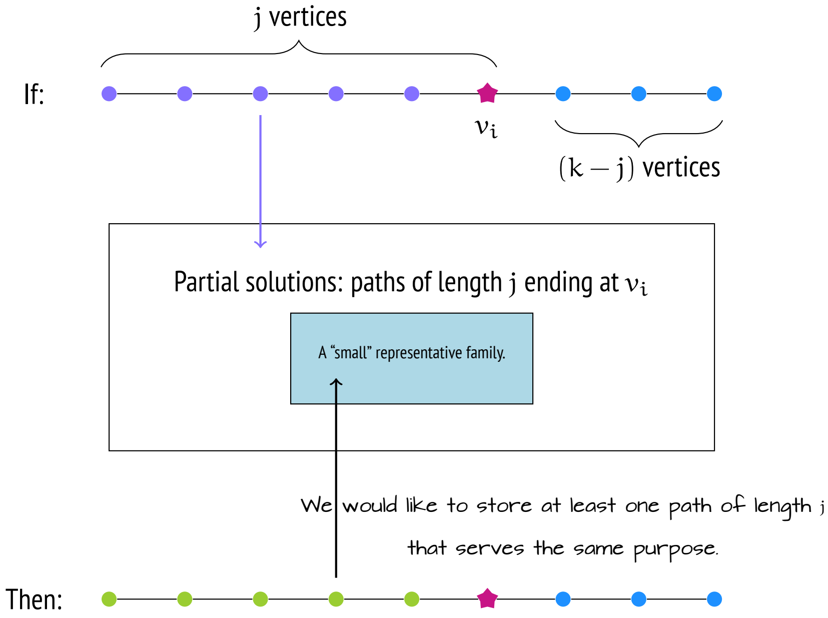

j vertices
If:
v_i
$(k - j)$ vertices
Partial solutions: paths of length j ending at v_i
A "small" representative family.
We would like to store at least one path of length j
that serves the same purpose.
Then:

Given: A (BIG) family $\mathcal{F}$ of p-sized subsets of $[n]$.

$$S_1, S_2, \ldots, S_t$$

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Want: A (small) subfamily $\widehat{\mathcal{F}}$ of $\mathcal{F}$ such that:

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Want: A (small) subfamily $\widehat{\mathcal{F}}$ of $\mathcal{F}$ such that:

For any $X \subseteq [n]$ of size $(k - p)$,

if there is a set S in $\mathcal{F}$ such that $X \cap S = \emptyset$,
then there is a set $\widehat{S}$ in $\widehat{\mathcal{F}}$ such that $X \cap \widehat{S} = \emptyset$.

Given: A (BIG) family $\mathcal{F}$ of p-sized subsets of $[n]$.

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The “second half” of a solution – can be any subset.

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This is a valid patch into X.

Given: A (BIG) family $\mathcal{F}$ of p-sized subsets of $[n]$.

$$S_1, S_2, \ldots, S_t$$

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This is a guaranteed replacement for S.

Given: A (BIG) family $\mathcal{F}$ of p-sized subsets of $[n]$.

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Want: A (small) subfamily $\widehat{\mathcal{F}}$ of $\mathcal{F}$ such that:

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Given: A $\leqslant \binom{n}{p}$ family $\mathcal{F}$ of p-sized subsets of $[n]$.

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Want: A (small) subfamily $\widehat{\mathcal{F}}$ of $\mathcal{F}$ such that:

For any $X \subseteq [n]$ of size $(k-p)$,

if there is a set S in $\mathcal{F}$ such that $X \cap S = \emptyset$,
then there is a set $\widehat{S}$ in $\widehat{\mathcal{F}}$ such that $X \cap \widehat{S} = \emptyset$.

Given: A $\leqslant \binom{n}{p}$ family $\mathcal{F}$ of p-sized subsets of $[n]$.

$$S_1, S_2, \ldots, S_t$$

Known: $\exists \binom{k}{p}$ subfamily $\widehat{\mathcal{F}}$ of $\mathcal{F}$ such that:

For any $X \subseteq [n]$ of size $(k - p)$,

if there is a set S in $\mathcal{F}$ such that $X \cap S = \emptyset$,
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Bolobás, 1965.

Given: A a matroid $(\mathcal{M}, \mathcal{I})$, and a family of p-sized subsets from $\mathcal{I}$:

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Want: A subfamily $\widehat{\mathcal{F}}$ of $\mathcal{F}$ such that:

For any $X \subseteq [n]$ of size at most q,

if there is a set S in $\mathcal{F}$ such that $X \cap S = \emptyset$ and $X \cup S \in \mathcal{I}$,
then there is a set $\widehat{S}$ in $\widehat{\mathcal{F}}$ such that $X \cap \widehat{S} = \emptyset$ and $X \cup \widehat{S} \in \mathcal{I}$.

Given: A a matroid $(\mathcal{M}, \mathcal{I})$, and a family of p-sized subsets from $\mathcal{I}$:

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There is a subfamily $\widehat{\mathcal{F}}$ of $\mathcal{F}$ of size at most $\binom{p+q}{p}$ such that:

For any $X \subseteq [n]$ of size at most q,

if there is a set S in $\mathcal{F}$ such that $X \cap S = \emptyset$ and $X \cup S \in \mathcal{I}$,
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Lovász, 1977

Given: A a matroid $(M, \mathcal{I})$, and a family of p-sized subsets from $\mathcal{I}$:

$$S_1, S_2, \ldots, S_t$$

There is an efficiently computable subfamily $\widehat{\mathcal{F}}$ of $\mathcal{F}$ of size at most $\binom{p+q}{p}$ such that:

For any $X \subseteq [n]$ of size at most q,

if there is a set S in $\mathcal{F}$ such that $X \cap S = \emptyset$ and $X \cup S \in \mathcal{I}$,
then there is a set $\widehat{S}$ in $\widehat{\mathcal{F}}$ such that $X \cap \widehat{S} = \emptyset$ and $X \cup \widehat{S} \in \mathcal{I}$.

Márx (2009) and Fomin, Lokshtanov, Saurabh (2013)

Summary.

We have at hand a p-uniform collection of independent sets, $\mathcal{F}$ and a number q. Let X be any set of size at most q. For any set $S \in \mathcal{F}$, if:

a X is disjoint from S, and

b X and S together form an independent set,

then a q-representative family $\widehat{\mathcal{F}}$ contains a set $\widehat{S}$ that is:

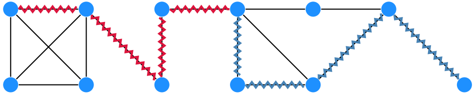
a disjoint from X, and

b forms an independent set together with X.

Such a subfamily is called a q-representative family for the given family.

REPRESENTATIVE SETS

Back to Why.



$1 \quad 2 \quad 3 \quad \cdots \quad i \quad \cdots \quad k-1 \quad k$

v_1

[RECALL]

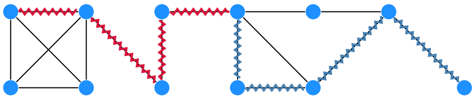
Worst case running time: $\mathcal{O}^\star\left(\binom{n}{k}\right)$

$\vdots$

v_j

$\vdots$

v_n



$1 \quad 2 \quad 3 \quad \cdots \quad i \quad \cdots \quad k-1 \quad k$

v_1

[RECALL]

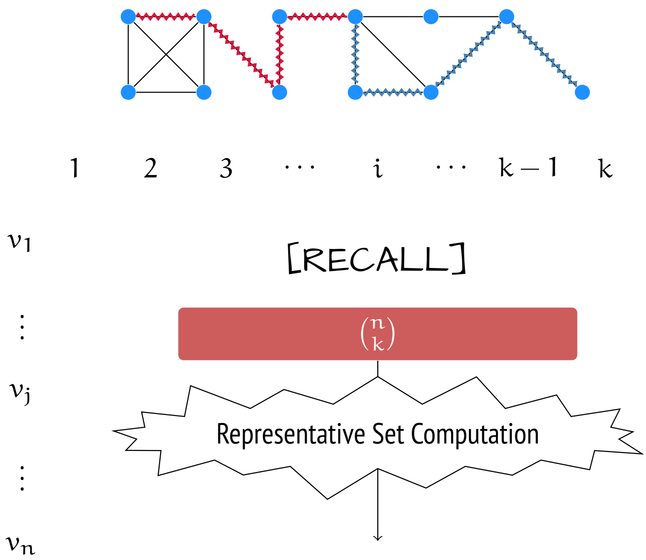
$\vdots$

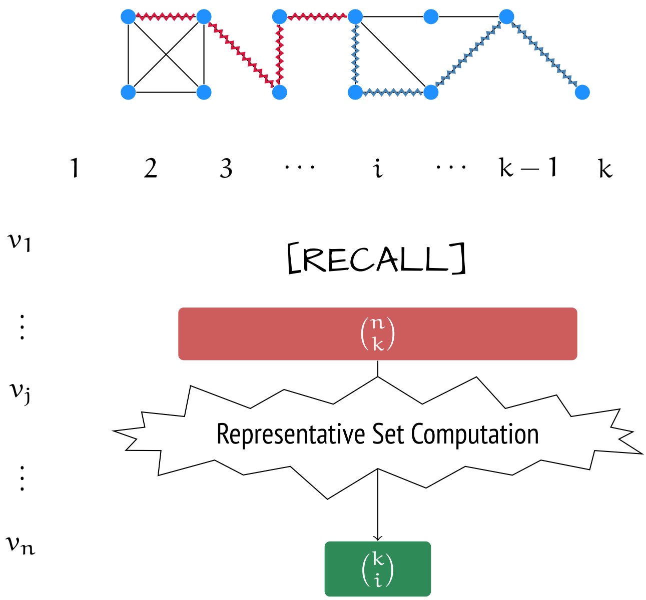
$\binom{n}{k}$

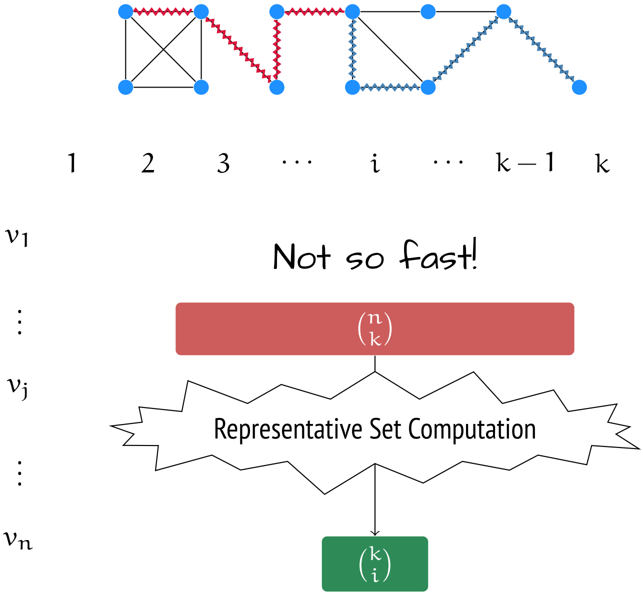
v_j

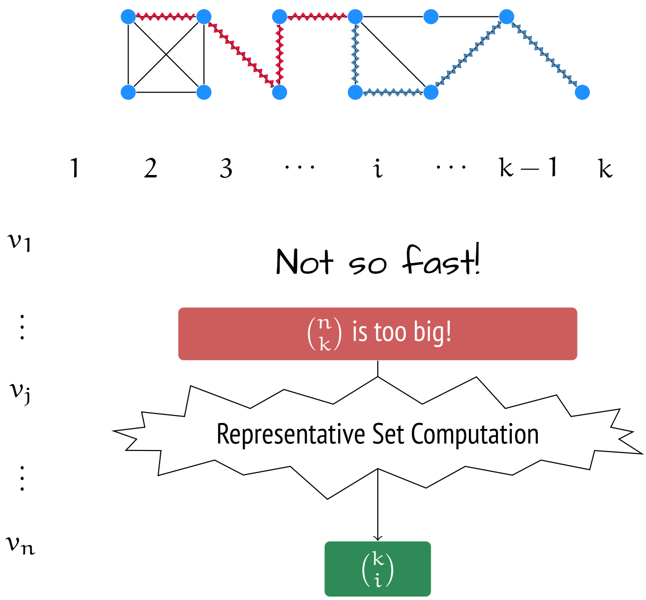
$\vdots$

v_n


1
2
3
⋯
i
⋯
k − 1
k
v_1
⋮
v_j
⋮
v_n
[RECALL]
$\binom{n}{k}$
Representative Set Computation


1
2
3
···
i
···
k − 1
k
v_1
⋮
v_j
⋮
v_n
[RECALL]
$\binom{n}{k}$
Representative Set Computation
$\binom{k}{i}$


1
2
3
· · ·
i
· · ·
k − 1
k
v1
⋮
vj
⋮
vn
Not so fast!
(n k)
Representative Set Computation
(k i)


1 2 3 ⋯ i ⋯ k − 1 k
v_1
⋮
v_j
⋮
v_n
Not so fast!
$\binom{n}{k}$ is too big!
Representative Set Computation
$\binom{k}{i}$

We are going to compute representative families at every intermediate stage of the computation.

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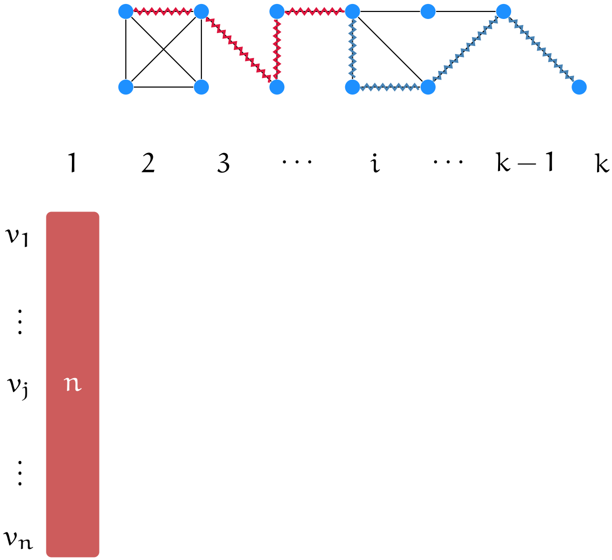
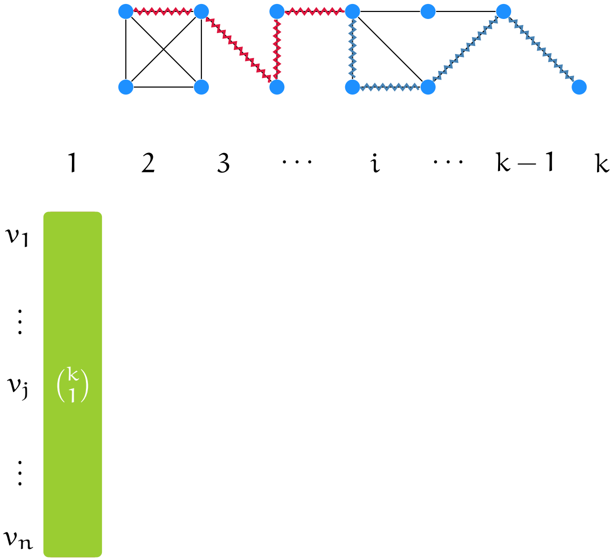
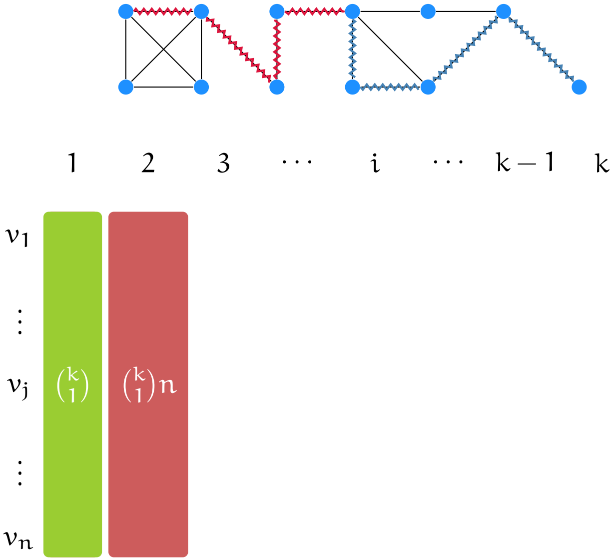
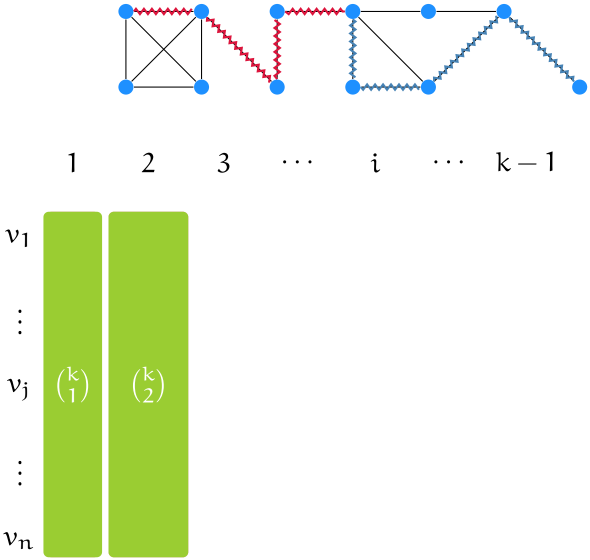
For instance, in the i^{th} column, we are storing i-uniform families.
Before moving on to column $(i+1)$, we compute $(k-i)$-representative families.

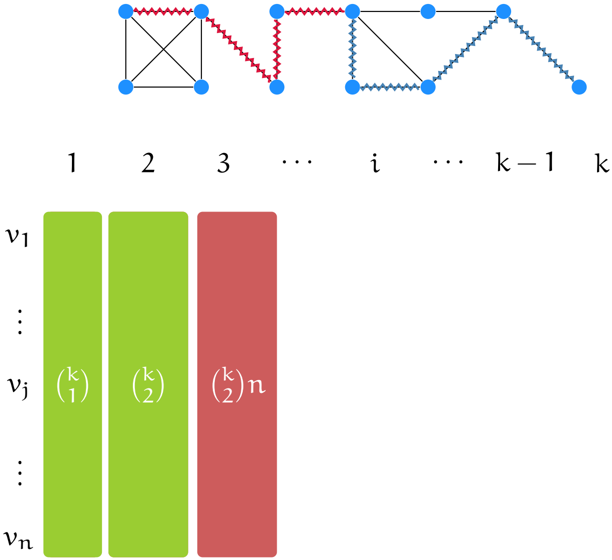
This keeps the sizes small as we go along.

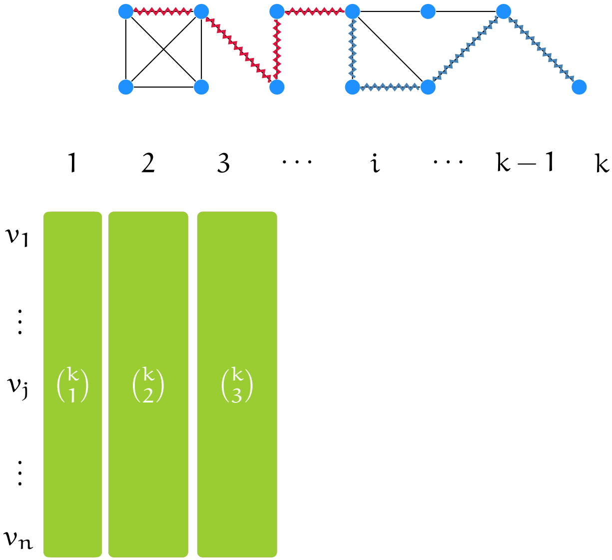

1 2 3 ··· i ··· k − 1 k
v_1
v_j
n
v_n


1
2
3
⋯
i
⋯
k − 1
k
v_1
⋮
v_j
$\binom{k}{1}$
⋮
v_n

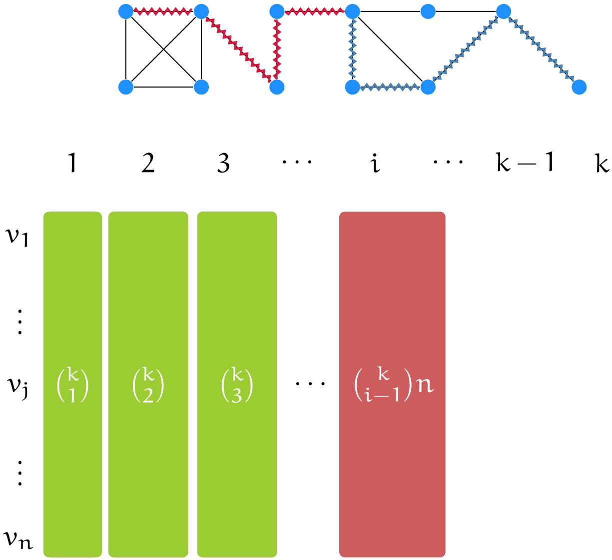

1
2
3
· · ·
i
· · ·
k − 1
k
v1
⋮
vj
⋮
vn
(k 1)
(k 1)n


1 2 3 ··· i ··· k − 1 k
v_1
v_j
v_n
$\binom{k}{1}$
$\binom{k}{2}$

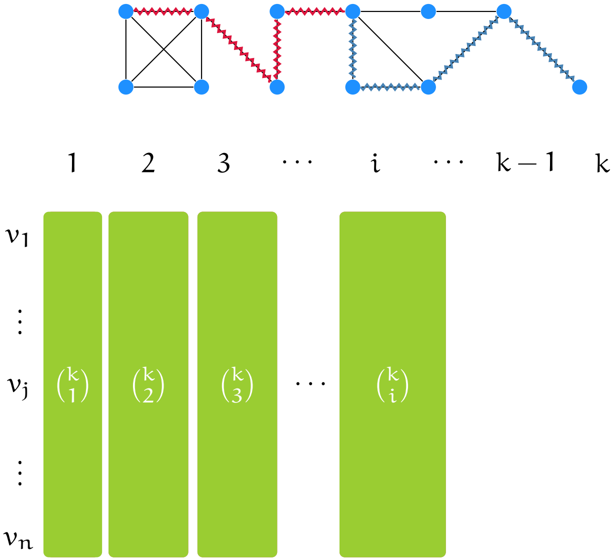

1
2
3
⋯
i
⋯
k − 1
k
v1
⋮
vj
⋮
vn
(k 1)
(k 2)
(k 2)n

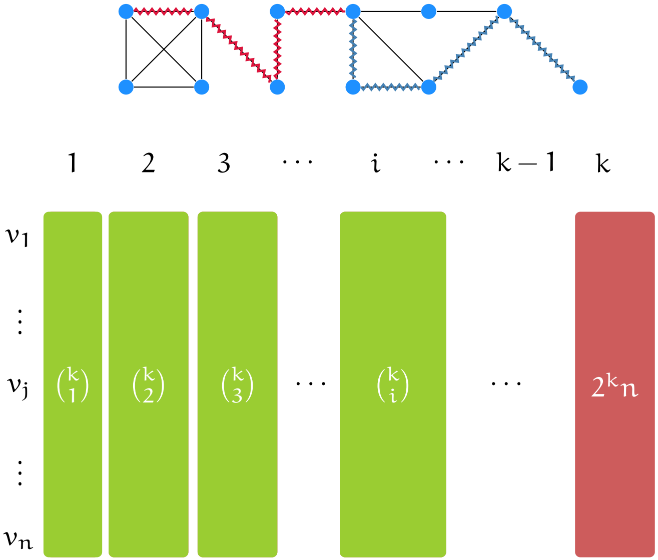


1
2
3
· · ·
i
· · ·
k − 1
k
v_1
⋮
v_j
⋮
v_n
$\binom{k}{1}$
$\binom{k}{2}$
$\binom{k}{3}$

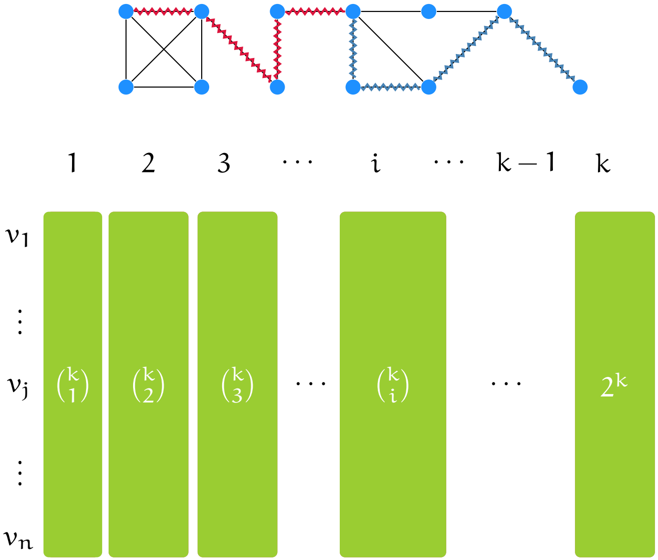


1
2
3
⋯
i
⋯
k − 1
k
v_1
v_j
v_n
$\binom{k}{1}$
$\binom{k}{2}$
$\binom{k}{3}$
⋯
$\binom{k}{i-1}n$


1
2
3
···
i
···
k − 1
k
v_1
⋮
v_j
⋮
v_n
$\binom{k}{1}$
$\binom{k}{2}$
$\binom{k}{3}$
···
$\binom{k}{i}$



1
2
3
· · ·
i
· · ·
k − 1
k
v_1
⋮
v_j
⋮
v_n
$\binom{k}{1}$
$\binom{k}{2}$
$\binom{k}{3}$
· · ·
$\binom{k}{i}$
· · ·
$2^k n$


1
2
3
⋯
i
⋯
k − 1
k
v1
vj
vn
(k 1)
(k 2)
(k 3)
⋯
(k i)
⋯
2^k

Let $\mathcal{P}_i^j$ be the set of all paths of length i ending at v_j.

It can be shown that the families thus computed at the i^{th} column, j^{th} row are indeed $(k-i)$-representative families for P_i^j.

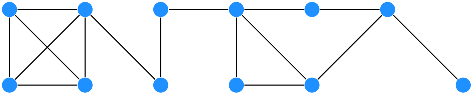
The correctness is implicit in the notion of a representative family.

REPRESENTATIVE SETS

A Different Why.

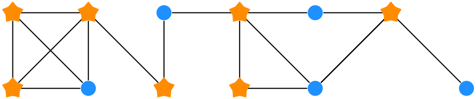
Vertex Cover

Can you delete k vertices to kill all edges?



Vertex Cover

Can you delete k vertices to kill all edges?



Let $(G = (V, E), k)$ be an instance of Vertex Cover.

Note that E can be thought of as a 2-uniform family over the ground set V.

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Note that E can be thought of as a 2-uniform family over the ground set V.

Goal: Kernelization.

In this context, we are asking if there is a small subset X of the edges such that

$G[X]$ is a YES-instance $\leftrightarrow$ G is a YES-instance.

Note: If G is a YES-instance, then $G[X]$ is a YES-instance for any subset $X \subseteq E$.

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We get one direction for free!

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It is the NO-instances that we have to worry about preserving.

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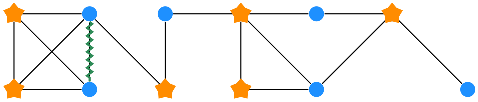
We get one direction for free!

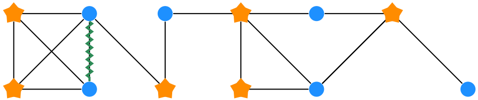
It is the NO-instances that we have to worry about preserving.

What is a NO-instance?



If G is a NO-instance:

For any subset S of size at most k,
there is an edge that is disjoint from S.



If G is a NO-instance:

For any subset S of size at most k,
there is an edge that is disjoint from S.

Ring a bell?

Recall.

We have at hand a p-uniform collection of independent sets, $\mathcal{F}$ and a number q. Let X be any set of size at most q. For any set $S \in \mathcal{F}$, if:

a X is disjoint from S, and

b X and S together form an independent set,

then a q-representative family contains a set $\widehat{S}$ that is:

a disjoint from X, and

b forms an independent set together with X.

Such a subfamily is called a q-representative family for the given family.

Claim: A k-representative family for E is in fact an $O(k^2)$ kernel for vertex cover.

$$E(G) = \{e_1, e_2, \ldots, e_m\}$$

Is there a Vertex Cover of size at most k?

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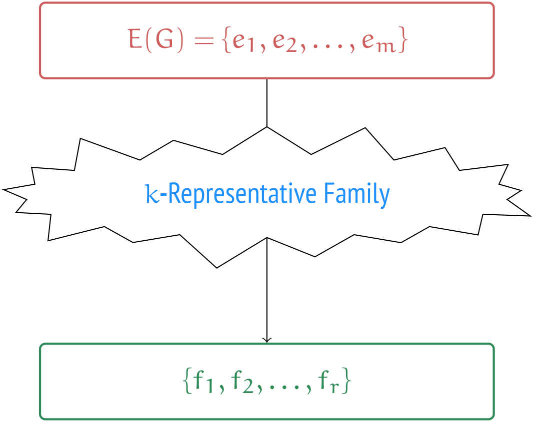
k-Representative Family

Is there a Vertex Cover of size at most k?



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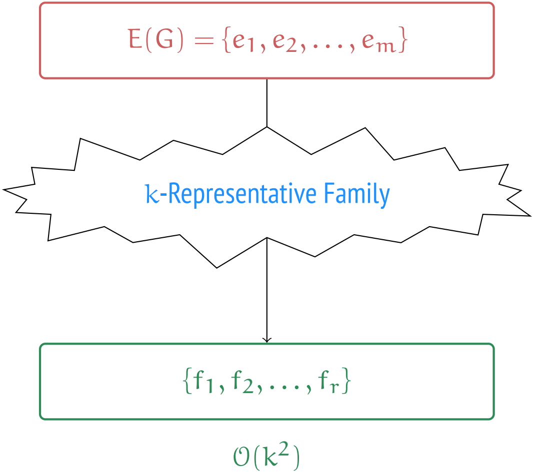
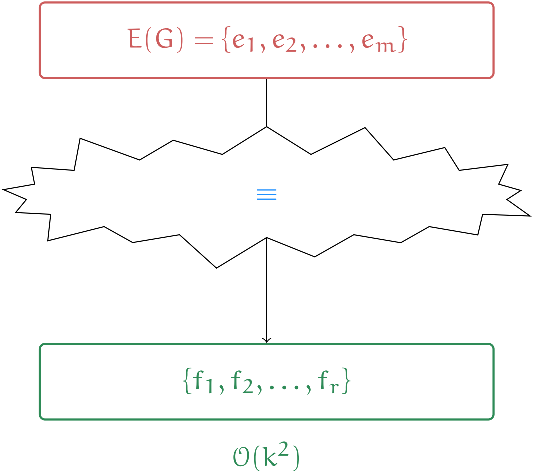


Is there a Vertex Cover of size at most k?



Is there a Vertex Cover of size at most k?

Let us show that if $G[X]$ is a YES-instance, then so is G.

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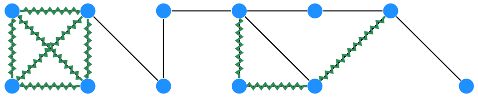
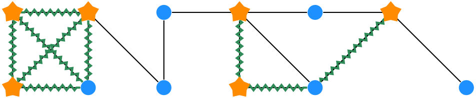
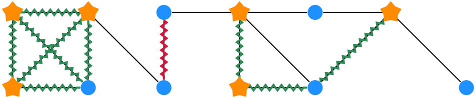
This time, by contradiction.

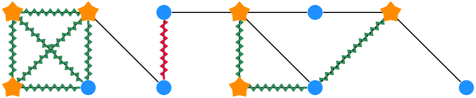




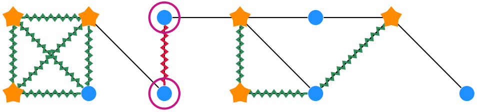
Try the solution for $G[X]$ on G.



Suppose there is an uncovered edge.

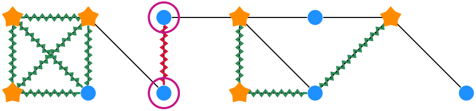


Since X is a k-representative family, for ANY $S \subseteq V$, where $|S| \leqslant k$:



Since X is a k-representative family, for ANY $S \subseteq V$, where $|S| \leqslant k$:

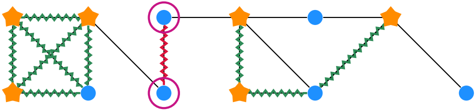
if there is a set e in E such that $e \cap S = \varnothing$,



Since X is a k-representative family, for ANY $S \subseteq V$, where $|S| \leqslant k$:

if there is a set e in E such that $e \cap S = \emptyset$,

then there is a set $\widehat{e}$ in X such that $\widehat{e} \cap S = \emptyset$.



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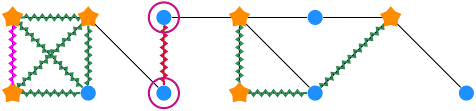
Note that the green edges denote $G[X]$.



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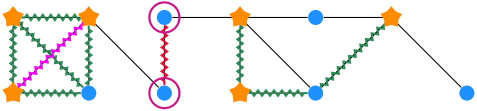

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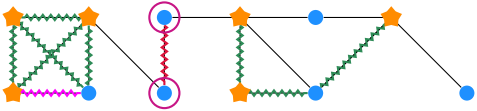

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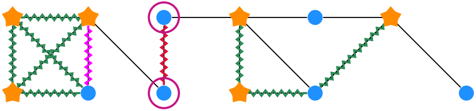

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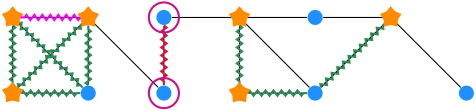

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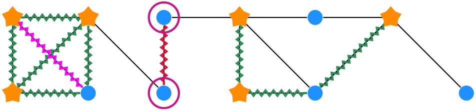

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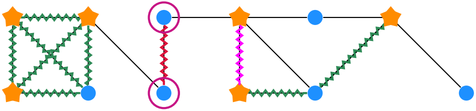

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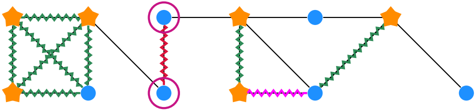

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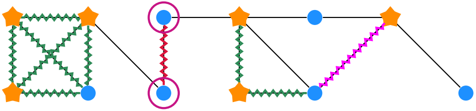
if there is a set e in E such that $e \cap S = \emptyset$,
then there is a set $\widehat{e}$ in X such that $\widehat{e} \cap S = \emptyset$.

Note that the green edges denote $G[X]$.



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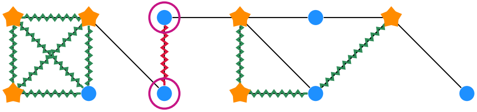

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Note that the green edges denote $G[X]$.

Contradiction!

A k-representative family for $E(G)$ is in fact an $O(k^2)$ instance kernel for Vertex Cover!

REPRESENTATIVE SETS

Why, What and How.

Notation

$\text{Det}(M) : [\![M]\!]$

Let M be a $m \times n$ matrix, and let $I \subseteq [m]$, $J \subseteq [n]$.

$M[I, J]$: M restricted to rows indexed by I and columns indexed by J

$M[\star, J]$: M restricted to all rows and columns indexed by J

$M[I, \star]$: M restricted to rows indexed by I and all columns

STANDARD LAPLACE EXPANSION

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

Fix a row and expand along the columns.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ \boxed{a_{41} \quad a_{42} \quad a_{43} \quad a_{44} \quad a_{45} \quad a_{46}} & & & & & \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

Fix a row and expand along the columns.

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Fix a row and expand along the columns.

$$\begin{bmatrix} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ & & & & & \\ & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

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$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

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$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

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$$\begin{bmatrix} a_{11} & & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & & a_{33} & a_{34} & a_{35} & a_{36} \\ & & & & & \\ a_{51} & & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & & a_{34} & a_{35} & a_{36} \\ & & & & & \\ a_{51} & a_{52} & & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & & a_{35} & a_{36} \\ & & & & & \\ a_{51} & a_{52} & a_{53} & & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & & a_{65} & a_{66} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

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$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{35} & a_{36} \\ a_{51} & a_{52} & a_{53} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{65} & a_{66} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{36} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{66} \end{bmatrix}$$

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Fix a row and expand along the columns.

$$\begin{bmatrix} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ & & & & & \\ & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix} \quad \begin{bmatrix} a_{11} & & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & & a_{33} & a_{34} & a_{35} & a_{36} \\ & & & & & \\ a_{51} & & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & & a_{34} & a_{35} & a_{36} \\ & & & & & \\ a_{51} & a_{52} & & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & & a_{35} & a_{36} \\ & & & & & \\ a_{51} & a_{52} & a_{53} & & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & & a_{65} & a_{66} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & & a_{36} \\ & & & & & \\ a_{51} & a_{52} & a_{53} & a_{54} & & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & & a_{66} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & \\ & & & & & \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & \end{bmatrix}$$

GENERALIZED LAPLACE EXPANSION

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

$$\mathsf{Det}(A) = \sum_{I \subseteq [n], |I| = |J|} \mathsf{Det}(A[\bar{I}, \bar{J}]) \cdot \mathsf{Det}(A[I, J]) \cdot (-1)^{\sum I + \sum J}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

Fix a set of columns, $J \subseteq [6]$.

$$\mathrm{Det}(A) = \sum_{I \subseteq [n], |I| = |J|} \mathrm{Det}(A[\bar{I}, \bar{J}]) \cdot \mathrm{Det}(A[I, J]) \cdot (-1)^{\sum I + \sum J}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

Fix a set of columns, $J \subseteq [6]$.

Iterate over all $I \subseteq [6]$ such that $|I| = |J|$.

$$\text{Det}(A) = \sum_{I \subseteq [n], |I| = |J|} \text{Det}(A[\bar{I}, \bar{J}]) \cdot \text{Det}(A[I, J]) \cdot (-1)^{\sum I + \sum J}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

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$$\begin{bmatrix} & & & & & \\ & & & & & \\ & a_{32} & & a_{34} & a_{35} & \\ & a_{42} & & a_{44} & a_{45} & \\ & a_{52} & & a_{54} & a_{55} & \\ & & & & & \end{bmatrix}$$

$\mathrm{Det}(A[\bar{I}, \bar{J}])$.

$$\mathrm{Det}(A) = \sum_{I \subseteq [n], |I| = |J|} \mathrm{Det}(A[\bar{I}, \bar{J}]) \cdot \mathrm{Det}(A[I, J]) \cdot (-1)^{\sum I + \sum J}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

Fix a set of columns, $J \subseteq [6]$.

Iterate over all $I \subseteq [6]$ such that $|I| = |J|$.

$$\begin{bmatrix} & & & & & \\ & & & & & \\ & a_{32} & & a_{34} & a_{35} & \\ & a_{42} & & a_{44} & a_{45} & \\ & a_{52} & & a_{54} & a_{55} & \\ & & & & & \end{bmatrix} \begin{bmatrix} a_{11} & & a_{13} & & & a_{16} \\ a_{21} & & a_{23} & & & a_{26} \\ & & & & & \\ & & & & & \\ & & & & & \\ a_{61} & & a_{63} & & & a_{66} \end{bmatrix}$$

$\mathrm{Det}(A[\bar{I}, \bar{J}])$. $\mathrm{Det}(A[I, J])$.

$$\mathrm{Det}(A) = \sum_{I \subseteq [n], |I| = |J|} \mathrm{Det}(A[\bar{I}, \bar{J}]) \cdot \mathrm{Det}(A[I, J]) \cdot (-1)^{\sum I + \sum J}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

Fix a set of columns, $J \subseteq [6]$.

Iterate over all $I \subseteq [6]$ such that $|I| = |J|$.

$$\begin{bmatrix} & & & & & \\ & & & & & \\ & a_{32} & & a_{34} & a_{35} & \\ & a_{42} & & a_{44} & a_{45} & \\ & a_{52} & & a_{54} & a_{55} & \\ & & & & & \end{bmatrix} \begin{bmatrix} a_{11} & & a_{13} & & & a_{16} \\ a_{21} & & a_{23} & & & a_{26} \\ & & & & & \\ & & & & & \\ & & & & & \\ a_{61} & & a_{63} & & & a_{66} \end{bmatrix} (-1)^{(1+3+6)+(1+2+6)}$$

$\mathrm{Det}(A[\bar{I}, \bar{J}])$. $\mathrm{Det}(A[I, J])$.

$$\mathrm{Det}(A) = \sum_{I \subseteq [n], |I| = |J|} \mathrm{Det}(A[\bar{I}, \bar{J}]) \cdot \mathrm{Det}(A[I, J]) \cdot (-1)^{\sum I + \sum J}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

Fix a set of columns, $J \subseteq [6]$.

Iterate over all $I \subseteq [6]$ such that $|I| = |J|$.

$$\begin{bmatrix} & & & & & \\ & & & & & \\ & a_{32} & & a_{34} & a_{35} & \\ & a_{42} & & a_{44} & a_{45} & \\ & a_{52} & & a_{54} & a_{55} & \\ & & & & & \end{bmatrix} \begin{bmatrix} a_{11} & & a_{13} & & & a_{16} \\ a_{21} & & a_{23} & & & a_{26} \\ & & & & & \\ & & & & & \\ & & & & & \\ a_{61} & & a_{63} & & & a_{66} \end{bmatrix} (-1)^{(1+3+6)+(1+2+6)}$$

$\mathrm{Det}(A[\bar{I}, \bar{J}])$. $\mathrm{Det}(A[I, J])$.

$$\mathrm{Det}(A) = \sum_{I \subseteq [n], |I| = |J|} \mathrm{Det}(A[\bar{I}, \bar{J}]) \cdot \mathrm{Det}(A[I, J]) \cdot (-1)^{\sum I + \sum J}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

Fix a set of columns, $J \subseteq [6]$.

Iterate over all $I \subseteq [6]$ such that $|I| = |J|$.

$$\begin{bmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & a_{42} & & a_{44} & a_{45} & \\ & a_{52} & & a_{54} & a_{55} & \\ & a_{62} & & a_{64} & a_{65} & \end{bmatrix} \begin{bmatrix} a_{11} & & a_{13} & & & a_{16} \\ a_{21} & & a_{23} & & & a_{26} \\ a_{31} & & a_{33} & & & a_{36} \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} (-1)^{(1+3+6)+(1+2+3)}$$

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$$\begin{bmatrix} a_{12} & a_{14} & a_{15} \\ a_{22} & a_{24} & a_{25} \\ a_{62} & a_{64} & a_{65} \end{bmatrix} \begin{bmatrix} a_{31} & a_{33} & a_{36} \\ a_{41} & a_{43} & a_{46} \\ a_{51} & a_{53} & a_{56} \end{bmatrix} (-1)^{(1+3+6)+(3+4+5)}$$

$\text{Det}(A[\bar{I}, \bar{J}])$. $\text{Det}(A[I, J])$.

$$\text{Det}(A) = \sum_{I \subseteq [n], |I| = |J|} \text{Det}(A[\bar{I}, \bar{J}]) \cdot \text{Det}(A[I, J]) \cdot (-1)^{\sum I + \sum J}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

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Recall: A Linear (or Representable) Matroid

$\mathcal{M} = (E, \mathcal{I})$, where $E = \{e_1, \ldots, e_n\}$ and $\mathcal{I} \subseteq 2^E$

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$$A_M = \begin{pmatrix} \overbrace{\qquad\qquad\qquad\qquad\qquad\qquad}^{\text{Columns indexed by elements of } E} \\ \\ \\ \end{pmatrix}$$

$\mathcal{M} = (E, \mathcal{I})$, where $E = \{e_1, \ldots, e_n\}$ and $\mathcal{I} \subseteq 2^E$

Columns indexed by elements of E

$$A_M = \begin{pmatrix} x_{e_1} & & & \end{pmatrix}$$

$$\mathcal{M} = (E, \mathcal{I}), \text{ where } E = \{e_1, \ldots, e_n\} \text{ and } \mathcal{I} \subseteq 2^E$$

Columns indexed by elements of E

$$A_M = \begin{pmatrix} x_{e_1} & x_{e_2} & & & \end{pmatrix}$$

$\mathcal{M} = (E, \mathcal{I})$, where $E = \{e_1, \ldots, e_n\}$ and $\mathcal{I} \subseteq 2^E$

$$A_M = \begin{pmatrix} \overbrace{\begin{matrix} | & | & | & & & & \\ x_{e_1} & x_{e_2} & \cdots & & & & \\ | & | & | & & & & \end{matrix}}^{\text{Columns indexed by elements of } E} \end{pmatrix}$$

$$\mathcal{M} = (E, \mathcal{I}), \text{ where } E = \{e_1, \ldots, e_n\} \text{ and } \mathcal{I} \subseteq 2^E$$

$$A_M = \left(\overbrace{\begin{matrix} \vdots & \vdots & \vdots & \vdots & & & \\ x_{e_1} & x_{e_2} & \cdots & x_{e_i} & & & \\ \vdots & \vdots & \vdots & \vdots & & & \end{matrix}}^{\text{Columns indexed by elements of } E} \right)$$

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$$A_M = \begin{pmatrix} \overbrace{x_{e_1} \quad x_{e_2} \quad \cdots \quad x_{e_i} \quad \cdots \quad x_{e_{n-1}} \quad\quad}^{\text{Columns indexed by elements of } E} \end{pmatrix}$$

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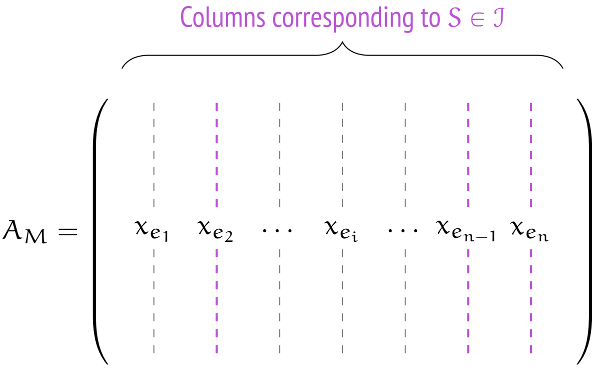
Columns indexed by elements of E

$$A_M = \begin{pmatrix} x_{e_1} & x_{e_2} & \cdots & x_{e_i} & \cdots & x_{e_{n-1}} & x_{e_n} \end{pmatrix}$$

$$\mathcal{M} = (E, \mathcal{I}), \text{ where } E = \{e_1, \ldots, e_n\} \text{ and } \mathcal{I} \subseteq 2^E$$



$\mathcal{M} = (E, \mathcal{I})$, where $E = \{e_1, \ldots, e_n\}$ and $\mathcal{I} \subseteq 2^E$

Columns corresponding to $S \in \mathcal{I}$

$$A_{\mathcal{M}} = \begin{pmatrix} x_{e_1} & x_{e_2} & \cdots & x_{e_i} & \cdots & x_{e_{n-1}} & x_{e_n} \end{pmatrix}$$

...are linearly independent.

$\mathcal{M} = (E, \mathcal{I})$, where $E = \{e_1, \ldots, e_n\}$ and $\mathcal{I} \subseteq 2^E$

Columns that are linearly independent...

$$A_M = \begin{pmatrix} x_{e_1} & x_{e_2} & \cdots & x_{e_i} & \cdots & x_{e_{n-1}} & x_{e_n} \end{pmatrix}$$

$$\mathcal{M} = (E, \mathcal{I}), \text{ where } E = \{e_1, \ldots, e_n\} \text{ and } \mathcal{I} \subseteq 2^E$$

Columns that are linearly independent...

$$A_{\mathcal{M}} = \begin{pmatrix} x_{e_1} & x_{e_2} & \cdots & x_{e_i} & \cdots & x_{e_{n-1}} & x_{e_n} \end{pmatrix}$$

...correspond to sets in $\mathcal{I}$.

$\mathcal{M} = (E, \mathcal{I})$, where $E = \{e_1, \ldots, e_n\}$ and $\mathcal{I} \subseteq 2^E$

$$A_M = \overbrace{\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{e_1} & x_{e_2} & \cdots & x_{e_i} & \cdots & x_{e_{n-1}} & x_{e_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}}^{\text{Columns indexed by elements of } E} \left.\vphantom{\begin{pmatrix} \vdots \\ x \\ \vdots \end{pmatrix}}\right\} \mathrm{rk}(M)$$

Given: A collection of p-sized independent sets[1]:

$$\mathcal{S} = \{S_1, \ldots, S_t\}.$$

[1] The rank of the underlying matroid is $(p + q)$.

Given: A collection of p-sized independent sets[1]:

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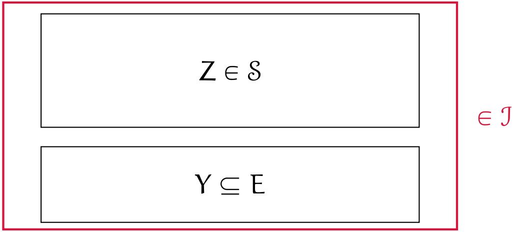
Want: A q-representative subfamily $\widehat{\mathcal{S}}$ of size $\leqslant \binom{p+q}{p}$.

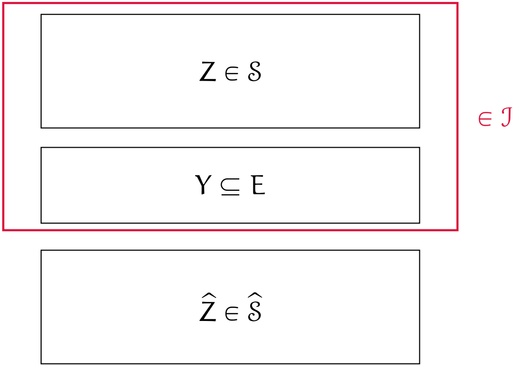
[1] The rank of the underlying matroid is $(p+q)$.

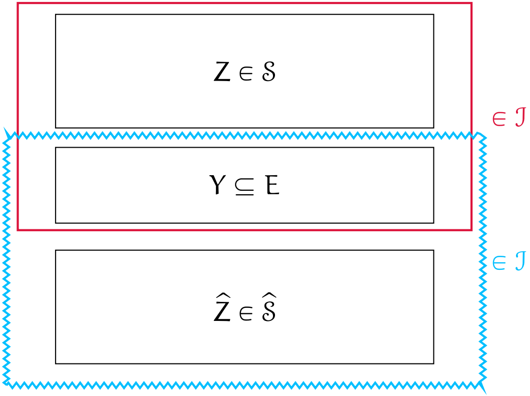
$$Z \in \mathcal{S}$$

$$Z \in \mathcal{S}$$

$$Y \subseteq E$$


Z ∈ 𝒮
Y ⊆ E
∈ ℐ


$Z \in \mathcal{S}$
$Y \subseteq E$
$\in \mathcal{I}$
$\widehat{Z} \in \widehat{\mathcal{S}}$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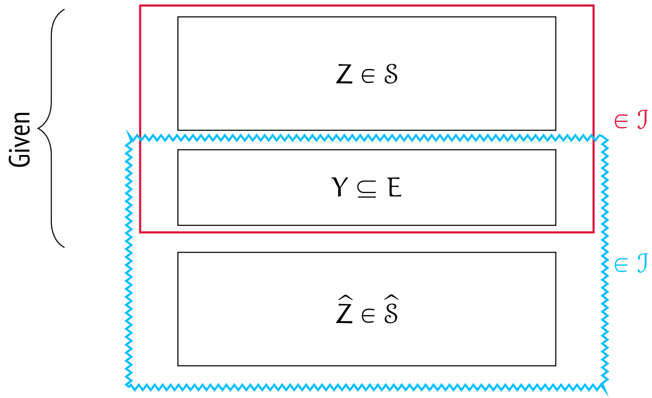
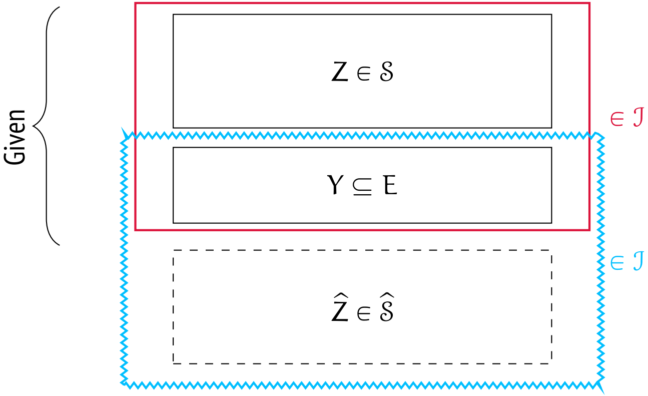


Given
$Z \in \mathcal{S}$
$Y \subseteq E$
$\hat{Z} \in \hat{\mathcal{S}}$
$\in \mathcal{I}$
$\in \mathcal{I}$


Given
$Z \in \mathcal{S}$
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$\in \mathcal{I}$
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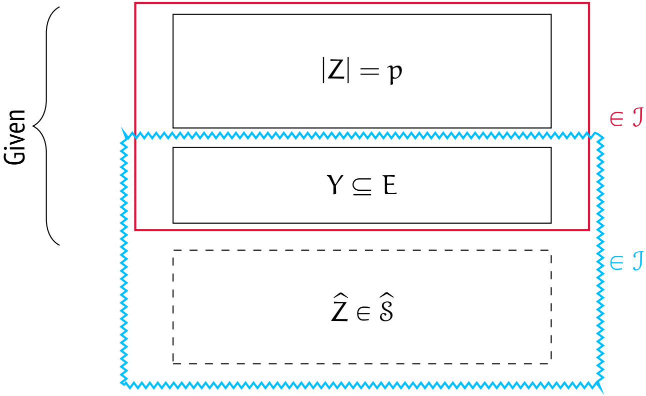
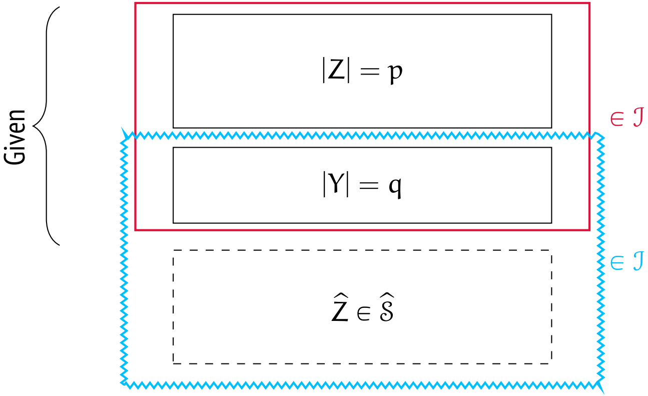
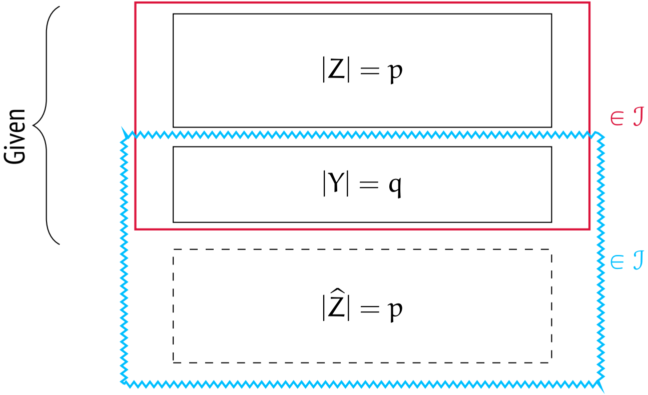

Given
|Z| = p
Y ⊆ E
∈ ℐ
∈ ℐ
Ẑ ∈ Ŝ

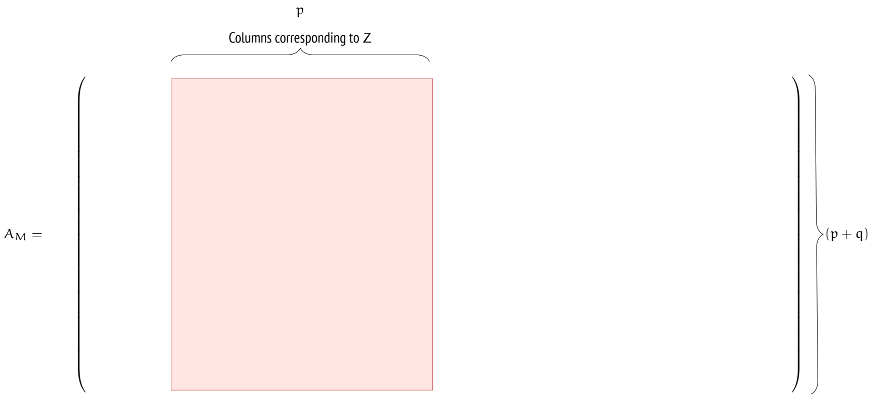


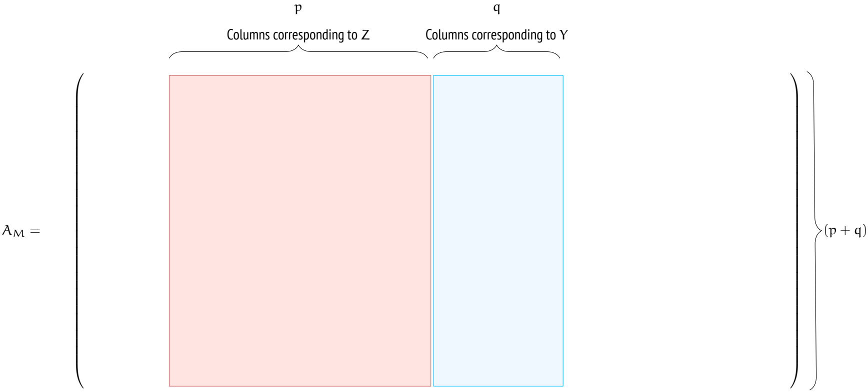
Given
$|Z| = p$
$|Y| = q$
$\widehat{Z} \in \widehat{\mathcal{S}}$
$\in \mathcal{I}$
$\in \mathcal{J}$



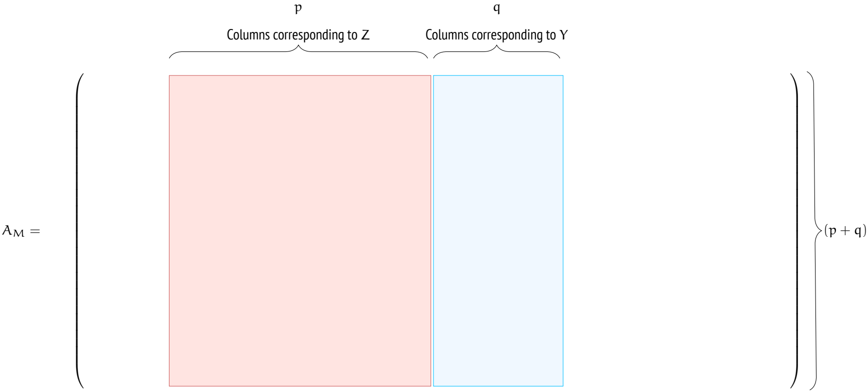
Given
|Z| = p
|Y| = q
|Ẑ| = p
∈ ℐ
∈ ℐ

$$A_M = \left(\qquad \right) \Bigg\} (p+q)$$


p
Columns corresponding to Z
$A_M =$
$(p + q)$


p
q
Columns corresponding to Z
Columns corresponding to Y
A_M =
(p + q)

$$\mathsf{Det}(A_M[\star, Z \cup Y])$$



$$0 \neq \mathrm{Det}(A_M[\star, Z \cup Y])$$

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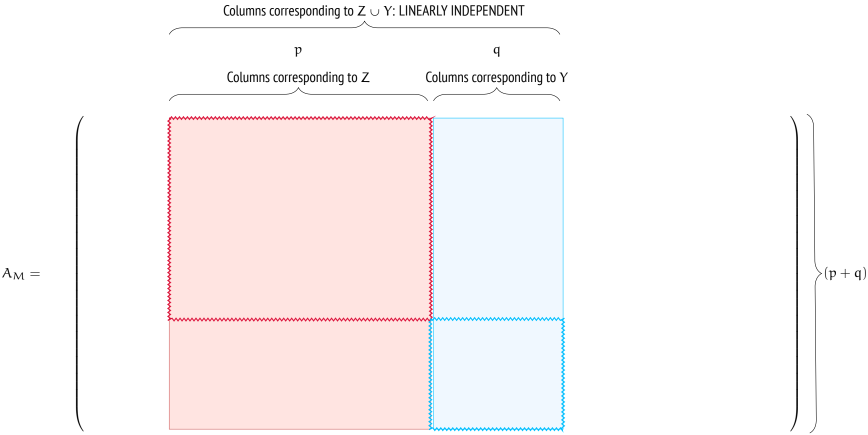
$$0 \neq \mathrm{Det}(A_M[\star, Z \cup Y]) = \sum_{I \subseteq [p+q], |I| = p} \mathrm{Det}(A[I, Z]) \cdot \mathrm{Det}(A[\bar{I}, Y]) \cdot \oslash$$



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$$v_Z := \left(\mathsf{Det}(A[I_0, Z]) \;, \ldots, \; \mathsf{Det}(A[I_j, Z]) \;, \ldots, \; \mathsf{Det}(A[I_r, Z]) \right)$$

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All subsets of size p of $(p+q)$.

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$$\vdots$$

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$$0 \neq \mathrm{Det}(A_M[\star, Z \cup Y]) = \sum_{I\subseteq[p+q],|I|=p} \mathrm{Det}(A[I, Z]) \cdot \mathrm{Det}(A[\bar{I}, Y]) \cdot \oslash$$

$$v_Z := \left(\mathrm{Det}(A[I_0, Z]) \;, \ldots, \; \mathrm{Det}(A[I_j, Z]) \;, \ldots, \; \mathrm{Det}(A[I_r, Z]) \right)$$

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All subsets of size p of $(p+q)$.

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$$\vdots$$

$$v_{S_t} := \left(\mathrm{Det}(A[I_0, S_t]) \;, \ldots, \; \mathrm{Det}(A[I_j, S_t]) \;, \ldots, \; \mathrm{Det}(A[I_r, S_t]) \right)$$

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$$v_Z := \big(\mathrm{Det}(A[I_0, Z]) \;, \ldots, \; \mathrm{Det}(A[I_j, Z]) \;, \ldots, \; \mathrm{Det}(A[I_r, Z]) \big)$$

All subsets of size p of $(p + q)$.

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$$\vdots$$

$$v_{S_i} := \big(\mathrm{Det}(A[I_0, S_i]) \;, \ldots, \; \mathrm{Det}(A[I_j, S_i]) \;, \ldots, \; \mathrm{Det}(A[I_r, S_i]) \big)$$

$$\vdots$$

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$$\chi(\mathcal{S}) := \{v_{S_1}, \ldots\ldots\ldots, v_{S_i}, \ldots\ldots\ldots, v_{S_t}\}$$

$$0 \neq \mathrm{Det}(A_M[\star, Z \cup Y]) = \sum_{I \subseteq [p+q], |I| = p} \mathrm{Det}(A[I, Z]) \cdot \mathrm{Det}(A[\bar{I}, Y]) \cdot \oslash$$

$$v_Z := \left(\mathrm{Det}(A[I_0, Z]) \;, \ldots, \; \mathrm{Det}(A[I_j, Z]) \;, \ldots, \; \mathrm{Det}(A[I_r, Z]) \right)$$

All subsets of size p of $(p+q)$.

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$\binom{p+q}{p}$-dimensional vectors.

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All subsets of size p of $(p+q)$.

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A basis of size $\leqslant \binom{p+q}{p}$ for

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$$v_Z = \lambda_1 v_{T_1} + \cdots + \lambda_r v_{T_r}$$

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$$\vdots$$

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A basis of size $\leqslant \binom{p+q}{p}$ for

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$$0 \neq \mathrm{Det}(A_M[\star, Z \cup Y]) = \sum_{i=1}^{r} \mathrm{Det}(A_M[\star, T_i \cup Y])$$

Note that at for at least one T_i, we have that:

$$\mathrm{Det}(A_M[\star, T_i \cup Y]) \neq 0$$

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For such a T_i, we know that:

1. $Y \cap T_i = \varnothing$ (easily checked: all terms that survive have this property),
2. $Y \cup T_i \in \mathcal{I}$ (since non-zero determinant $\longrightarrow$ linearly independent columns).

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Thus, the sets corresponding to the basis vectors, $T_1, \ldots, T_r$, do form a q-representative family.

$$0 \neq \mathrm{Det}(A_M[\star, Z \cup Y]) = \sum_{i=1}^{r} \mathrm{Det}(A_M[\star, T_i \cup Y])$$

$$v_Z := \begin{pmatrix} \mathrm{Det}(A[I_0, Z]) & ,\dots, & \mathrm{Det}(A[I_j, Z]) & ,\dots, & \mathrm{Det}(A[I_r, Z]) \end{pmatrix}$$

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A basis of size $\binom{p+q}{p}$ for

$$\chi(\mathcal{S}) := \{v_{S_1}, \dots\dots\dots, v_{S_i}, \dots\dots\dots, v_{S_t}\}$$

Computing $T_1, \ldots, T_r$.

We form a matrix with the vectors $\{v_{S_1}, \ldots, v_{S_i}, \ldots, v_{S_t}\}$ as the columns:

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$$\begin{pmatrix} \vdots & & & & \\ v_{S_1} & & & & \\ \vdots & & & & \end{pmatrix}$$

Computing $T_1, \ldots, T_r$.

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$$\begin{pmatrix} \vdots & \vdots & & & & \\ v_{S_1} & v_{S_2} & & & & \\ \vdots & \vdots & & & & \end{pmatrix}$$

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We form a matrix with the vectors $\{v_{S_1}, \ldots, v_{S_i}, \ldots, v_{S_t}\}$ as the columns:

$$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & & & \\ v_{S_1} & v_{S_2} & \cdots & v_{S_i} & & & \\ \vdots & \vdots & \vdots & \vdots & & & \end{pmatrix}$$

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$$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \\ v_{S_1} & v_{S_2} & \cdots & v_{S_i} & \cdots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

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We form a matrix with the vectors $\{v_{S_1}, \ldots, v_{S_i}, \ldots, v_{S_t}\}$ as the columns:

$$\begin{pmatrix} v_{S_1} & v_{S_2} & \cdots & v_{S_i} & \cdots & v_{S_{t-1}} \end{pmatrix}$$

Computing $T_1, \ldots, T_r$.

We form a matrix with the vectors $\{v_{S_1}, \ldots, v_{S_i}, \ldots, v_{S_t}\}$ as the columns:

$$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{S_1} & v_{S_2} & \cdots & v_{S_i} & \cdots & v_{S_{t-1}} & v_{S_t} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Computing $T_1, \ldots, T_r$.

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...and compute a column basis.

$$
\begin{pmatrix}
[\![A[I_0, S_1]]\!] & [\![A[I_0, S_2]]\!] & \dots & [\![A[I_0, S_i]]\!] & \dots & [\![A[I_0, S_t]]\!] \\
[\![A[I_1, S_1]]\!] & [\![A[I_1, S_2]]\!] & \dots & [\![A[I_1, S_i]]\!] & \dots & [\![A[I_1, S_t]]\!] \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
[\![A[I_j, S_1]]\!] & [\![A[I_j, S_2]]\!] & \dots & [\![A[I_j, S_i]]\!] & \dots & [\![A[I_j, S_t]]\!] \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
[\![A[I_r, S_1]]\!] & [\![A[I_r, S_2]]\!] & \dots & [\![A[I_r, S_i]]\!] & \dots & [\![A[I_r, S_t]]\!]
\end{pmatrix}
$$

$$
\overbrace{\begin{pmatrix}
[\![A[I_0,S_1]]\!] & [\![A[I_0,S_2]]\!] & \dots & [\![A[I_0,S_i]]\!] & \dots & [\![A[I_0,S_t]]\!] \\
[\![A[I_1,S_1]]\!] & [\![A[I_1,S_2]]\!] & \dots & [\![A[I_1,S_i]]\!] & \dots & [\![A[I_1,S_t]]\!] \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
[\![A[I_j,S_1]]\!] & [\![A[I_j,S_2]]\!] & \dots & [\![A[I_j,S_i]]\!] & \dots & [\![A[I_j,S_t]]\!] \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
[\![A[I_r,S_1]]\!] & [\![A[I_r,S_2]]\!] & \dots & [\![A[I_r,S_i]]\!] & \dots & [\![A[I_r,S_t]]\!]
\end{pmatrix}}^{\text{t columns}}
$$

$$
\overbrace{\begin{pmatrix}
[\![A[I_0,S_1]]\!] & [\![A[I_0,S_2]]\!] & \dots & [\![A[I_0,S_i]]\!] & \dots & [\![A[I_0,S_t]]\!] \\
[\![A[I_1,S_1]]\!] & [\![A[I_1,S_2]]\!] & \dots & [\![A[I_1,S_i]]\!] & \dots & [\![A[I_1,S_t]]\!] \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
[\![A[I_j,S_1]]\!] & [\![A[I_j,S_2]]\!] & \dots & [\![A[I_j,S_i]]\!] & \dots & [\![A[I_j,S_t]]\!] \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
[\![A[I_r,S_1]]\!] & [\![A[I_r,S_2]]\!] & \dots & [\![A[I_r,S_i]]\!] & \dots & [\![A[I_r,S_t]]\!]
\end{pmatrix}}^{t \text{ columns}}
\left.\vphantom{\begin{matrix}a\\a\\a\\a\\a\\a\end{matrix}}\right\} \binom{p+q}{q} \text{ rows}
$$

$t \cdot \binom{p+q}{q}$ Determinant Computations.

Let $\mathcal{M}$ be a linear matroid of rank $p + q = k$, $\mathcal{S} = \{S_1, \ldots, S_t\}$ be a p-family of independent sets. Then there exists a q-representative of size at most $\binom{p+q}{q}$.

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Moreover, given a representation of $\mathcal{M}$ over a field $\mathbb{F}$, we can find such a representative family in $O\left(\binom{p+q}{q}tp^{\omega} + t\binom{p+q}{q}^{\omega-1}\right)$ operations over $\mathbb{F}$.

REPRESENTATIVE SETS

And that will be all!