Obstructions for Fine-grained Relationships

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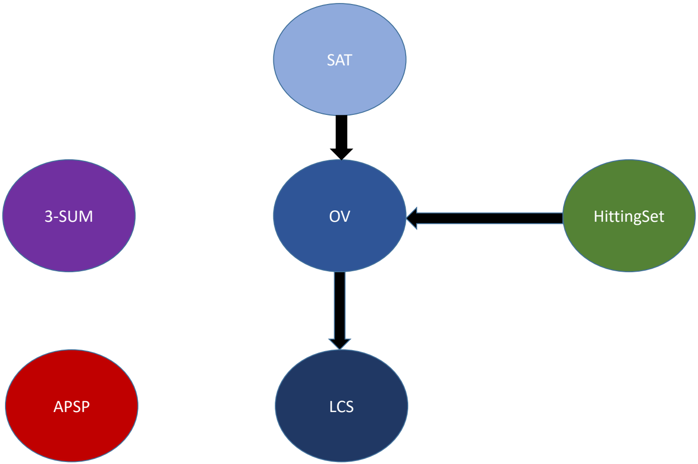
Outline

Problems Fine-grained Complexity Relationships

- Cnf-Sat, Orthogonal Vectors, 3-sum, Hitting Set, Longest Common Subsequence, All-Pairs Shortest Paths, $\cdots$
- Fine-grained reductions: $(L_1, T_1) \leq_{FGR} (L_2, T_2)$ implies polynomial improvements in the conjectured complexity T_2 of L_2 will produce a polynomial improvement in the conjectured complexity T_1 of L_1.

Problems and Fine-grained Relationships



Obstructions for a Reduction from CNF-SAT to 3-SUM and other problems

3-SUM Problem

3-SUM: Given a sequence of integers $x_1, x_2, \ldots, x_n$ where $x_i \in [-n^d, n^d]$, determine if there exist i, j, k such that $x_i + x_j + x_k = 0$.

Plan

- Fine-grained reductions
- A subquadratic co-nondeterministic algorithm for 3-SUM.
- Nondeterministic SETH (**NSETH**)
- Reduction of k-SAT to formulas of the form $\exists\vec{x}\forall\vec{y}\phi$ where ϕ is a 3-DNF.

Fine-grained Reductions

Definition (Fine-Grained Reductions ($\leq_{FGR}$))

Let L_1 and L_2 be languages, and let T_1 and T_2 be time bounds. We say that (L_1, T_1) fine-grained reduces to (L_2, T_2) (denoted $(L_1, T_1) \leq_{FGR} (L_2, T_2)$) if for all $\varepsilon > 0$, there is a $\delta > 0$ and a deterministic Turing reduction $\mathcal{M}^{L_2}$ from L_1 to L_2 satisfying the following conditions.

(a) The time complexity of the Turing reduction without counting the oracle calls is bounded by $T_1^{1-\delta}$.

$$\mathbf{TIME}[\mathcal{M}] \leq T_1^{1-\delta}$$

(b) Let $\tilde{Q}(\mathcal{M}, x)$ denote the set of queries made by $\mathcal{M}$ to the oracle on an input x of length n. The query lengths obey time bound: $\sum_{q \in \tilde{Q}(\mathcal{M},x)} (T_2(|q|))^{1-\epsilon} \leq (T_1(n))^{1-\delta}$

Basic Properties of Deterministic Fine-grained Reductions

Lemma

Fine-grained reductions are closed under composition.

Lemma

Let $(L_1, T_1) \leq_{FGR} (L_2, T_2)$ and $L_2 \in \mathbf{DTIME}(T_2^{1-\varepsilon}(n))$ for some $\varepsilon > 0$. Then there exists a $\delta > 0$ such that $L_1 \in \mathbf{DTIME}(T_1^{1-\delta}(n))$.

- Deterministic fine-grained reductions translate savings in deterministic time.

Basic Properties of Deterministic Fine-grained Reductions

Lemma

Let $(L_1, T_1) \leq_{FGR} (L_2, T_2)$ *and*
$L_2 \in \mathbf{NTIME}(T_2^{1-\varepsilon}(n)) \cap \mathbf{coNTIME}(T_2^{1-\varepsilon}(n))$ *for some* $\varepsilon > 0$*.*
Then there exists a $\delta > 0$ *such that*
$L_1 \in \mathbf{NTIME}(T_1^{1-\delta}(n)) \cap \mathbf{coNTIME}(T_1^{1-\delta}(n))$*.*

- Deterministic fine-grained reductions translate savings when there is savings in both nondeterministic time and co-nondeterministic time.

Efficient Algorithm for Proving 3-SUM Freeness

Theorem

3-SUM *can be solved in* $(\mathbf{NTIME} \cap \mathbf{coNTIME})(\tilde{O}(n^{1.5}))$.

- In other words, there is a subquadratic proof to verify the existence of a triple (i, j, k) such that $x_i + x_j + x_k = 0$ and
- there is a subquadratic proof to verify that there is no triple (i, j, k) such that $x_i + x_j + x_k = 0$.

Proof

- $S = \{[(i,j,k),p] | x_i + x_j + x_k = 0 \mod p\}$.
- $|S| \leq O(dn^3 \log n)$.
- The number of triples is at most n^3. The sum corresponding to any triple is at most $3n^d$ which has at most $O(d \log n)$ prime factors.
- $S_p = \{(i,j,k) | x_i + x_j + x_k = 0 \mod p\}$.
- There exist a prime p among the first $n^{1.5}$ primes such that $|S_p| \leq \tilde{O}(n^{1.5})$ by pigeon-hole principle.
- The size of such a prime is at most $\tilde{O}(n^{1.5})$.

Proof of 3-SUM Freeness

- Nondeterministically select a number $p = O(n \log n)$.
- Verify that p is a prime. If not, reject.
- Nondeterministically guess the size t of $S_p = \{(i, j, k) | x_i + x_j + x_k = 0 \mod p\}$.
- If $t > dn^{1.5} \log n$, reject.
- Compute $|S_p|$ using FFT.
- Form $P(z) = \sum_i z^{x_i \mod p}$.
- Compute $P^3(z)$ using FFT in time $\tilde{O}(n^{1.5})$ and extract the coefficients of z^0, z^p and z^{2p} and sum them.
- If $t \neq |S_p|$, reject.
- Nondeterministically select t triples (i_l, j_l, k_l) for $1 \leq l \leq t$.
- For each $1 \leq l \leq t$, check if $x_{i_l} + x_{j_l} + x_{k_l} = 0 \mod p$ and $x_{i_l} + x_{j_l} + x_{k_l} \neq 0$.
- If any check fails, reject.
- Otherwise, accept.

Analysis

- Time complexity is $\tilde{O}(n^{1.5})$.
- If there is a triple (i, j, k) such that $x_i + x_j + x_k = 0$, the algorithm will reject assuming that p and t are selected such that $p \in \tilde{O}(n^{1.5})$ and $t = |S_p|$.
- If not, the algorithm will reject any way.

Co-nondeterministic Algorithms for Other Problems

Theorem

Hitting Set *can be solved in* $(\mathbf{NTIME} \cap \mathbf{coNTIME})(O(m)$ *where* m *is the size of the input.*

Theorem

Maximum Matching *can be solved in* $(\mathbf{NTIME} \cap \mathbf{coNTIME})(O(n+m)$ *where* n *is the number of vertices and* m *is the number of edges.*

Theorem

All-Pairs Shortest Paths *can be solved in* $(\mathbf{NTIME} \cap \mathbf{coNTIME})(O(n^{(3+\omega)/2})$*.*

Nondeterministic **SETH** (**NSETH**)

Definition

NSETH: For every $\varepsilon > 0$, there exists a k so that k-TAUT is not in **NTIME**$(2^{n(1-\varepsilon)})$ where k-TAUT is the language of all k-CNF which are tautologies.

Plausibility:

- Research in designing proof systems for k-TAUT hit an impasse similar to the one in the case of research in developing efficient algorithms for k-SAT. The best proof systems require 2^n time in the limit as k increases.
- Restricted proof systems (for example, tree-like resolution, regular resolution) provably require 2^n time.
- If **NSETH** is false, we can obtain circuit lower bounds.

Consequences of NSETH

Theorem

If **NSETH** *and* $L \in \mathbf{NTIME}(T_L) \cap \mathbf{coNTIME}(T_L)$ *for some language* L*, then* $(\textsc{Cnf-Sat}, 2^n) \not\leq_{FGR} (L, T_L^{(1+\varepsilon)})$ *for any* $\varepsilon > 0$

Proof:

- If $(\textsc{Cnf-Sat}, 2^n) \leq_{FGR} (L, T_L^{(1+\varepsilon)})$, there exists $\delta > 0$ such that $\textsc{Cnf-Sat} \in \mathbf{NTIME}(2^{n(1-\delta)}) \cap \mathbf{coNTIME}(2^{n(1-\delta)})$.
- since deterministic fine-grained reductions pass on improvements in nondeterministic and co-nondeterministic algorithms.
- which violates **NSETH**.

Consequences of **NSETH**

Corollary

If **NSETH**, *there is no deterministic fine-grained reductions from* CNF-SAT *to the following problems with the conjectured complexities for any* $\gamma > 0$.

- 3-SUM *with the conjectured complexity* $n^{1.5+\gamma}$.
- HITTING SET *with the conjectured complexity* $n^{1+\gamma}$.
- MAXFLOW *and* MAXIMUM MATCHING *with the conjectured complexity* $n^{1+\gamma}$.
- ALL-PAIRS SHORTEST PATHS *with the conjectured complexity* $n^{(3+\omega)/2+\gamma}$ *where* ω *is the matrix multiplication constant.*

Status of Fine-grained Relationships

- Consider the problems: OV, HS, LCS, 3-$\textsc{sum}$
- Known fine-grained reductions: $\mathrm{HS} \leq_{FGR} \mathrm{OV} \leq_{FGR} \mathrm{LCS}$.
- **NSETH** obstruction: $\mathrm{OV} \rightarrow \mathrm{HS}$; $\mathrm{LCS} \rightarrow \mathrm{HS}$; $\mathrm{OV} \rightarrow$ 3-$\textsc{sum}$; $\mathrm{LCS} \rightarrow$ 3-$\textsc{sum}$
- Circuit satisfiability obstruction: $\mathrm{LCS} \rightarrow \mathrm{OV}$;
- Open: $\mathrm{HS} \rightarrow$ 3-$\textsc{sum}$; 3-$\textsc{sum} \rightarrow \mathrm{HS}$; 3-$\textsc{sum} \rightarrow \mathrm{OV}$; 3-$\textsc{sum} \rightarrow \mathrm{LCS}$;

Other Open Problems

- Does **SETH** imply **NSETH**?

Quantified Boolean Formulas

Show that the satisfiability of formulas of the form $\forall\vec{x}\exists\vec{y}\phi(\vec{x},\vec{y})$ where ϕ is a 3-CNF or of the form $\exists\vec{x}\forall\vec{y}\phi(\vec{x},\vec{y})$ where ϕ is a 3-DNF cannot be solved in time $2^{n(1-\varepsilon)}$ for any $\varepsilon > 0$ assuming **SETH**. Here n, the number of variables, is the sum of the number of variables in $\vec{x}$ and $\vec{y}$.

Hint: Reduce k-CNF to formulas of the form $\exists\vec{x}\forall\vec{y}\phi(\vec{x},\vec{y})$ where ϕ is a 3-DNF. Use Sparsification Lemma and the following minimally unsatisfiable formula to reduce k-CNF. The minimally unsatisfiable formula is a conjunction of the following disjunctions.

- $\bigvee_{i=1}^{m} p_i$
- $p_i \rightarrow q_j$ for all $1 \leq i, j \leq m$.
- $\bigvee_{i=1}^{m} \bar{q}_i$

Thank You