Optimal Algorithms for Bounded Weighted Edit Distance

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Based on joint works with:

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PennState · University of Maryland · UC San Diego

Alejandro Cassis and Philip Wellnitz

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(Weighted) Edit Distance

Edit distance ed(X, Y) — Levenshtein distance

Minimum number of character insertions, deletions, and substitutions that transform X to Y.

X : `b b a b a b b a a b`

Y : `b a b a b a a a b b`

$\mathrm{ed}(X, Y) = 3$

(Weighted) Edit Distance

Edit distance $\mathrm{ed}(X, Y)$ — Levenshtein distance

Minimum number of character insertions, deletions, and substitutions that transform X to Y.

w	ε	a	b
ε	0	1	1
a	1	0	1
b	1	1	0

X : b b a b a b b a a b

Y : b a b a b a a a b b

$\mathrm{ed}^w(X, Y) = 3$

Weighted edit distance $\mathrm{ed}^w(X, Y)$ — $w : (\Sigma \cup \{\varepsilon\}) \times (\Sigma \cup \{\varepsilon\}) \to \mathbb{R}_{\geq 0}$

The minimum cost of transforming X to Y using character edits, where:

- inserting b costs $w(\varepsilon, b)$;
- deleting a costs $w(a, \varepsilon)$;
- substituting a for b costs $w(a, b)$.

(Weighted) Edit Distance

Edit distance ed(X, Y) — Levenshtein distance

Minimum number of character insertions, deletions, and substitutions that transform X to Y.

w	ε	a	b
ε	0	1	3
a	1	0	2
b	3	2	0

X : b b a b a b b a a b

Y : b a b a b a a a b b

$\mathrm{ed}^w(X, Y) \leq 8$

Weighted edit distance $\mathrm{ed}^w(X, Y)$ — $w : (\Sigma \cup \{\varepsilon\}) \times (\Sigma \cup \{\varepsilon\}) \to \mathbb{R}_{\geq 0}$

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w	ε	a	b
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a	1	0	2
b	3	2	0

X : b b a b a b b a a b

Y : b a b a b a a a b b

$\mathrm{ed}^w(X, Y) = 6$

Weighted edit distance $\mathrm{ed}^w(X, Y)$ — $w : (\Sigma \cup \{\varepsilon\}) \times (\Sigma \cup \{\varepsilon\}) \to \mathbb{R}_{\geq 0}$

The minimum cost of transforming X to Y using character edits, where:

- inserting b costs $w(\varepsilon, b)$;
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Computing Edit Distance

Reference	Time	Remarks
Vin68,NW70,Sel74,WF74	$\mathcal{O}(n^2)$	
BI18	$\Omega(n^{2-o(1)})$	unweighted; conditioned on SETH/OV

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Bounded edit distance: $\mathrm{ed}^w(X, Y) \leq k$ for some input threshold k

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Assumption: Normalized weight function, that is, $w(a, b) \ge 1$ for all distinct $a, b \in \Sigma \cup \{\varepsilon\}$.

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Ukk85,Mye86	$\mathcal{O}(nk)$	
LV88	$\mathcal{O}(n + k^2)$	unweighted only
folklore	$\Omega(n + k^{2-o(1)})$	unweighted; conditioned on SETH/OV

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DGH**K**S23	$\mathcal{O}(n + k^5)$	

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C**K**W23	$\widetilde{\mathcal{O}}(n + \sqrt{nk^3})$	

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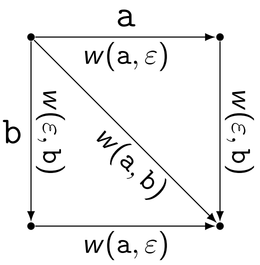
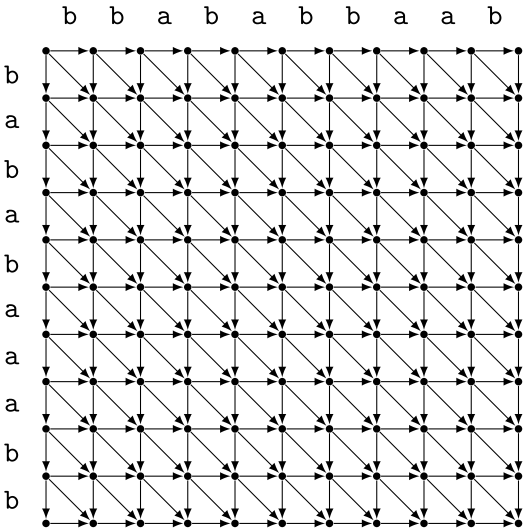
DGH**K**S23	$\mathcal{O}(n + k^5)$	
C**K**W23	$\tilde{\mathcal{O}}(n + \sqrt{nk^3})$	
C**K**W23	$\Omega(n + \sqrt{nk^{3-o(1)}})$	conditioned on APSP, $\sqrt{n} \le k \le n$

Alignment Graph and Dynamic-Programming Algorithms





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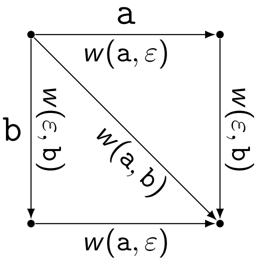

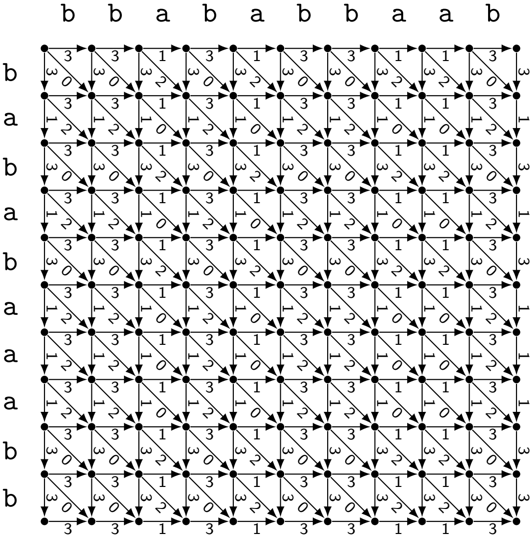

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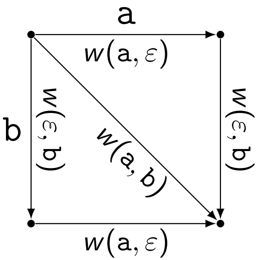

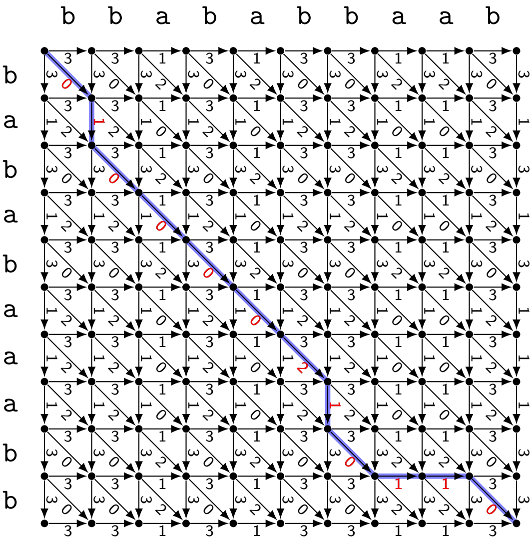

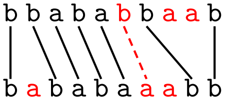

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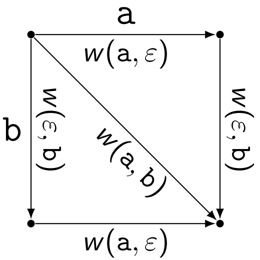
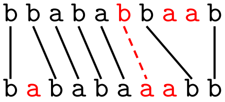


		b	b	a	b	a	b	b	a	a	b
	0	3	6	7	10	11	14	17	18	19	22
b	3	0	3	4	7	8	11	14	15	16	19
a	4	1	2	3	6	7	10	13	14	15	18
b	7	4	1	2	3	4	7	10	11	12	15
a	8	5	2	1	4	3	6	9	10	11	14
b	11	8	5	4	1	2	3	6	7	8	11
a	12	9	6	5	2	1	4	5	6	7	10
a	13	10	7	6	3	2	3	6	5	6	9
a	14	11	8	7	4	3	4	5	6	5	8
b	17	14	11	10	7	6	3	4	5	6	5
b	20	17	14	13	10	9	6	3	4	5	6



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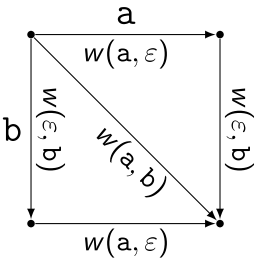
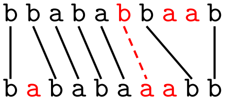


		b	b	a	b	a	b	b	a	a	b
	0	3	6	7	10	11	14	17	18	19	22
b	3	0	3	4	7	8	11	14	15	16	19
a	4	1	2	3	6	7	10	13	14	15	18
b	7	4	1	2	3	4	7	10	11	12	15
a	8	5	2	1	4	3	6	9	10	11	14
b	11	8	5	4	1	2	3	6	7	8	11
a	12	9	6	5	2	1	4	5	6	7	10
a	13	10	7	6	3	2	3	6	5	6	9
a	14	11	8	7	4	3	4	5	6	5	8
b	17	14	11	10	7	6	3	4	5	6	5
b	20	17	14	13	10	9	6	3	4	5	6



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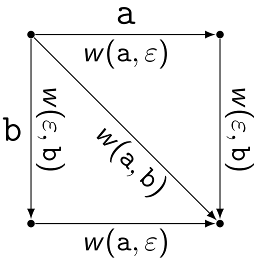
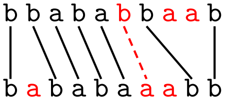


		b	b	a	b	a	b	b	a	a	b
	0	3	6	7	10	11	14				
b	3	0	3	4	7	8	11	14			
a	4	1	2	3	6	7	10	13	14		
b	7	4	1	2	3	4	7	10	11	12	
a	8	5	2	1	4	3	6	9	10	11	14
b	11	8	5	4	1	2	3	6	7	8	11
a	12	9	6	5	2	1	4	5	6	7	10
a		10	7	6	3	2	3	6	5	6	9
a			8	7	4	3	4	5	6	5	8
b				10	7	6	3	4	5	6	5
b					10	9	6	3	4	5	6



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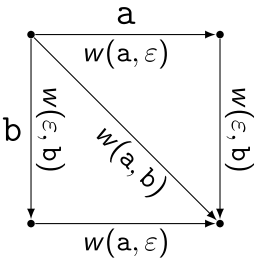
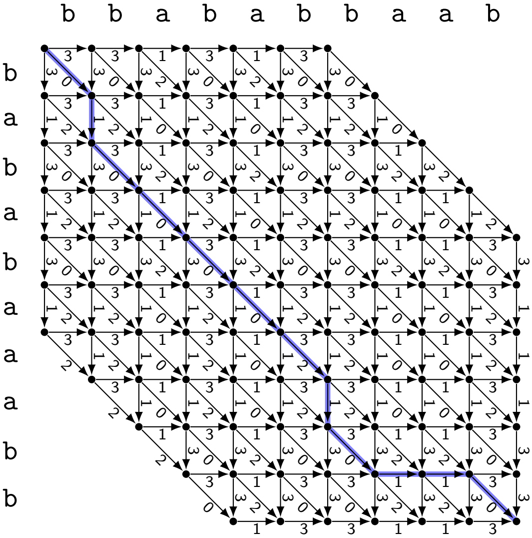
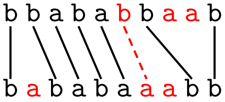






Why Is Weigheted Edit Distance Harder?

Unweighted case:
The values along diagonals may only increase.

		b	b	a	b	a	b	b	a	a	b
	0	1	2	3	4	5	6	7	8	9	10
b	1	0	1	2	3	4	5	6	7	8	9
a	2	1	1	1	2	3	4	5	6	7	8
b	3	2	1	2	1	2	3	4	5	6	7
a	4	3	2	1	2	1	2	3	4	5	6
b	5	4	3	2	1	2	1	2	3	4	5
a	6	5	4	3	2	1	2	2	2	3	4
a	7	6	5	4	3	2	2	3	2	2	3
a	8	7	6	5	4	3	3	3	3	2	3
b	9	8	7	6	5	4	3	3	4	3	2
b	10	9	8	7	6	5	4	3	4	4	3

Why Is Weigheted Edit Distance Harder?

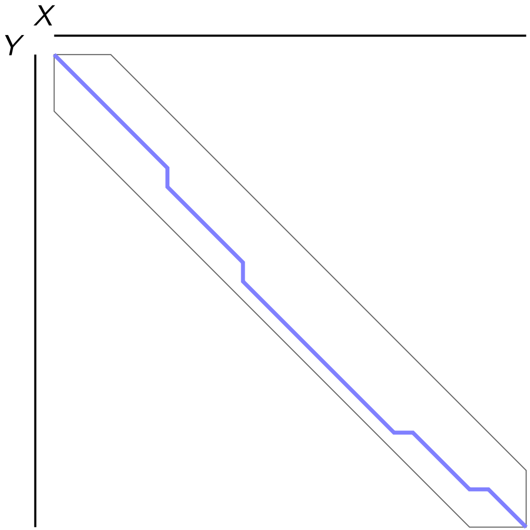
Unweighted case:
The values along diagonals may only increase.

Weighted case:
The values along diagonals may both increase and decrease.

		b	b	a	b	a	b	b	a	a	b
	0	3	6	7	10	11	14	17	18	19	22
b	3	0	3	4	7	8	11	14	15	16	19
a	4	1	2	3	6	7	10	13	14	15	18
b	7	4	1	2	3	4	7	10	11	12	15
a	8	5	2	1	4	3	6	9	10	11	14
b	11	8	5	4	1	2	3	6	7	8	11
a	12	9	6	5	2	1	4	5	6	7	10
a	13	10	7	6	3	2	3	6	5	6	9
a	14	11	8	7	4	3	4	5	6	5	8
b	17	14	11	10	7	6	3	4	5	6	5
b	20	17	14	13	10	9	6	3	4	5	6

Synchronized Fragments

Normalization implies $\mathrm{ed}(X, Y) \leq \mathrm{ed}^w(X, Y)$, so we build an optimal unweighted alignment.



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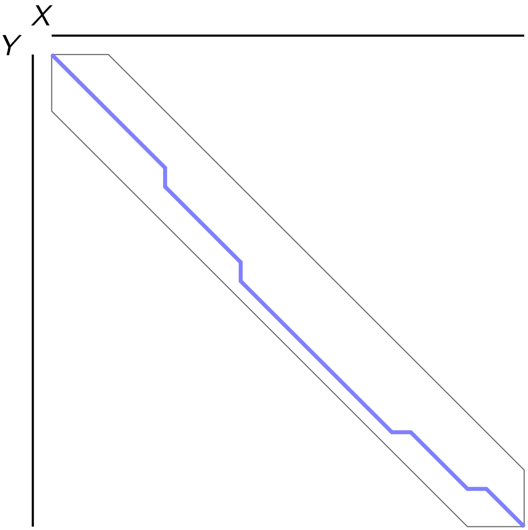
Normalization implies $\mathrm{ed}(X, Y) \le \mathrm{ed}^w(X, Y)$, so we build an optimal unweighted alignment.

The alignment decomposes X and Y into $\mathcal{O}(k)$ characters and synchronized fragments:

Synchronized fragments:

Two fragments $X[x \mathinner{.\,.} x')$ and $Y[y \mathinner{.\,.} y')$ are **k-synchronized** if

1. $|x - y| \le k$ and
2. $X[x \mathinner{.\,.} x') = Y[y \mathinner{.\,.} y')$.



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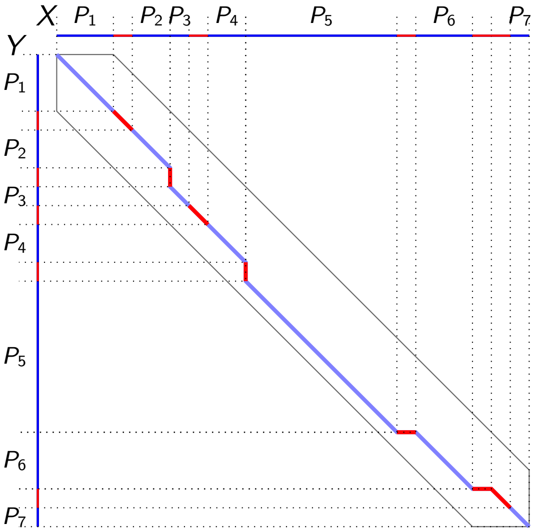
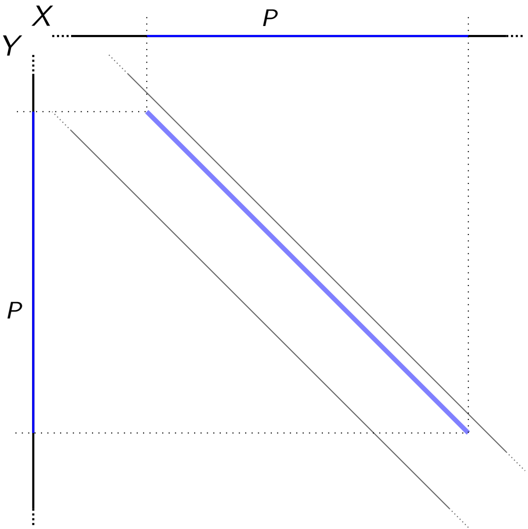


Synchronized Fragments vs Optimal Alignments

How can an optimal weighted alignment interact with synchronized fragments?



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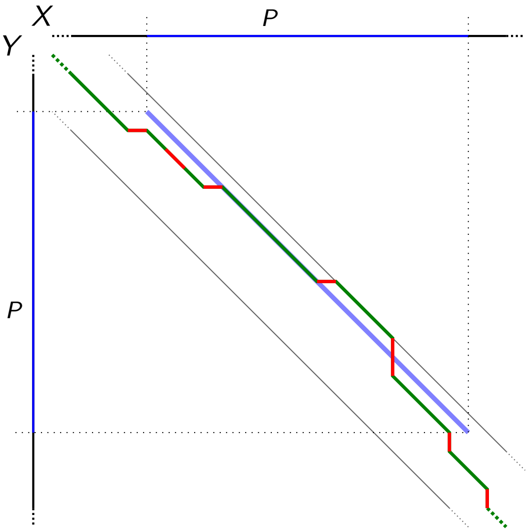

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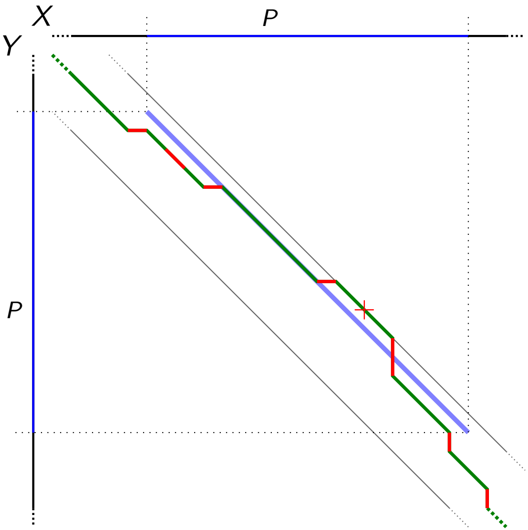

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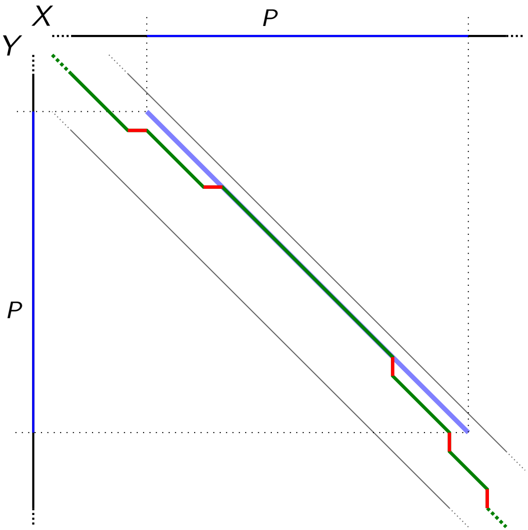
How can an optimal weighted alignment interact with synchronized fragments?

- The alignment touches the corresponding segment at most once.

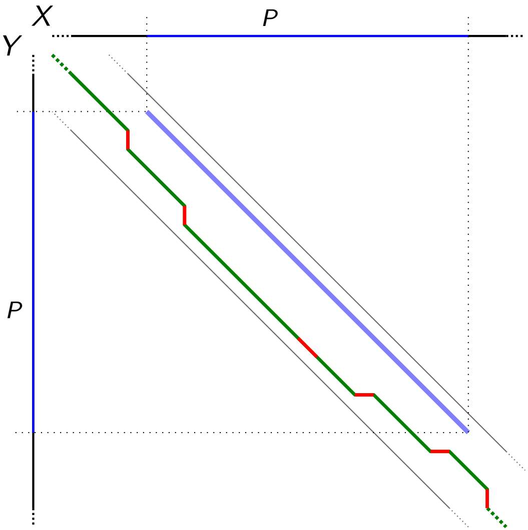


Synchronized Fragments vs Optimal Alignments

How can an optimal weighted alignment interact with synchronized fragments?

- The alignment touches the corresponding segment at most once.

What if the alignment never touches the segment?



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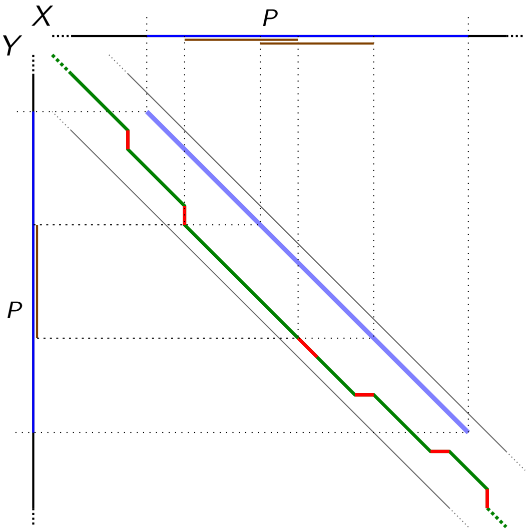
- Each piece of Y matched perfectly occurs in X twice, $\leq 2k$ positions apart.



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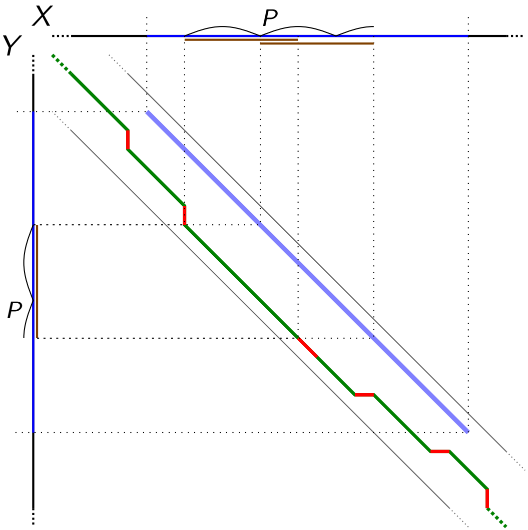


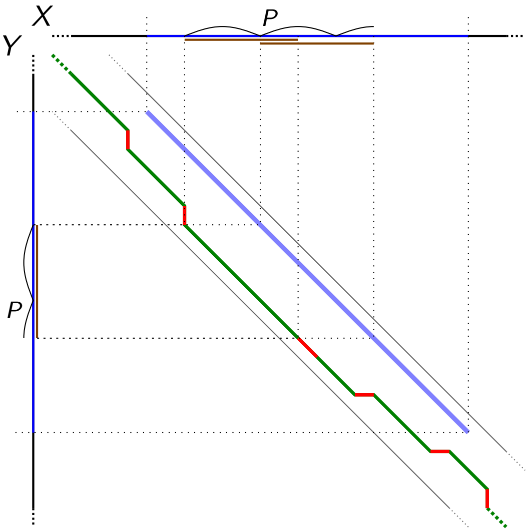
Synchronized Fragments vs Optimal Alignments

How can an optimal weighted alignment interact with synchronized fragments?

- The alignment touches the corresponding segment at most once.

What if the alignment never touches the segment?

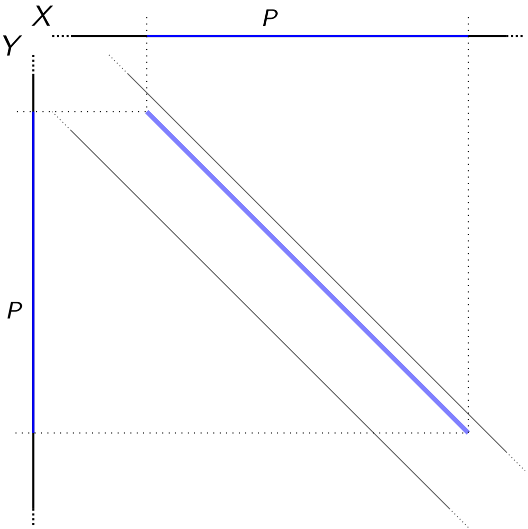
- Each piece of Y matched perfectly occurs in X twice, $\leq 2k$ positions apart.
- The synchronized fragments consist of $\leq 3k$ pieces with **periods** $\leq 2k$.



First Reduction

Lemma

An *optimal weighted alignment* *matches* *synchronized fragments*, *except for a prefix and a suffix of* $\le 3k$ *pieces* *with periods* $\le 2k$.



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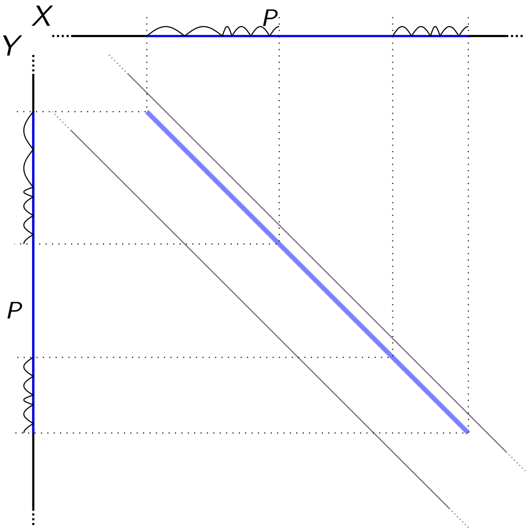

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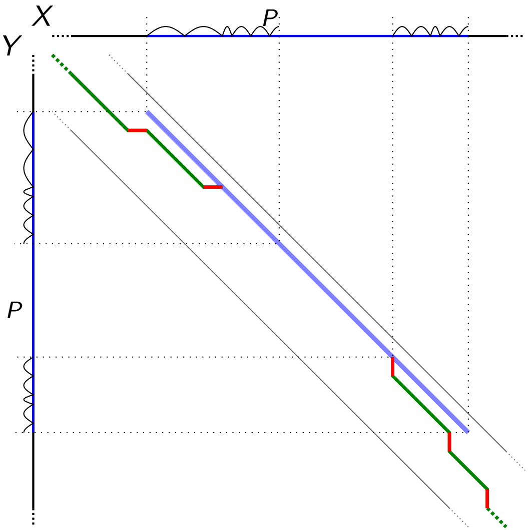

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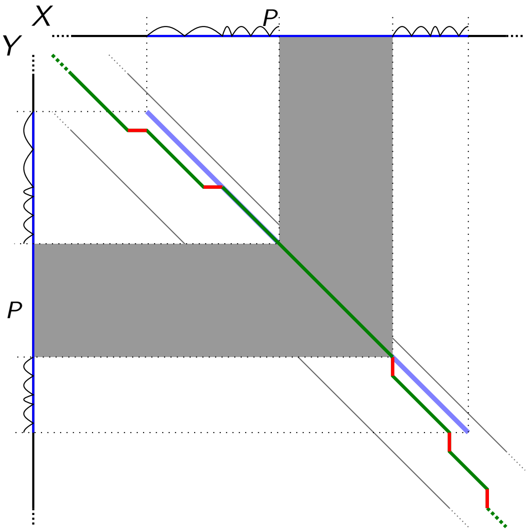
An optimal weighted alignment matches synchronized fragments, except for a prefix and a suffix of $\leq 3k$ pieces with periods $\leq 2k$.

The characters in the middle, if any, can be removed without affecting

$$\mathrm{ed}^w_{\leq k}(X,Y) = \begin{cases} \mathrm{ed}^w(X,Y) & \text{if } \mathrm{ed}^w(X,Y) \leq k, \\ \infty & \text{otherwise.} \end{cases}$$



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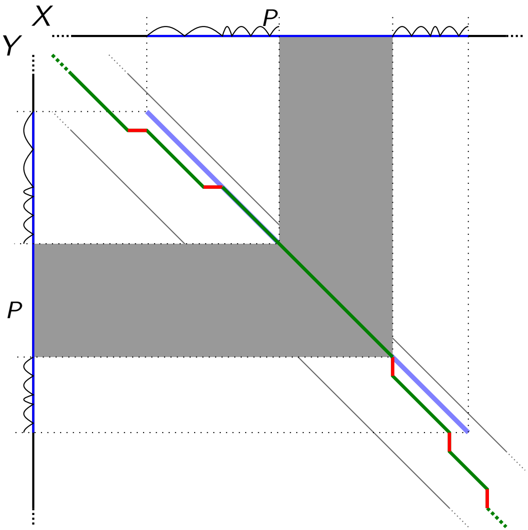
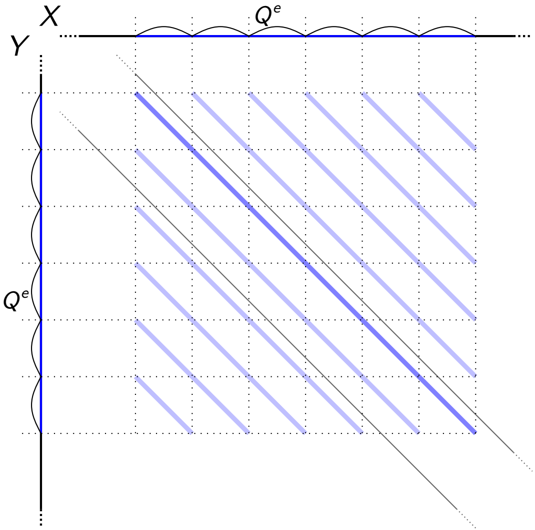
We can assume that X, Y can be decomposed into $\mathcal{O}(k^2)$ characters and synchronized fragments with periods $\leq 2k$.



Periodic Synchronized Fragments vs Optimal Alignments

How can an optimal weighted alignment interact with synchronized occurrences of Q^e?



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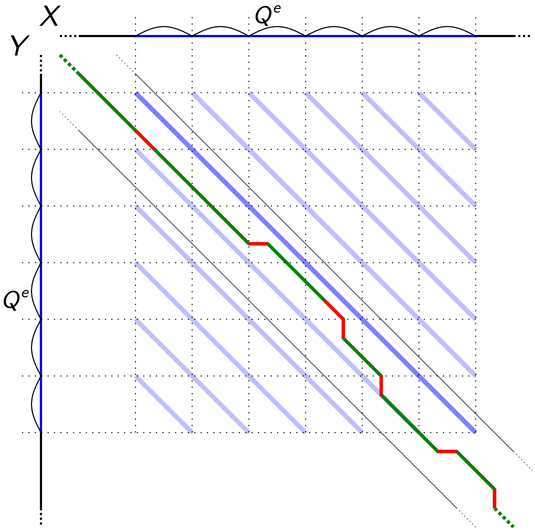

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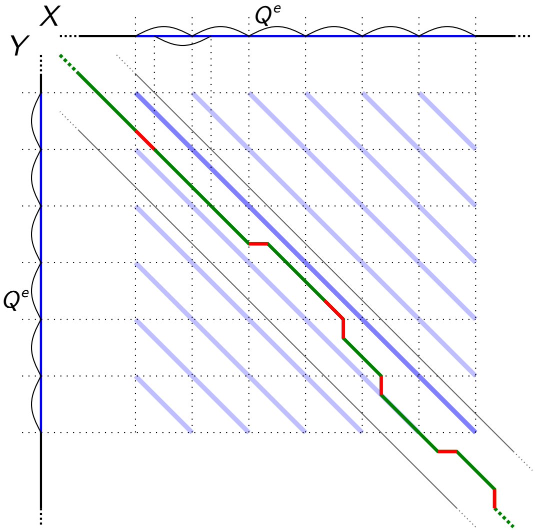

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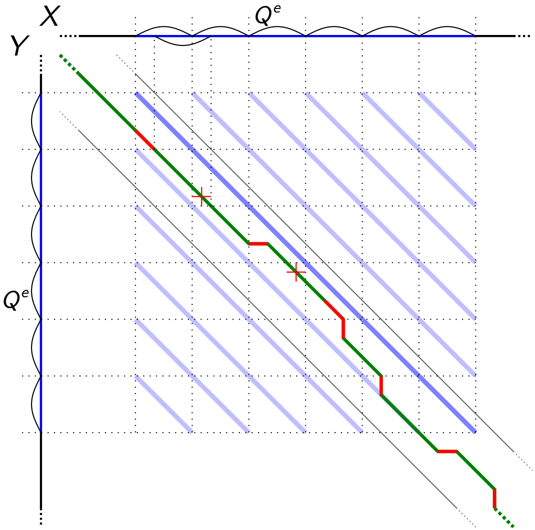

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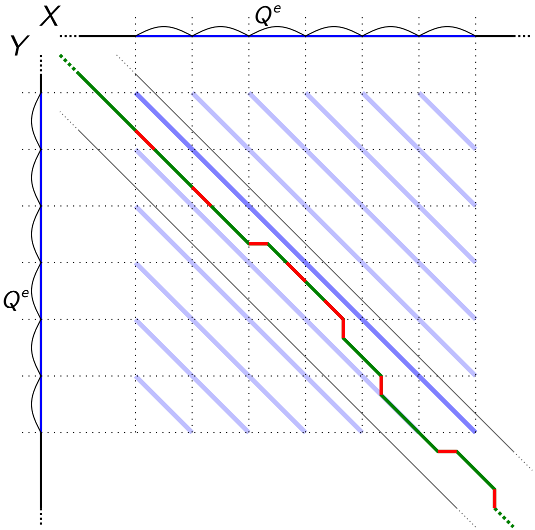

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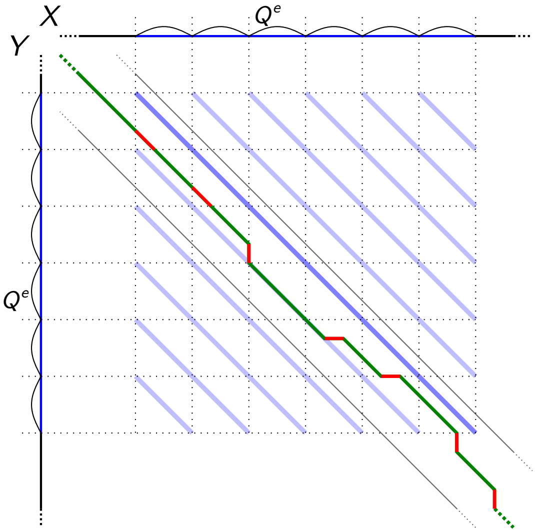

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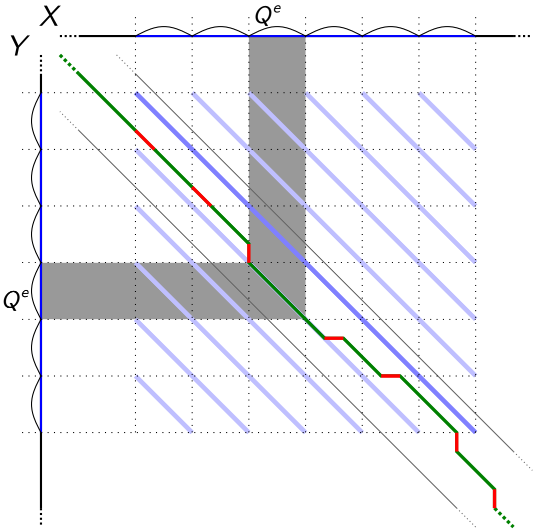
If $e > 4k$, we can **reduce the exponent** to $4k$ without affecting $\mathrm{ed}^w_{\leq k}(X, Y)$.



Universal Kernel

Theorem (DGHK**S, STOC'23)**

There is an $\mathcal{O}(n)$ *time algorithm that, given strings* $X, Y \in \Sigma^{\leq n}$ *and an integer* $k > 0$*, constructs strings* X', Y' *of length* $\mathcal{O}(k^4)$ *such that* $\mathrm{ed}^w_{\leq k}(X', Y') = \mathrm{ed}^w_{\leq k}(X, Y)$ *holds for every normalized weight function* $w : (\Sigma \cup \{\varepsilon\})^2 \to \mathbb{R}_{\geq 0}$.

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1. Use the optimal unweighted alignment to decompose X and Y into $\mathcal{O}(k)$ characters and synchronized fragments.
2. For each pair of synchronized fragments:
   1. Find the longest prefix of $3k$ pieces with periods $\le 2k$.
   2. Find the longest suffix of $3k$ pieces with periods $\le 2k$.
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$\mathcal{O}(k^4)$-size kernel $+$ $\mathcal{O}(nk)$-time dynamic-programming $\rightsquigarrow$ $\mathcal{O}(n + k^5)$-time algorithm

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Thank you for your attention!