ADFOCS 2024, MPI Summer School
Exercise Set: Prophet Inequalities

Paul Dütting

August 2024

Morning Sessions

Question 1. Consider the problem of designing a posted-price mechanism for n buyers and a single item. Suppose that the values $v_1, \ldots, v_n$ are drawn from $U[0,1]$ (uniform distribution on the $[0,1]$ interval). A posted-price mechanism sets a sequence of thresholds $\tau_1, \ldots, \tau_n$ and accepts v_i, conditionally on reaching step i, if $v_i \geq \tau_i$. Consider a strategic buyer, that only buys the item if it's in her best interest. That is, conditionally on reaching step i, assume that buyer i buys the item if $v_i \geq \tau_i$. The welfare of such a mechanism is v_i if buyer i buys the item. The revenue in this case is τ_i.

Note: Our definition of a posted-price mechanism is more general than the definition of a threshold algorithm from the first lecture.

(a) Considering the goal of maximizing expected welfare, compute the expected welfare the prophet can achieve for $n = 1, 2, 3, 4$ buyers.

(b) Considering the goal of maximizing expected revenue, compute the optimal thresholds and corresponding expected revenue for $n = 1, 2, 3$ buyers.

(c) Prove that, for any $n \geq 1$, the first n thresholds that are optimal for the welfare objective with $n+1$ buyers coincide with the first n thresholds that are optimal for the revenue objective with one less buyer (i.e., with n buyers).

(d) (Exploratory) Find another distribution, that is a distribution that is different from the uniform distribution, which satisfies the property established in (c). Characterize / identify a class of distributions with this property.

Question 2. Consider the online contention resolution scheme (OCRS) approach introduced and discussed in the lecture. Specifically, consider the case where $v_i = x_i$ with probability 0 and $v_i = 0$ otherwise. Recall the ex-ante relaxation:

$$
\begin{aligned}
\text{maximize} \quad & \sum_{i \in [n]} y_i \cdot x_i \\
\text{subject to} \quad & \sum_{i \in [n]} y_i \leq 1 \qquad \forall i \in [n] \\
& y_i \in [0, p_i] \qquad \forall i \in [n].
\end{aligned}
$$

(a) Show that the ex-ante relaxation is truly a relaxation of the prophet's problem by giving a concrete example in which the objective value of this LP is strictly larger than the expected value achievable by the prophet.

(b) Let $y_i^\star$ for $i \in [n]$ be an optimal solution to the ex-ante relation. Show that the algorithm, which conditioned on arriving in step i and $v_i = x_i$, accepts v_i with probability $y_i^\star/2$ yields a 4-competitive prophet inequality.

Afternoon Sessions

Question 3. Consider the online combinatorial auction problem with n buyers and m items. Suppose that each buyer $i \in [n]$ has a valuation function $v_i : 2^{[m]} \to \mathbb{R}_+$ drawn independently from a known distribution $\mathcal{D}_i$.

(a) Show that any fractionally subadditive (XOS) valuation function $v_i : 2^{[m]} \to \mathbb{R}_+$ admits balanced prices.

Next consider the following relaxation of balanced prices. A valuation function $v_i : 2^{[m]} \to \mathbb{R}_+$ admits (α, β)-balanced prices if for every set of items $U \subseteq [m]$ there exists item prices p_j for $j \in U$ such that for all $T \subseteq U$:

1. $\sum_{j \in T} p_j \geq \frac{1}{\alpha} \cdot (v_i(U) - v_i(U \setminus T))$, and
2. $\sum_{j \in U \setminus T} p_j \leq \beta \cdot v_i(U \setminus T)$.

As shown in [Dütting et al. 2017], the existence of (α, β)-balanced prices implies the existence of a $(1 + \alpha \cdot \beta)$-competitive prophet inequality. This prophet inequality is attained by a posted-price mechanism that uses (static, anonymous) item pricing.

For the following question you may use, without proof, that for any subadditive valuation function $v_i : 2^{[m]} \to \mathbb{R}_+$ and any set $T \subseteq [m]$ there exists an additive function $a_i : 2^T \to \mathbb{R}_+$ such that

(i) $\sum_{j \in S} a_i(j) \leq v_i(S)$ for all $S \subseteq T$, and

(ii) $\sum_{j \in T} a_i(j) \geq \frac{1}{\gamma} v_i(T)$, where $\gamma = \mathcal{H}_m = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m} \in O(\log m)$.

(b) Use these ingredients to show that subadditive valuations admit a $O(\log m)$-approximate prophet inequality via (static, anonymous) item pricing.

Question 4. Consider the problem of designing an online algorithm that establishes a prophet inequality for the setting where we face a sequence of n independent draws from a single distribution $\mathcal{D}$, when we have access to $n-1$ independent samples $s_1, \ldots, s_{n-1}$ from distribution $\mathcal{D}$. Suppose we set a single threshold equal to $\tau = \max\{s_1, \ldots, s_{n-1}\}$, and accept the first value v_i such that $v_i \geq \tau$.

(a) Show that for any $i \in [n]$, conditional on stopping at step i, the expected value collected by the algorithm is at least $\mathbb{E}[\max\{v_1, \ldots, v_n\}]$.

(b) Show that the probability with which we select some value (stop at some step i, with $i \leq n$) is at least $1/2$.

(c) Use the assertions in (a) and (b) to argue that this algorithm is 2-competitive against the prophet benchmark.

Question 5. (Open, presumably hard) Show or disprove that the optimal competitive ratio for the free order prophet inequality problem is equal to that of the prophet inequality problem with identical distributions.