Problem sets ADFOCS 2024

Topics in mechanism design

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Set 1

1. Consider the problem of related machines scheduling in which machines are agents with private values for their speeds $s_1, \ldots, s_n$, and we want to minimize the makespan. Consider the mechanism that selects an arbitrary optimal solution. Is this mechanism truthful for 2 machines? For 3 machines?

   Solution.

   A mechanism is truthful if the allocation is monotone: if the speed of a machine decreases, the machine will not get a larger load.

   For 2 machines the mechanism is monotone, by simple case analysis.

   For 3 machines the mechanism is not truthful. Example: the machines have speed 1,2,10 and the tasks are 1,2,100. An optimal allocation is to allocate the i-th task to the i-th machine, for $i = 1, 2, 3$. Furthermore, the allocation remains optimal even if agent 1 gets the first two tasks. Now when the speed of machine 1 is reduced and the mechanism switches from the first allocation to the second allocation, it is not monotone for agent 1.

2. Give a polynomial-time truthful deterministic mechanism with an approximation ratio 3/2 for the related scheduling problem with 2 machines. Recall that in the related scheduling problem there are two machines with speeds (s_1, s_2), the input is a set of jobs $\{w_1, \ldots, w_n\}$ and we want a monotone mechanism to minimize the makespan.

   Solution.

   The following is a monotone scheduling algorithm. By symmetry, assume that $s_1 \leq s_2$:

   - If $s_2/s_1 \leq 2$, the algorithm partitions the jobs into almost equal loads — this can be done by a PTAS. It then allocates the smaller load to the first machine and the larger one to the second machine.
   - If $s_2/s_1 > 2$, it assigns all jobs to the second machine.

   We can argue by case analysis that the algorithm is monotone, i.e., if a machine decreases its speed, it will not get a higher load. For example, if s_1 decreases and the allocation switches from the first case to the second case, the load of the machine decreases (it changes from a positive value to 0).

   We now argue that the approximation ratio is at most 3/2. Assume without loss of generality that the optimum makespan is 1, so the total load is at most $s_1 + s_2$. In the first case, the approximation ratio is at most $\frac{s_1+s_2}{2} \cdot \frac{1}{s_1} \leq \frac{1+s_2/s_1}{2} \leq 3/2$. In the second, the approximation ratio is $(s_1 + s_2) \cdot \frac{1}{s_2} \leq 1 + \frac{s_1}{s_2} \leq 3/2$.

3. Let's consider a two-sided auction. There are two agents: a seller and a buyer with valuations (v_1, v_2), respectively, for a single item. Assume that agent 1 owns the item.

   We want a mechanism that trades the item, i.e., it decides whether to leave the item to agent 1 or move it to agent 2.

   Describe the VCG mechanism with Clarke pivot rule for this setting. What are the payments? Does the mechanism have the "budget-balance" property?

   Analyze the situation when there is another buyer with value v_3.

   Solution.

   If $v_1 \geq v_2$, there is no trade and the payments are 0. Otherwise, the mechanism will give the item to agent 2. Agent 2 should pay v_1 (their additional contribution to the social welfare is $v_2 - v_1$) and agent 1 should receive v_2 (their presence increases the social welfare from 0 to v_2). This does not have the budget-balance property.

   If there is a third agent, with $v_1 \leq v_2 \leq v_3$, the third agent will get the item, but she will pay v_2 (as in the second-price auction). Similarly for the other cases.

4. Consider the case of a simple item to be allocated to one of n agents with nonnegative values $v_1, \ldots, v_n$. A mechanism is truthful for an agent i if and only if their payment p_i depends only on the values v_{-i} of the other agents.

   (a) Suppose that $p_i(v_{-i})$ are arbitrary functions. Explain why this mechanism may not be valid.

   (b) Suppose that there are only two agents and that the mechanism is defined by a function $p_1(v_2)$ as follows: if $v_1 \geq p_1(v_2)$, the item is allocated to agent 1, otherwise it is allocated to agent 2. What property must $p_1(v_2)$ satisfy for the mechanism to be truthful?

   (c) Let's generalize the mechanism to multiple agents: there are payment functions $p_1(v_{-1})$,. . ., $p_{n-1}(v_{-(n-1)})$, $p_n(v_{-n}) = 0$ and the mechanism allocates the item to the first agent i with $v_i \geq p_i(v_{-i})$. For which functions $p_i(v_{-i})$ is the mechanism truthful?

   Solution.

   (a) The mechanism is not valid because it may give the item to multiple agents.

   (b) The allocation is always valid. Agent 1 is truthful for every $p_1(v_2)$. The allocation to agent 2 is monotone — and therefore truthful — if and only if $p_1(v_2)$ is non-increasing in v_2.

   (c) Similarly for many agents, the mechanism is truthful if and only if $p_i(v_{-i})$ is non-increasing in v_j for $i < j$.

5. Consider the following mechanism for selling $k > 1$ identical items to unit-demand bidders. The bidders with the k highest bids get an item and pay as follows: The highest bidder pays the second highest bid, the second highest bidder pays the third highest bid, etc. Is this mechanism truthful for all bidders?

   Solution.

Not in general. The bidders with the highest $k-1$ bids have a reason to declare a value slightly above the $(k+1)$-st bid. There is an extensive literature about a variant of this untruthful auction, called GSP (generalized second price auction). Google was using this auctions for ad placement.

Set 2

1. Consider the scheduling problem with two unrelated machines and two tasks. VCG for this setting tries to minimize the social cost, i.e., it computes

$$\arg\min(t_{1,1}+t_{1,2},\ t_{1,1}+t_{2,2},\ t_{2,1}+t_{1,2},\ t_{2,1}+t_{2,2}),$$

and gives the two tasks accordingly (for example, if the minimum comes from the third value $t_{2,1}+t_{1,2}$, machine 1 gets the second task and machine 2 gets the first task). Now consider changing the first expression from $t_{1,1}+t_{1,2}$ to $t_{1,1}+t_{1,2}-1$. Argue that this mechanism is an affine minimizer. Show the partition of the space into allocations of the first machine, when the second machine has values $t_2=(3,2)$. Show that this mechanism has unbounded approximation ratio, when the objective is the makespan.

Solution.

The mechanism is an affine minimizer because it only changes a γ value (see the definition of affine minimizers). To show that the approximation ratio is unbounded consider the input

$$t_1=(1-\epsilon,0), t_2=(0,0),$$

for some $\epsilon>0$. The mechanism will allocate both tasks to machine 1, with makespan $1-\epsilon$, while the optimal makespan is 0.

2. Consider the scheduling problem with two unrelated machines and two tasks. Consider an affine minimizer with $\lambda_i=1$ for $i=1,2$. What are the conditions on payments (or equivalently on γ's in the definition of affine minimizers) so that the mechanism is quasi-bundling? quasi-flipping? task-independent?

Solution.

The expression

$$x=p_1((0,0),t_2)+p_1((1,1),t_2)-p_1((0,1),t_2)-p_1((1,0),t_2)=\gamma(0,0)+\gamma(1,1)-\gamma(0,1)-\gamma(1,0)$$

determines whether the mechanism is quasi-bundling ($x>0$), quasi-flipping ($x<0$) or task-independent ($x=0$). Interestingly, the property (quasi-bundling / quasi-flipping / task-independent mechanism) does not depend on the values of t_2.

3. Analyze the approximation ratio of the Hybrid mechanism for a star of 2 leaves.

   Can you suggest another truthful mechanism with better approximation ratio?

Solution.

The approximation ratio of the Hybrid mechanism is 2. Consider values on the edges $(1-\epsilon, 1)$ and $(1, 2-\epsilon)$, where the first value is the value of the root and ϵ is a small positive value. The optimal allocation is 1 (first edge to leaf and second edge to root), while Hybrid allocates both edges to the leaves, with makespan $2-\epsilon$. As ϵ tends to 0, the approximation ratio tends to 2.

To get a better mechanism, we generalize the Hybrid mechanism as follows. Instead of considering the maximum value of the leaves, we consider a weighted maximum. Specifically, assume that the values of the leaves are $l_1 \leq l_2 \leq \cdots \leq l_m$ (here we consider stars with m leaves), and the corresponding values of the root are $r_1, \ldots, r_m$. The weighted Hybrid mechanism with weights $\alpha_1, \alpha_2, \ldots$, allocates a set $T = \{1, \ldots, k\}$ of edges to the leaves where k belongs to

$$\arg \min_{0 \leq k \leq m} \sum_{j>k} r_j + \alpha_k l_k,$$

where by convention $l_0 = 0$. The (unweighted) Hybrid mechanism has $\alpha_k = 1$ for all k.

If the sequence $\alpha_1, \alpha_2, \ldots$ is nonnegative and non-decreasing, the mechanism is truthful. For two leaves, it achieves an approximation ratio of $1.618\ldots$ (equal to the golden ratio).

4. Recall the SQUARE mechanism, which is a fractional task-independent mechanism that allocates fractions inversely proportional to the square of the values. Prove that it has approximation ratio $(n+1)/2$.

Solution.

See paper: George Christodoulou, Elias Koutsoupias, Annamária Kovács: "Mechanism design for fractional scheduling on unrelated machines." ACM Trans. Algorithms 6(2): 38:1-38:18 (2010).